# Quantum representations of mapping class groups

Progress report

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March 13, 2012

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# Introduction

**Note**: This is the online version of my progress report, compiled March 13, 2012, which differs slightly from the version I handed in May 27, 2011. Feel very free to comment on the contents and to write me about misprints and mistakes. The newest version of this document is available at http://home.imf.au.dk/pred.

This progress report is the culmination of my first two years as a PhD scholar at Aarhus Graduate School of Science, studying under supervision of Jørgen Ellegaard Andersen at the Centre for Quantum Geometry of Moduli Spaces. In essence, it accounts for the progress of my PhD project, entitled *Quantum representations of mapping class groups*, by introducing much of the relevant background material that I have picked up during these two years, and by explaining concretely in which direction the project is heading.

The ultimate aim of the project is to combine techniques from the two somewhat disjoint areas of mathematics that are gauge theory and quantum topology, and as such my attempt to create a self-contained introduction to both – as well as to my own work in progress – has resulted in this perhaps rather lengthy document. Yet, it is naive to provide a full treatment of the subjects at hand at any reasonable[1] length, and I should note for anyone picking up this progress report that some prior knowledge of knot and surgery theory, complex differential geometry, and the theory of vector bundles on manifolds will come in handy.

That being said, it is my hope that the progress report provides all necessary references for obtaining a further understanding of the theory, itself outlining the relevant mathematical context and explaining the notions central for understanding the present PhD project.

The report is structured as follows: It is split into four chapters, the first three of which contain known results and constructions relevant for our own studies. New results have all been collected in the final chapter.

In **Chapter 1**, we give an elementary description of the mapping class groups of surfaces. The Dehn–Lickorish theorem tells us that these groups are generated by certain mapping classes called Dehn twists, which will be of particular importance to us in the final chapter of the report. These will be discussed in more detail, and we review several of their algebraic properties. We end the chapter by introducing a classification of mapping classes due to Nielsen and Thurston which will form the basis of one part of the project.

We begin **Chapter 2** by giving a brief outline of gauge theory and the study of connections in principal bundles over manifolds, mainly in order to fix the notation used for later sections. We then turn to the concept of geometric

[1] "Reasonable" referring here to the AGSoS progress report guide lines.

quantization and discuss in particular the part of it known as prequantization. Whereas prequantization can be discussed in the general framework of symplectic geometry, we will be focussing primarily on the example of the moduli space of flat $\mathrm{SU}(n)$-connections in a principal bundle over a surface and explain how this space is prequantizable through a discussion of Chern–Simons theory.

**Chapter 3** contains the main theoretical bulk of the report. We begin by very briefly relating the project at hand to mathematical physics and topological quantum field theory. The latter term is due to Witten who first constructed such a theory using mathematically ill-defined path integrals. From a mathematical point of view, this is of course a major defect, which mathematicians have sought to remedy for decades.

A topological quantum field theory (in $2+1$ dimensions) consists, in essence, of a topological invariant of 3-manifolds which from a physical point of view arises as the partition function of a quantum field theory. Furthermore, it immediately gives rise to finite-dimensional representations of mapping class groups, which are exactly the quantum representations we seek to study.

We provide an axiomatic mathematically precise description of topological quantum field theory and turn to two concrete constructions of such. The first of these is due to Reshetikhin and Turaev who – using quantum groups and the method of surgery along links in 3-manifolds – gave the historically first mathematical definition of the desired 3-manifold invariant. We will see how this fits into the general theory of modular categories. The second construction we will describe is due to Blanchet, Habegger, Masbaum and Vogel and is more combinatorial in nature and more suited for our purposes; concrete calculations of the quantum representations arising from this theory turn out to be very feasible, and so we describe various features of the theory in greater detail.

Finally, in **Chapter 4**, we collect the various constructions made throughout the report and explain in detail how to construct from them quantum representations of mapping class groups. As a warm-up, we describe some of the main properties of the representations and put up several unsolved conjectures involving the algebraic properties of the representations as well as their connection with geometric quantization of moduli spaces and mathematical physics. Most of our attention is devoted to understanding the representations of Dehn twists on a torus, the Chern–Simons theory of the associated mapping torus, its associated moduli space and the semi-classical behaviour of the partition function. Main new results are Proposition 4.7, Theorem 4.20, Corollary 4.22, and Proposition 4.24. Most of what is contained in this chapter is work having only just begun, and we end the report by giving a breakdown of future perspectives and expectations for the PhD project.

Let me end this introduction by first of all thanking my supervisor Jørgen Ellegaard Andersen for presenting to me all of the problems discussed in the report as well as for being a great source of motivation and keeping up with all of the questions and problems I have come across in the process. Let me also thank all of my fellow students at the Centre for Quantum Geometry for Moduli Spaces for many good discussions on our various projects, and in particular Jens-Jakob Kratmann Nissen and Jens Kristian Egsgaard for carefully proofreading this report.

Aarhus, May 2011 Søren Fuglede Jørgensen

# Contents

# Chapter 1

# Mapping class groups

## 1.1 Definition and preliminary remarks

The main algebraic object under scrutiny in this progress report is the mapping class group of a surface. Very roughly, one thinks of the mapping class group as the group of symmetries of a given surface. Let $\Sigma = \Sigma_{g,n}$ be a compact surface of genus $g \geq 0$ with $n \geq 0$ boundary components. Let $\operatorname{Homeo}(\Sigma, \partial\Sigma)$ be the group of orientation-preserving homeomorphisms restricting to the identity on $\partial\Sigma$, and let $\operatorname{Homeo}_0(\Sigma, \partial\Sigma)$ denote the normal subgroup of those homeomorphisms that are isotopic (i.e. homotopic through homeomorphisms relative to the boundary) to the identity. The *mapping class group* of $\Sigma$ is the quotient

$$\Gamma(\Sigma) = \operatorname{Homeo}(\Sigma, \partial\Sigma) / \operatorname{Homeo}_0(\Sigma, \partial\Sigma),$$

or, equivalently, $\Gamma(\Sigma) = \pi_0(\operatorname{Homeo}(\Sigma, \partial\Sigma))$. The class of a homeomorphism in $\Gamma(\Sigma)$ is called its *mapping class*. Obviously, homeomorphic surfaces have isomorphic mapping class groups, and we will often simply write $\Gamma_{g,n} = \Gamma(\Sigma)$. Also, we will write $\Gamma_g = \Gamma_{g,0}$.

Several variations on this theme exist. It is common to define the mapping class group of a surface as the group $\pi_0(\operatorname{Diff}(\Sigma, \partial\Sigma))$ of orientation-preserving diffeomorphisms of $\Sigma$ rather than homeomorphisms. It is a non-trivial fact that any homeomorphism is isotopic to a diffeomorphism, and that isotopy can be replaced by smooth isotopy, so we obtain an isomorphic group, and we will use the two interchangeably.

We will also occasionally be considering surfaces $\Sigma_{g,n}^m$ with $m \geq 0$ punctures, i.e. $m$ points removed from the interior of the surface, and consider homeomorphisms of the resulting non-compact surface. Equivalently, one could consider the surface with a set of $m \geq 0$ marked points, and require that homeomorphisms and isotopies fix this set. The resulting mapping class group will be denoted $\Gamma_{g,n}^m$. Similarly, some definitions ease the condition on the behaviour on the boundary and consider instead homeomorphisms and isotopies preserving the boundary setwise rather than pointwise.

## 1.2 Examples and generators

A guiding example in what follows will be the closed torus $\Sigma_1$. Homeomorphisms of the torus act by determinant 1 automorphisms on the first homology

$H_1(\Sigma_1, \mathbb{Z}) \cong \mathbb{Z}^2$ of the torus. In fact, any element $M$ of $\mathrm{SL}(2,\mathbb{Z})$ defines a homeomorphism of the torus, viewed as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, whose action on homology is exactly $M$. Likewise, it follows from general $K(G,1)$ theory, that any such homomorphism arises from a (based) map on the torus, unique up to homotopy. We thus obtain the following (and refer to [FM11, Thm. 2.5] for the details).

**Theorem 1.1.** *The homomorphism $\Gamma_1 \to \mathrm{SL}(2,\mathbb{Z})$ given by the action on homology is an isomorphism.*

*Remark* 1.2. One might conjecture that general mapping class groups are linear, i.e. admit injective representations, as in the torus case. The case of $\Gamma_2$ was handled in [BB01], where an explicit 64-dimensional representation is constructed, but the same question for higher genus (closed) surfaces is still open.

### 1.2.1 Dehn twists

The most important class of examples of mapping classes for our purpose are the Dehn twists about simple closed curves, which we intuitively think of as obtained by cutting the surface along a curve, giving one of the resulting boundary components a $2\pi$ left twist, and gluing the boundary components back together (see Figure 1.1 for a Dehn twist on the closed torus). More precisely, consider the annulus $A = S^1 \times [0,1]$ considered as an oriented surface in $\mathbb{R}^2$ via the map $(\theta, r) \mapsto (\theta, r+1)$ with the orientation induced by the orientation of $\mathbb{R}^2$. Define a map $t : A \to A$ by

$$t(\theta, r) = (\theta + 2\pi r, r),$$

as illustrated in Figure 1.2. Now let $\gamma$ be a simple closed curve in an oriented surface $\Sigma$, and let $N$ be a regular neighbourhood of $\gamma$. Choose an orientation-preserving homeomorphism $\varphi : A \to N$, and define the *Dehn twist about* $\gamma$, denoted $t_\gamma : \Sigma \to \Sigma$, by $t_\gamma = \varphi \circ t \circ \varphi^{-1}$ on $N$, and $t_\gamma = \mathrm{id}$ on $\Sigma \setminus N$. This defines an orientation-preserving homeomorphism on $\Sigma$. The mapping class of $t_\gamma$ depends neither on the choice of regular neighbourhood, nor the homeomorphism $\varphi$. Furthermore, the mapping class is determined by the isotopy class of $\gamma$. If $a$ is the isotopy class of $\gamma$, we write $t_a$ for the resulting mapping class. We will often make a slight abuse of notation, writing $t_\gamma$ for the mapping class as well.

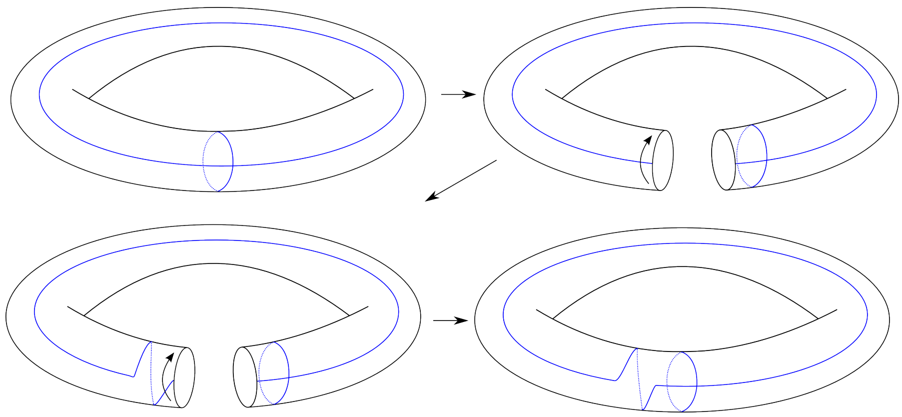

Figure 1.1: The action on two simple closed curves in a torus of the Dehn twist about a meridian.

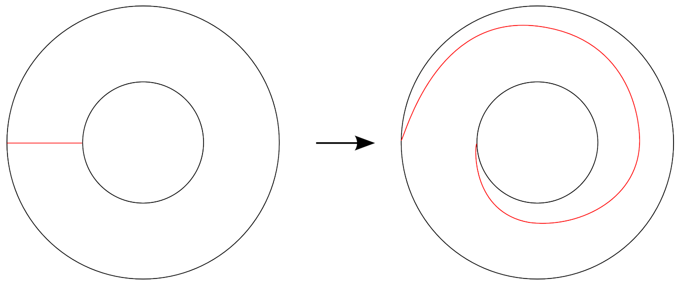

Figure 1.2: The action of the twist map $t : A \to A$ on a horizontal line in the annulus.

*Remark* 1.3. What we defined above is really a *left Dehn twist*. Similarly, one could have used the map $t : A \to A$ given by $t(\theta, r) = (\theta - 2\pi r, r)$ to obtain instead a *right Dehn twist*. The mapping class of the resulting homeomorphism would be the inverse to the one obtained above.

The importance of Dehn twists stems from the fact that they generate the mapping class groups. Before discussing exactly how, we note some of their algebraic properties. In the following, let $\Sigma$ be any surface. The *intersection number*, denoted $i(a, b)$, between two isotopy classes of curves $a$ and $b$ in $\Sigma$, is the minimal number of intersections between representative curves. A simple closed curve curve in $\Sigma$ is called *essential*, if it is not homotopic to a point, a boundary component or a marked point. We will need the following non-trivial fact. For a proof, see [FM11, Prop. 3.2].

**Proposition 1.4.** *Let $a$ and $b$ be isotopy classes of essential closed curves, and let $k \in \mathbb{Z}$. Then $i(t_a^k(b), b) = |k| i(a, b)^2$.*

Note that a Dehn twist about a simple closed curve homotopic to a point is trivial in the mapping class group. In general, however, Dehn twists are non-trivial:

**Corollary 1.5.** *Let $a$ be the isotopy class of a simple closed curve $\alpha$ not homotopic to a point or a puncture in $\Sigma$. Then $t_a$ has infinite order.*

*Proof.* By Proposition 1.4, it is enough to find an isotopy class $b$, such that $i(a, b) > 0$. Assume first that $\Sigma$ has no boundary components. Then this is possible by the so-called change of coordinates principle: It follows from the classification of surfaces that there is an orientation-preserving homeomorphism of $\Sigma$ taking one simple closed curve to another if and only if the two results of cutting the surface along the two curves will be homeomorphic surfaces. In other words, up to homeomorphism there is only one non-separating curve and finitely many separating ones, and we may assume that $\alpha$ is one of the curves in Fig. 1.3 (the separating curve might of course enclose more holes, punctures or boundary components). In both cases, the existence of the isotopy class $b$ is obvious. In the case where $\Sigma$ has boundary, using the same method as above, it remains to prove that Dehn twists about boundary components have infinite order – this is proved by a similar argument. □

In the following, let $\Gamma$ be the mapping class group of the surface $\Sigma$, and let $a$ and $b$ denote isotopy classes of simple closed curves in $\Sigma$.

**Lemma 1.6.** *If $t_a = t_b$, then $a = b$.*

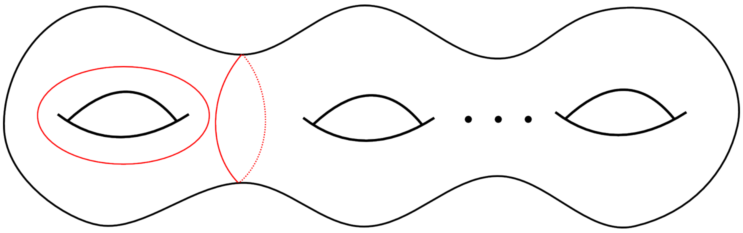

Figure 1.3: Using the change of coordinate principle to simplify $\alpha$.

*Proof.* Assume that $a \neq b$. Using the change of coordinates principle as above, considering all the various cases, it is possible to find an isotopy class $c$, such that $i(a, c) = 0$, $i(b, c) \neq 0$. By Proposition 1.4,

$$i(t_a(c), c) = i(a, c)^2 = 0 \neq i(b, c)^2 = i(t_b(c), c),$$

so $t_a(c) \neq t_b(c)$, and $t_a \neq t_b$. □

**Lemma 1.7.** *For $f \in \Gamma(\Sigma)$, we have $t_{f(a)} = f t_a f^{-1}$.*

Note here that when writing a product of mapping classes, we always apply them from right to left.

*Proof.* Let $\varphi$ be a representative of $f$, and let $\gamma$ a representative of $a$. Then $\varphi^{-1}$ takes a regular neighbourhood of $\varphi(\gamma)$ to a regular neighbourhood of $\gamma$. Using this neighbourhood to define $t_\gamma$, we obtain $t_{\varphi(\gamma)} = \varphi t_\gamma \varphi^{-1}$. □

**Lemma 1.8.** *Dehn twists about two simple closed curves commute if and only if the isotopy classes of the curves have zero intersection number.*

*Proof.* That Dehn twists about non-intersecting curves commute is obvious. It follows from Lemma 1.6 and Lemma 1.7 that a given mapping class $f$ commutes with a Dehn twist $t_a$, if and only if $f$ fixes $a$. Thus, if $t_a t_b = t_b t_a$ for isotopy classes $a$ and $b$ of simple closed curves, we obtain $t_a(b) = b$, and by Proposition 1.4 $i(a, b)^2 = i(t_a(b), b) = 0$. □

**Lemma 1.9** (Braid relation)**.** *If $i(a, b) = 1$ for isotopy classes of simple closed curves $a$ and $b$, then $t_a t_b t_a = t_b t_a t_b$.*

*Proof.* We prove first that $t_a t_b(a) = b$. By using the change of coordinates principle, we assume that $a$ and $b$ are represented by curves $\alpha$ and $\beta$ as in Figure 1.4 for which the equation is seen to hold by the sequence of mappings in the figure. It follows that $t_{t_a t_b(a)} = t_b$, and from Lemma 1.7, we obtain $(t_a t_b) t_a (t_a t_b)^{-1} = t_b$. □

We consider now the effect on the mapping class group of a surface when adding boundary components to it. When $\Sigma$ is a topologically closed subsurface of $\Sigma'$, we define a homomorphism $\Gamma(\Sigma) \to \Gamma(\Sigma')$ as follows: For a homeomorphism $\varphi \in \mathrm{Homeo}(\Sigma, \partial\Sigma)$ representing a mapping class $f \in \Gamma(\Sigma)$, we extend $\varphi$ to a homeomorphism $\varphi \in \mathrm{Homeo}(\Sigma', \partial\Sigma')$ by letting it act identically on $\Sigma' \setminus \Sigma$. The induced map $\Gamma(\Sigma) \to \Gamma(\Sigma')$ is well-defined.

In the special case where $\Sigma'$ is obtained from $\Sigma$ by *capping* a boundary component, that is, $\Sigma' \setminus \Sigma$ is a once-punctured disk with a boundary curve $\beta$, the resulting homomorphism $\Gamma(\Sigma) \to \Gamma(\Sigma')$ fits into a short exact sequence

$$1 \to \langle t_\beta \rangle \to \Gamma(\Sigma) \to \Gamma(\Sigma') \to 1.$$

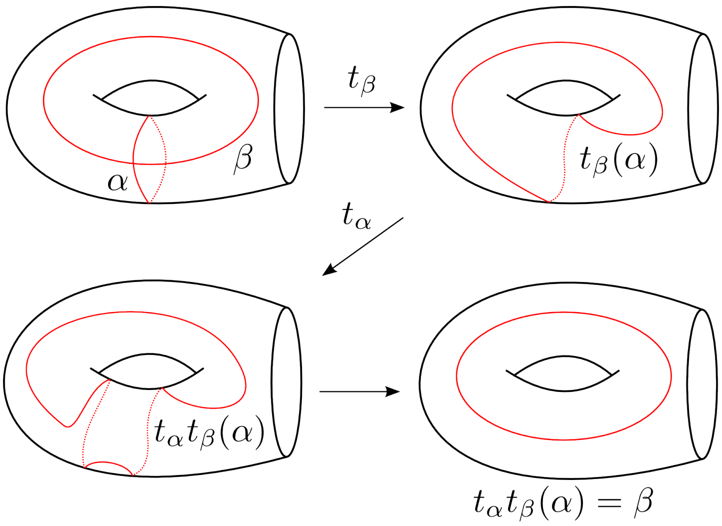


Figure 1.4: The curves $\alpha$ and $\beta$, and the equation $t_\alpha t_\beta(\alpha) = \beta$. The last arrow is a simple isotopy.

As an example, we consider again the torus. Using the so-called Alexander trick, one can prove that the mapping class group of the once punctured torus $\Sigma_{1,0}^1$ is once again given by its action on homology, so $\Gamma(\Sigma_{1,0}^1) \cong \mathrm{SL}(2,\mathbb{Z})$. As above, the mapping class group of the torus with one boundary component $\Sigma_{1,1}$ thus fits into the exact sequence

$$1 \to \mathbb{Z} \to \Gamma(\Sigma_{1,1}) \to \Gamma(\Sigma_{1,0}^1) \to 1.$$

We can describe this mapping class group as follows: Recall that $\mathrm{SL}(2,\mathbb{Z})$ has a presentation $\mathrm{SL}(2,\mathbb{Z}) \cong \langle a, b \mid aba = bab, (ab)^6 \rangle$, explicitly given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto a, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \mapsto b.$$

Consider the braid group on 3 strands, $B_3 \cong \langle a, b \mid aba = bab \rangle$. From the presentations, we get a homomorphism $B_3 \to \mathrm{SL}(2,\mathbb{Z})$ with kernel $\langle (ab)^6 \rangle \cong \mathbb{Z}$. There are maps $\mathrm{SL}(2,\mathbb{Z}) \to \Gamma(\Sigma_{1,0})$, $\mathrm{SL}(2,\mathbb{Z}) \to \Gamma(\Sigma_{1,0}^1)$, and $B_3 \to \Gamma(\Sigma_{1,1})$ given by mapping the generators $a, b$ to Dehn twists about meridian and longitude curves respectively. These fit into the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & B_3 & \longrightarrow & \mathrm{SL}(2,\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma(\Sigma_{1,1}) & \longrightarrow & \Gamma(\Sigma_{1,0}^1) & \longrightarrow & 1. \end{array}$$

Using the five-lemma, we obtain the following result:

**Proposition 1.10.** *The mapping class group of a torus with one boundary component is* $\Gamma(\Sigma_{1,1}) \cong B_3$.

### 1.2.2 Generators of mapping class groups

In the examples above, we saw that the mapping class groups could be generated by particular Dehn twists in the case of the closed torus, the once punctured torus and the torus with one boundary component. The case of the closed torus is a special case of the Dehn–Lickorish theorem (sometimes also called the Lickorish twist theorem).

**Theorem 1.11** (Dehn–Lickorish)**.** *For $g \geq 1$, the group $\Gamma_g$ is generated by $3g-1$ Dehn twists about nonseparating simple closed curves (see Figure 1.5).*

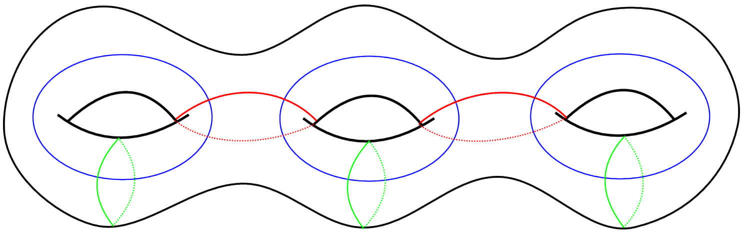

Figure 1.5: The $3g-1$ curves appearing in the Dehn–Lickorish theorem in the case $g = 3$.

In fact, Humphries [Hum79] has proved that the minimal (and realizable) number of Dehn twists required to generate $\Gamma_g$, $g > 1$, is $2g + 1$. In the case where the surface has boundary components, the picture changes but we remark that Dehn twists still generate the mapping class group.

Since we will need it later, we note that $\Gamma_2$ has the following presentation, due to Birman and Hilden, [BH73]:

$$\begin{aligned}\Gamma_2 \cong \langle a_1, \ldots, a_5 \,|\, & a_i a_j = a_j a_i, \ |i-j| \geq 2, \\ & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \\ & (a_1 a_2 a_3 a_4 a_5)^5 = 1, \\ & (a_1 \ldots a_5 a_5 \ldots a_1)^2 = 1, \\ & [a_1 \ldots a_5 a_5 \ldots a_1, a_1] = 1, \rangle\end{aligned}$$

As in the torus case, we can realize the generators $a_1, \ldots, a_5$ as Dehn twists about the five curves shown in Figure 1.6.

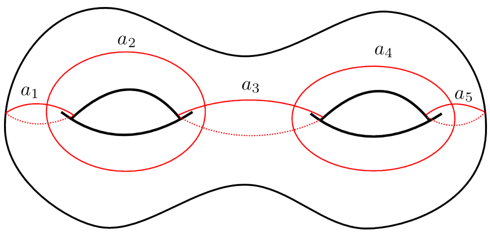


Figure 1.6: The Dehn twists generating $\Gamma_2$.

### 1.2.3 Finite order elements

Whereas non-trivial Dehn twists have infinite order in the mapping class group, we will also need to discuss finite order mapping classes. In the case of the torus, $\Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$, and examples of order 2, 3, 4, and 6 are

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

respectively. An example of an order 2 mapping class in $\Gamma_2$ is given by the element $a_1 \dots a_5 a_5 \dots a_1$ described above. More generally, the mapping class of the homeomorphism obtained by an angle $\pi$ rotation about the axis shown in Figure 1.7 has order 2 and is called *hyperelliptic involution* $H_g$. In fact,

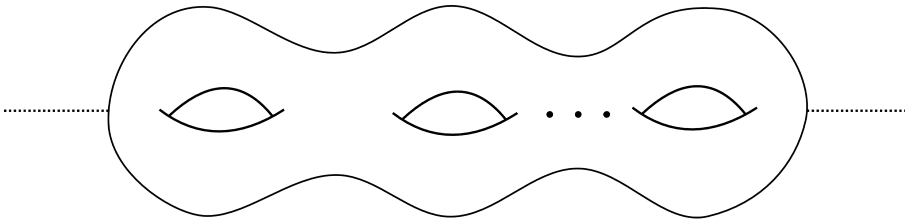

Figure 1.7: The hyperelliptic involution as a rotation of a surface.

hyperelliptic involutions are the only possible central elements of mapping class groups of closed surfaces. We have the following.

**Theorem 1.12.** *The center $Z(\Gamma_g)$ of $\Gamma_g$ is isomorphic to $\mathbb{Z}_2$ for $g = 1, 2$ and trivial otherwise.*

While we will not go into all details of the proof of this, one way of understanding this is as follows: As in the proof of Lemma 1.8, any central element will fix the isotopy class of every simple closed curve. A combinatorial argument using the so-called Alexander method shows that no non-trivial elements can do this when $g \geq 3$ and leaves $H_1$ and $H_2$ as the only possible central elements in $\Gamma_1$ and $\Gamma_2$. To prove that these two elements are in fact central, it suffices to check that they fix the isotopy classes of the curves giving the Dehn twists used to generate the respective mapping class groups in Theorem 1.11.

This theorem is no longer true when the surface is not closed. For example, the Dehn twist about a boundary component will always be central.

## 1.3 The Nielsen–Thurston classification

The examples considered above are in some sense the simplest. Finite order mapping classes can be realized by homeomorphisms of finite order, and while Dehn twists have infinite order, they will still have a simple action on certain curves (namely they fix the isotopy class of the curve used to define them). We end this chapter with a brief discussion of the Nielsen–Thurston classification and in particular we discuss the notion of a pseudo-Anosov homeomorphism. One of the main goals of the project at hand will be to analyze the behaviour of quantum representations under the trichotomy of the classification.

To define a pseudo-Anosov homeomorphism we need the notion of a transverse measure in a singular foliation.

**Definition 1.13.** A *singular foliation* on a surface $\Sigma$ is a decomposition of the surface into a disjoint union of *leaves* such that all but finitely many *singular points* in $\Sigma$ will have smooth charts $U \to \mathbb{R}^2$ taking leaves to horizontal lines. The singular points have smooth charts taking leaves to $k$-prong singularities as in Figure 1.8. Punctures are allowed to have 1-prong singularities as in Figure 1.9. We require also that every boundary component has at least one singularity, and that boundary components are unions of leaves connecting the singularities. Two singular foliations are called *transverse* if they have the same singular points and have transverse leaves at all other points (see Figure 1.10).

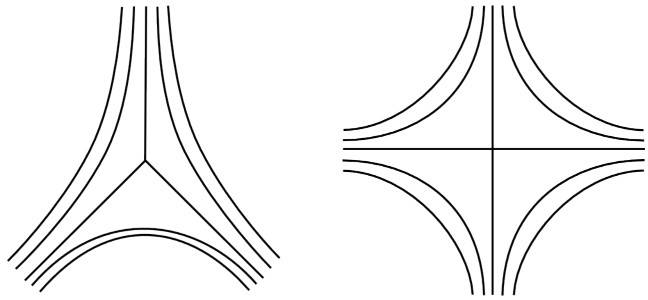

Figure 1.8: The $k$-prong singularities for $k = 3, 4$.

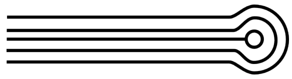

Figure 1.9: The 1-prong singularity for a punctured surface.

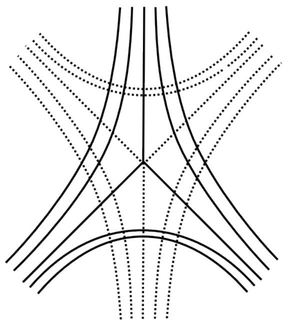

Figure 1.10: A pair of transverse singular foliations.

**Definition 1.14.** Let $\mathcal{F}$ be a singular foliation. A smooth arc $\alpha$ is called *transverse* to $\mathcal{F}$, if it misses all singular points and is transverse to the leaves. A *transverse measure* $\mu$ in a singular foliation $\mathcal{F}$ defines on every arc transverse to $\mathcal{F}$ a (non-negative) Borel measure $\mu(\alpha)$ such that:

1. If $\beta$ is a subarc of $\alpha$, then $\mu(\beta)$ is the restriction of $\mu(\alpha)$ to $\beta$.
2. If two arcs $\alpha_0, \alpha_1 : I \to \Sigma$ are related by a homotopy $H : I \times I \to \Sigma$ such that $H(I \times \{0\}) = \alpha_0$, $H(I \times \{1\}) = \alpha_1$, and such that $H(\{a\} \times I)$ is contained in a single leaf for each $a \in I$, then $\mu(\alpha_0) = \mu(\alpha_1)$, identifying here the leaves using the homotopy.

A singular foliation together with a transverse measure is called a *measured foliation.*

Homeomorphisms of $\Sigma$ act on measured foliations by $\varphi\cdot(\mathcal{F}, \mu) = (\varphi(\mathcal{F}), \varphi_*\mu)$, where $\varphi_*\mu(\alpha) = \mu(\varphi^{-1}(\alpha))$. We can now state the main result of this section (see [BC88, Thm. 6.3], [FLP79, Exposé 1, Thm. 5]).

**Theorem 1.15** (The Nielsen–Thurston classification)**.** *A mapping class $f$ in $\Gamma_g$, $g \geq 0$, has exactly one of the following three properties.*

1. *The class $f$ has finite order in $\Gamma_g$.*
2. *The class $f$ has infinite order but is* reducible. *That is, some power of $f$ preserves the isotopy class of an essential simple closed curve.*

3. *The class $f$ is* pseudo-Anosov *meaning that there exist transverse measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$, and $\lambda > 1$ real, such that $f$ is represented by a homeomorphism $\varphi$ (which we will also call pseudo-Anosov) satisfying*

$$\varphi \cdot (\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \quad \varphi \cdot (\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

In the pseudo-Anosov case, $\mathcal{F}^s$ and $\mathcal{F}^u$ are called the *stable* and *unstable* foliations respectively, and the number $\lambda$ which turns out to depend only on $f$ is called the *stretch factor* or *dilatation* of $f$.

### 1.3.1 Constructing pseudo-Anosov homeomorphisms

We give an explicit construction of pseudo-Anosov elements due to Penner, [Pen88], generalizing work by Thurston, [Thu88]. Thurston's idea was the following: It turns out that every pseudo-Anosov homeomorphism carries a so-called train track on the surface (see below for details). Train tracks provide a way of translating the a priori non-linear problem of determining the action of pseudo-Anosovs on curves to a linear and combinatorial problem; associated to the pseudo-Anosov homeomorphism and the train track carried by it is a so-called incidence matrix. It turns out that this matrix is Perron–Frobenius; that is, it satisfies the conditions of the following theorem.

**Theorem 1.16** (Perron–Frobenius)**.** *Let $A$ be an $n \times n$ matrix with integer entries. If $A$ has a power whose entries are positive, then $A$ has a unique eigenvector $v$ of unit length with non-negative entries. The eigenvalue $\lambda$ corresponding to $v$ is larger in absolute value than all other eigenvalues.*

The eigenvalue of the incidence matrix coming from this theorem turns out to be exactly the stretch factor of the pseudo-Anosov map being considered. Conversely, under certain conditions, if a given homeomorphism carries a particular train track, and the associated incidence matrix is Perron–Frobenius, then the given homeomorphism is pseudo-Anosov. By explicitly constructing train tracks for a certain class of homeomorphisms, Penner immediately constructs a large family of pseudo-Anosovs.

**Definition 1.17.** A *multicurve* in a surface $\Sigma$ is a collection of disjoint simple closed curves in $\Sigma$. We say that two multicurves $A = \{\alpha_1, \dots, \alpha_n\}$, $B = \{\beta_1, \dots, \beta_m\}$ *fill* $\Sigma$, if the isotopy class of any essential simple closed curve has non-zero intersection with the isotopy class of one of the $\alpha_i$ or $\beta_j$.

**Theorem 1.18** ([Pen88])**.** *Assume that $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_m\}$ fill $\Sigma$. Then any product of positive powers of $t_{\alpha_i}$ and negative powers of $t_{\beta_j}$, with all curves appearing at least once, is pseudo-Anosov.*

*Remark* 1.19. Penner conjectures that all pseudo-Anosovs arise this way. More precisely, he conjectures that for any pseudo-Anosov $\psi$, there exists $A$ and $B$ as in the theorem, and $n > 0$, such that $\psi^n$ is a word in positive powers of Dehn twists of curves from $A$ and negative powers of those from $B$.

As already mentioned, Penner proves this by giving an explicit construction of which we give an example relevant to our later studies. Recall the presentation of $\Gamma_2$ given in Section 1.2.2 and consider the mapping class $w = a_1 a_2^{-1} a_3 a_4^{-1} a_5$. The two multicurves giving rise to the Dehn twists $a_1, a_3, a_5$ and $a_2, a_4$ respectively fill the surface, so by Penner's theorem $w$ is pseudo-Anosov. As described above, we can determine the stretch factor of the mapping class using the theory of train tracks.

**Definition 1.20.** A *train track* on a surface $\Sigma$ is a finite graph $\tau$ embedded in $\Sigma$ satisfying the following:

1. Each edge is the smooth image of an interval. Edges are called *branches* of $\tau$.

2. If $b_1$ and $b_2$ are branches meeting a vertex, their one-sided tangents in the vertex point either coincide or differ by a rotation by an angle $\pi$ of the tangent plane. Vertices are called *switches*.

3. No component of $\Sigma \setminus \tau$ is an embedded null-gon, mono-gon, bi-gon, once punctured null-gon, or annulus.

Here, an *n-gon* is a disc embedded in $\Sigma$ with $n$ discontinuities in the tangent in the boundary of the disc. We say that a train track $\tau$ *carries* another track $\tau'$, if there is a $C^1$ map $\Phi : \Sigma \to \Sigma$ homotopic to the identity with $\Phi(\tau') \subseteq \tau$ and such that $d\Phi_p$ has non-zero restriction to tangents of $\tau'$, for all $p \in \tau'$. In this case we define the *incidence matrix* $M$ of $\Phi$ as follows. For every branch $b_i$ of $\tau$, choose a point $x_i$ in the interior of $b_i$, and define $M_{ij} = \#\{\Phi^{-1}(x_i) \cap c_j\}$, where the $c_j$ range over branches of $\tau'$.

Let us return to the mapping class $w \in \Gamma_2$ defined above. One defines a train track on $\Sigma_2$ by "smoothing" as in Figure 1.11. It is now easy to see that the train track is carried by each of the elements in the word $w$, and the carrying property is clearly transitive, so $w$ carries the train track. Penner proves that this procedure works in general.

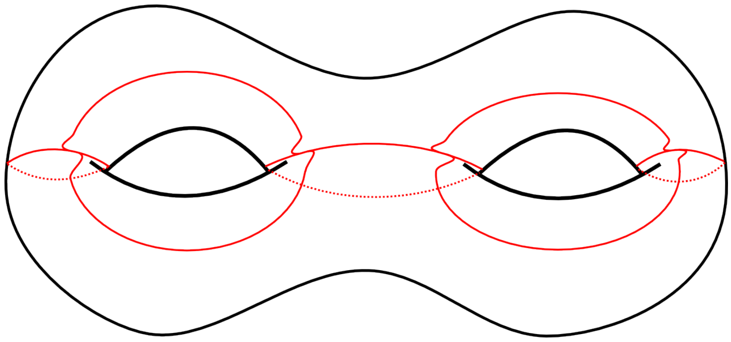

Figure 1.11: The train track carried by $w$.

The incidence matrix in question will be $12 \times 12$. In practice it might be easier to consider measured train tracks. A *measure* on a train track $\tau$ is an assignment of non-negative integers called *weights* to branches of $\tau$ such that these satisfy the *switch condition*: The branches of $\tau$ are divided into two sets by condition 2. above. We require that the sums of weights of branches in the two sets agree. In our example, the weights of the branches in the 12-branch train track are determined by 5 of the weights, and one could define an incidence matrix using only these 5 branches. In this particular case, an incidence matrix is

$$\begin{pmatrix} 2 & 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

and the stretch factor is $\frac{1}{2}(5 + \sqrt{21}) \approx 4.79$.

# Chapter 2

# Geometric quantization

In this chapter, we develop the framework that will later result in particular representations of the mapping class group.

In general, geometric quantization is one of several schemes involving the passage from a classical physical theory to a quantum mechanical analogue. The phase space of a classical system is a certain symplectic manifold $(M, \omega)$, and observables correspond to smooth functions on $M$. Geometric quantization associates to $M$ a complex line bundle $\mathcal{L} \to M$ and a Hilbert space $\mathcal{H}$ of states consisting of certain sections of $\mathcal{L}$. To observables on $M$ it associates self-adjoint operators on $\mathcal{H}$. The process of geometric quantization is typically divided into three parts; *prequantization* which is concerned with the associations above, *polarization* which restricts the collection of quantizable observables through a choice of a certain distribution on $M$, and finally *metaplectic correction* which involves repairs to the quantization which are necessary, for example in order to obtain the correct energy values for the harmonic oscillator. We will largely avoid discussion of physics and in order to obtain our desired representations, all we need is an understanding of the line bundles arising from prequantization. Details on the entire method can be found in e.g. [Woo92], [AE05].

## 2.1 Preliminaries

### 2.1.1 Connections in principal bundles

**Definition 2.1.** Let $M$ be a manifold and $G$ a Lie group. A principal $G$-bundle over $M$ is a manifold $P$ satisfying the following:

1. There is a free right action of $G$ on $P$ such that $M$ is the quotient space of $P$ under this action, and the quotient $\pi : P \to P/G = M$ is smooth.

2. Furthermore, $P$ is locally trivializable; that is, every point of $M$ has a neighbourhood $U$ with an equivariant diffeomorphism $\pi^{-1}(U) \to U \times G$ covering the identity on $M$.

Let $\pi : P \to M$ be a principal $G$-bundle over a manifold $M$. For $p \in P$, let $i_p : G \to P$ denote the map $i_p(g) = p \cdot g$, and similarly let $r_g : P \to P$ denote the map $i_g(p) = p \cdot g$. For every $X = X_e \in \mathfrak{g} \cong T_e(G)$, let $X_p^* = (di_p)_e(X_e)$.

This defines a vector field on $P$ which is vertical in the sense that $d\pi(X^*) = 0$. In other words, for every $p \in P$, we have a short exact sequence

$$0 \to \mathfrak{g} \overset{(di_p)_e}{\to} T_pP \overset{d\pi_p}{\to} T_{\pi(p)}M \to 0.$$

Now, a connection on $P$ defines a notion of horizontality in the principal bundle through a smooth choice of $G$-equivariant splittings of this exact sequence for all $p \in P$. More precisely, we consider the following:

**Definition 2.2.** A *connection* on a principal bundle $G \to M$ is a $\mathfrak{g}$-valued 1-form $A$ on $P$ such that

1. $A(X^*) = X$ for all $X \in \mathfrak{g}$.
2. $A$ is $G$-equivariant in the sense that for all $g \in G$, we have $r_g^*(A) = \mathrm{Ad}_{g^{-1}} A$.

Thus, if we let $V_p = \ker(d\pi) \subseteq T_pP$ denote the space of all vertical vectors in $T_pP$, then $T_pP = V_p \oplus H_p$, where $H_p = \ker(A_p)$ is the *horizontal* subspace of $T_pP$. Conversely, for every $n$-dimensional distribution $H$ on $P \to M$ satisfying that $d\pi_p|_{H_p} : H_p \to T_{\pi(p)}M$ is an isomorphism and that $H_{p\cdot g} = d(r_g)H_g$, there is a unique connection $A$ on $P \to M$ with $\ker(A_p) = H_p$ for all $p$.

Throughout the rest of this report, let $\mathcal{A}_P$ denote the set of connections on the principal bundle $P \to M$.

### 2.1.2 Curvature of connections

We now introduce the concept of curvature of a principal bundle connection. We do this by instead considering it as an affine connection in an associated vector bundle, and describing the curvature as the failure of a certain sequence to be a chain complex.

**Definition 2.3.** Let $P \to M$ be a principal $G$-bundle, and let $\rho : G \to \mathrm{GL}(V)$ be a representation of $G$ on a finite dimensional vector space $V$. The *vector bundle* associated to $P$ by $\rho$ is the quotient space

$$P \times_\rho V = P \times V / \sim,$$

where $(p, v) \sim (p \cdot g, \rho(g^{-1})v)$ with projection $\pi : P \times_\rho V \to M$ given by $\pi([(p, v)]) = p$. In particular, for the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the associated vector bundle is denoted $\mathrm{Ad}_P$ and called the *adjoint bundle*.

*Remark* 2.4. Let $E = P \times_\rho V$ be the vector bundle associated to a principal $G$-bundle $P$ by $\rho$. Associate to a $G$-equivariant map $f : P \to V$, i.e. a map satisfying $f(p \cdot g) = \rho(g^{-1})f(p)$, a section $\varphi_f$ of $E$ by letting $\varphi_f(x) = [(p, f(p))]$ for some $p \in \pi^{-1}(x)$. This association is a bijection.

This observation allows us to associate to every connection $A$ in $P \to M$ an affine connection in the adjoint bundle $\mathrm{Ad}_P$ as follows: Let $\varphi : P \to V$ be the $G$-equivariant map corresponding to a section of $\mathrm{Ad}_P$, and let $X$ be a vector field on $M$. Let $p \in P$, and let $\tilde{X}_p$ be the unique horizontal lift of $X_{\pi(p)}$ to $T_pP$, given by the connection $A$. Now, define a $G$-equivariant map $\nabla_X^A \varphi : P \to V$ by letting

$$\nabla_X^A \varphi(p) = d\varphi_p(\tilde{X}_p) \in T_{\varphi(p)}V \cong V.$$

Recall that any affine connection $\nabla : C^\infty(M, E) \to \Omega^1(M, E)$ in a vector bundle $E \to M$ extends to a map $\nabla : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ (see e.g. [Wel08, p. 74]) and gives a sequence

$$\cdots \to \Omega^{k-1}(M, E) \overset{\nabla}{\to} \Omega^k(M, E) \overset{\nabla}{\to} \Omega^{k+1}(M, E) \to \cdots . \tag{2.1}$$

In a local frame $f$ over $U \subseteq M$ for $E$, this extension is given by

$$\nabla\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f),$$

where $\xi \in \Omega^p(U, E)$, and $\theta(f) = \theta(\nabla, f) \in \Omega^1(U, \mathrm{Hom}(E, E))$ is the connection matrix associated with $\nabla$. In general, we will not have $\nabla \circ \nabla = 0$, and we let $F_\nabla \in \Omega^2(M, \mathrm{Hom}(E, E))$ denote the *curvature* of $\nabla$ given by

$$F_\nabla \varphi = \nabla\nabla\varphi$$

for $\varphi \in C^\infty(M, E)$. In a local frame $f$, the curvature is given in terms of the connection matrix: Let $\xi \in C^\infty(U, E)$. Ignoring the notational dependence on $f$, we find that

$$\begin{aligned} F_\nabla \xi &= (d + \theta)(d + \theta)\xi = (d + \theta)(d\xi + \theta \cdot \xi) \\ &= d^2\xi + \theta \cdot (d\xi) + d(\theta \cdot \xi) + \theta \wedge (\theta \cdot \xi) \\ &= \theta \cdot (d\xi) + d\theta \cdot \xi - \theta \cdot (d\xi) + (\theta \wedge \theta)\xi \\ &= d\theta \cdot \xi + (\theta \wedge \theta) \cdot \xi. \end{aligned}$$

That is, we have the following:

**Lemma 2.5.** *In a local frame $f$,*

$$F_\nabla(f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

We are now in a position to describe the curvature of a connection in a principal $G$-bundle.

**Definition 2.6.** Let $A$ be a connection in $P \to M$, and let $\nabla^A$ be the induced connection in $\mathrm{Ad}_P$. The *curvature* $F_A$ of $A$ is the curvature $F_{\nabla^A} \in \Omega^2(M, \mathrm{Hom}(\mathrm{Ad}_P, \mathrm{Ad}_P))$.

We end this section by giving a few alternative descriptions of the curvature of $A$. Let $\Omega^k(P; \mathfrak{g}) = \Omega^k(P \times \mathfrak{g})$ be the space of $\mathfrak{g}$-valued $k$-forms on $P$, and let $\Omega^k(P; \mathfrak{g})^G$ be the subset of those $k$-forms that are $G$-equivariant. The pullback of the adjoint bundle $\mathrm{Ad}_P$ under $\pi : P \to M$ is the trivial bundle $P \times \mathfrak{g} \to P$. The pullback $\tilde{\nabla}^A$ of $\nabla^A$ to $P \times \mathfrak{g}$ in fact restricts to the $G$-equivariant forms on $P$ and gives rise to a sequence

$$\cdots \to \Omega^{k-1}(P; \mathfrak{g})^G \xrightarrow{\tilde{\nabla}^A} \Omega^k(P; \mathfrak{g})^G \xrightarrow{\tilde{\nabla}^A} \Omega^{k+1}(P; \mathfrak{g})^G \to \cdots. \tag{2.2}$$

**Proposition 2.7.** *We have*

$$\tilde{\nabla}^A \tilde{\nabla}^A \varphi = [(dA + \tfrac{1}{2}[A \wedge A]) \wedge \varphi]$$

*for $\varphi \in \Omega^k(P; \mathfrak{g})$, where $[\cdot, \cdot]$ denotes the bracket on $\mathfrak{g}$. Let $F_{\tilde{\nabla}^A}$ be the curvature of $\tilde{\nabla}^A$, i.e. $F_{\tilde{\nabla}^A} = \tilde{\nabla}^A \circ \tilde{\nabla}^A$. Then $\pi^*(F_A) = F_{\tilde{\nabla}^A}$.*

Finally, using the $G$-equivariance of $A$, it is possible to view the curvature $F_A$ as an element of $\Omega^2(M, \mathrm{Ad}_P)$. Namely, for $q \in M$, $p \in \pi^{-1}(q)$, and vector fields $X, X'$ on $M$, define

$$\tilde{F}_A(X_q, X'_q)(p) = (dA + \tfrac{1}{2}[A \wedge A])((d\pi|_{H_p})^{-1} X_q, (d\pi|_{H_p})^{-1} X'_q).$$

This (well-)defines a $G$-equivariant map $P \to \mathfrak{g}$, and by Remark 2.4 we can consider $\tilde{F}_A \in \Omega^2(M, \mathrm{Ad}_P)$. In the following proposition, $[\cdot, \cdot] : \mathrm{Ad}_P \otimes \mathrm{Ad}_P \to \mathrm{Ad}_P$ is defined by $[(p, v), (p, v')] = (p, [v, v'])$ on representatives $(p, v)$ and $(p, v')$.

**Proposition 2.8.** *Let $\tilde{F}_A \in \Omega^2(M, \mathrm{Ad}_P)$ be the 2-form defined above. Then for every $\varphi \in \Omega^k(M, \mathrm{Ad}_P)$, we have*

$$\nabla^A \nabla^A \varphi = [F_A \wedge \varphi],$$

*and $\pi^*(\tilde{F}_A) = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P; \mathfrak{g})^G$.*

Conversely, $d\mathrm{Ad} : \mathfrak{g} \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ induces a map

$$\Omega^k(M, \mathrm{Ad}_P) \to \Omega^k(M, \mathrm{Hom}(\mathrm{Ad}_P, \mathrm{Ad}_P))$$

mapping the $\tilde{F}_A$ of Proposition 2.8 to the $F_A$ of Definition 2.6. For a proof of the above results, see [Him10]. Because of this, we will abuse notation and also refer to the element $\pi^*(\tilde{F}_A) \in \Omega^2(P; \mathfrak{g})^G$ as the curvature of $A$ and simply denote it $F_A$.

**Definition 2.9.** A connection $A$ on a principal $G$-bundle $P \to M$ is called *flat*, if $F_A = 0$ pointwise. The space of all flat connections on $P$ is denoted $\mathcal{F}_P$.

Notice that we could have simply defined $F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P; \mathfrak{g})^G$, but we stress the fact that the curvature of $A$ is exactly the obstruction to the sequence (2.1) being a complex.

## 2.2 Prequantum line bundles

Throughout this section, let $(M, \omega)$ be a symplectic manifold with a symplectic form $\omega \in H^2(M, \mathbb{R})$.

**Definition 2.10.** A *prequantum line bundle* on $(M, \omega)$ is a triple $(\mathcal{L}, \nabla, (\cdot, \cdot))$ consisting of a complex line bundle $\mathcal{L} \to M$ with a Hermitian structure $(\cdot, \cdot)$, and a compatible connection $\nabla$ satisfying the prequantum condition

$$F_\nabla = \frac{i}{2\pi}\omega.$$

A necessary and sufficient condition for the existence of a prequantum line bundle on $M$ is that $[\omega] \in \mathrm{Im}(H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}))$. See e.g. [Woo92].

Let $(\mathcal{L}, \nabla, (\cdot, \cdot))$ be a prequantum line bundle on a compact symplectic manifold $M$, and let $\mathcal{H}^k = C^\infty(M, \mathcal{L}^{\otimes k})$ be the space of all sections of $\mathcal{L}^{\otimes k}$. This is an inner product space with an inner product

$$\langle s_1, s_2 \rangle = \frac{1}{n!} \int_M (s_1, s_2)_k \omega^n,$$

where $2n$ is the dimension of $M$, and $(\cdot, \cdot)_k$ is the Hermitian structure on $\mathcal{L}^{\otimes k}$ induced by that on $\mathcal{L}$. The relevant Hilbert space in this context is the $L^2$-completion of $\mathcal{H}^k$. For our purpose though, the main point of interest lies in a certain finite-dimensional subspace of $\mathcal{H}^k$, constructed as follows.

Assume that $\mathcal{T}$ is a smooth manifold smoothly parametrizing Kähler structures on $M$. That is, assume that there is a map $I : \mathcal{T} \to C^\infty(M, \mathrm{End}(TM))$ mapping $\sigma \mapsto I_\sigma$ such that for every $\sigma \in \mathcal{T}$, $(M, \omega, I_\sigma)$ is Kähler, and such that $I$ is smooth in the sense that it defines a smooth section of the pullback bundle $\pi_M^*(\mathrm{End}(TM))$ over $\mathcal{T} \times M$, where $\pi_M : \mathcal{T} \times M \to M$ denotes the projection onto $M$. We denote by $M_\sigma$ the Kähler manifold $(M, \omega, I_\sigma)$. Using the complex structure on $M_\sigma$, $\omega$ has type $(1, 1)$, and by the prequantum condition, the line bundles $\mathcal{L}^{\otimes k}$ canonically obtain the structures of holomorphic line bundles (see [Kob97, Prop. 3.7]) denoted $\mathcal{L}_\sigma^{\otimes k}$. Now, let

$$H_\sigma^k = H^0(M_\sigma, \mathcal{L}_\sigma^{\otimes k}) = \{s \in \mathcal{H}_k \mid (\nabla_{I_\sigma}^{0,1})s = 0\}$$

be the space of holomorphic sections of $\mathcal{H}^k$. Using the splitting

$$T^*M \otimes \mathbb{C} = T_M^* \oplus \bar{T}_M^*,$$

of the complexified cotangent bundle into eigenspaces of $I_\sigma$, the operator $\nabla_{I_\sigma}^{0,1}$ is the composition

$$C^\infty(M, \mathcal{L}^{\otimes k}) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes \mathcal{L}^{\otimes k}) \xrightarrow{\pi_{I_\sigma}^{0,1}} C^\infty(M, \bar{T}_M^* \otimes \mathcal{L}^{\otimes k}),$$

which can be identified with the operator $\bar{\partial}_{I_\sigma}$ giving rise to the holomorphic structure on $\mathcal{L}^{\otimes k}$.

The process of restricting the space of sections using Kähler structures on $M$ is known as *Kähler quantization*.

## 2.3 Hitchin's connection

Our next objective is to understand the dependence of $H_\sigma^k$ on the complex structures $\sigma$. More precisely, consider the trivial bundle $\mathcal{T} \times \mathcal{H}^k \to \mathcal{T}$. It is a non-trivial fact that the vector spaces $H_\sigma^k$ are all finite-dimensional, and we now assume that they form a finite rank subbundle $\mathcal{V}_k$ of $\mathcal{T} \times \mathcal{H}^k$. Our goal is to find a connection in $\mathcal{T} \times \mathcal{H}^k$ preserving the subbundle $\mathcal{V}_k$. Let $\boldsymbol{\nabla}^t$ denote the trivial connection in the vector bundle $\mathcal{T} \times C^\infty(M, \mathcal{L}^{\otimes k})$. Then the composition of $\boldsymbol{\nabla}^t$ with the fibre-wise projection $\pi_\sigma^k : \mathcal{H}^k \to H_\sigma^k$ defines a connection which preserves $\mathcal{V}_k$ by construction. However, this connection is non-flat and thus not suited for our purpose.

Hitchin's idea was to instead consider the connection $\boldsymbol{\nabla}^H$ in $\mathcal{T} \times \mathcal{H}^k$ defined by $\boldsymbol{\nabla}_V^H = \boldsymbol{\nabla}_V^t - u(V)$ for vector fields $V$ on $\mathcal{T}$, where $u(V)_\sigma \in \mathrm{Diff}^{(2)}(M, \mathcal{L}^{\otimes k})$ is a second order differential operator, and to analyze under which conditions on $u$ the connection $\boldsymbol{\nabla}^H$ will preserve $\mathcal{V}_k$. Whereas this is not always the case, under certain conditions one can find explicit formulas for $u$; see e.g. [And06b].

A natural question concerns the flatness of the connection. We will see that in certain special cases, and indeed in the cases relevant for us, it is possible to find a projectively flat connection in the bundle $\mathcal{V}_k$. Here, a connection is called *projectively flat*, if parallel transport defines isomorphisms of fibres up to scalar multiplication. In other words, if the projectivization $\mathbb{P}\mathcal{V}_k$ admits a flat connection.

## 2.4 Toeplitz operators

Whereas we will not need it directly in this chapter, the theory of Toeplitz operators turns out to give a convenient tool to describe asymptotics of quantum representations, as we will see later.

The inner product on $H_\sigma^k$ determines an operator norm $\|\cdot\|$ on $\mathrm{End}(H_\sigma^k)$.

**Definition 2.11.** Let $f \in C^\infty(M)$. The *Toeplitz operator* $T_{f,\sigma}^k : H_\sigma^k \to H_\sigma^k$ is defined by

$$T_{f,\sigma}^k(s) = \pi_\sigma^k(f \cdot s).$$

In what follows, we suppress the dependence on the complex structure $\sigma$ for Toeplitz operators and simply write $T_f^k$ for $T_{f,\sigma}^k$. As we will primarily be interested in the Toeplitz operators because of their abilities to describe asymptotics, we list the folowing properties, due to Bordemann, Meinrenken and Schlichenmaier [BMS94], and Schlichenmaier [Sch98] respectively.

**Lemma 2.12.** *Let $f \in C^\infty(M)$. Then*

$$\lim_{k\to\infty} \|T_f^k\| = \sup_{x\in M} |f(x)|.$$

**Theorem 2.13.** *Let $f_1, f_2 \in C^\infty(M)$. Then*

$$T_{f_1}^k T_{f_2}^k \sim \sum_{l=0}^{\infty} T_{c_l(f_1,f_2)}^k k^{-l},$$

*where $c_l(f_1,f_2) \in C^\infty(M)$ are uniquely determined functions, and $c_0(f_1,f_2) = f_1 f_2$. Here, $\sim$ means that*

$$\|T_{f_1}^k T_{f_2}^k - \sum_{l=0}^{L} T_{c_l(f_1,f_2)}^k k^{-l}\| = O(k^{-(L+1)})$$

*for all positive integers $L$.*

*Remark* 2.14. The coefficients $c_l$ closely relates geometric quantization to deformation quantization as they define a star product on $M$. We will not need this fact and refer to [KS01] for the details.

## 2.5 Quantization of moduli spaces

The space we will be interested in quantizing is the space of flat connections in a trivializable principal $G$-bundle on a given surface. Before going into the details, we review the general picture and some of the central results.

### 2.5.1 The moduli space of flat connections

Let $\pi : P \to M$ be a principal $G$-bundle. We first describe how, given a connection $A$ in $P \to M$, any curve in $M$ can be lifted to a unique horizontal curve in $P$.

**Lemma 2.15.** *Let $A$ be a connection in $P \to M$. Let $\alpha : [0,1] \to M$ be a smooth curve with starting point $\alpha_0$, and let $p_0 \in \pi^{-1}(\alpha_0)$. Then there exists a unique smooth lift $\beta : [0,1] \to P$ of $\alpha$ with starting point $p_0$, such that $\dot\beta_t$ is a horizontal lift of $\dot\alpha_t$.*

*Proof.* Let $\gamma : [0,1] \to P$ be any lift of $\alpha$. Then we want to find a smooth curve $\beta_t = \gamma_t \cdot g_t : I \to P$, where $g_t : [0,1] \to G$, so that $A(\dot\beta_t) = 0$ for all $t \in [0,1]$. We see that

$$A(\dot\beta_t) = A((dr_{g_t})\dot\gamma_t + (di_{\gamma_t})\dot g_t) = \mathrm{Ad}_{g_t^{-1}}(\dot\gamma_t) + (dl_{g_t^{-1}})(\dot g_t),$$

where $l_g : G \to G$ denotes left multiplication in $G$. Thus, $A(\dot\beta_t) = 0$ if and only if

$$A(\dot\gamma_t) = -(dr_{g_t^{-1}})\dot g_t.$$

Solving this equation with initial condition $\gamma_0 \cdot g_0 = p_0$ gives the desired horizontal lift $\beta_t$. $\square$

If $\alpha : [0,1] \to M$ is a loop, $\alpha(0) = \alpha(1) = x_0$, the starting and ending points of the lift $\beta$ defined above are both in the fibre $P_{x_0}$ over $x_0$. Thus there is a $g$ so that $\beta(1) = \beta(0) \cdot g$. This $g$ is called the *holonomy of $A$ along $\alpha$ with respect to* $p_0$ and is denoted $g = \mathrm{hol}_{A,p_0}(\alpha)$. This defines a map

$$\mathrm{hol}_{A,p_0} : \mathrm{Loops}(M, x_0) \to G$$

for any given $p_0 \in \pi^{-1}(x_0)$.

The space $\mathcal{A}_P$ of all connections in the principal $G$-bundle $P \to M$ is too big for our purposes, and in this section we restrict it using the natural symmetry arising from the $G$-action.

**Definition 2.16.** A *principal bundle homomorphism* between two principal $G$-bundles $P$ and $P'$ is a $G$-equivariant bundle homomorphism. If $P = P'$ it is called a *gauge transformation* of the bundle. Denote by $\mathcal{G}_P$ the group of all gauge transformations $P \to P$.

*Remark* 2.17. To every $G$-equivariant map $u : P \to G$, $p \mapsto u_p$, we associate a gauge transformation $\Phi : P \to P$ by letting $\Phi(p) = p \cdot u_p$. Here, $g \in G$ acts on itself on the right by $h \mapsto g^{-1}hg$. This association is a bijection.

The group $\mathcal{G}_P$ acts on $\mathcal{A}_P$ via pullback, and the action preserves $\mathcal{F}_P$. For a $G$-equivariant map $u : P \to G$, we write this action $A \mapsto A \cdot u$.

**Definition 2.18.** The *moduli space of flat connections on a principal $G$-bundle* $P \to M$ is the space $\mathcal{M}_P = \mathcal{F}_P / \mathcal{G}_P$.

Before indulging in the question on how to quantize $\mathcal{M}_P$, we give a group theoretical description of it using the holonomy map. For a proof of the following results, see Prop. 3.10.1 and Thm. 3.10.4 of [Him10].

**Proposition 2.19.** *Let $A$ be a flat connection in $P$, and assume that $M$ is connected. Let $x_0 \in M$, let $p_0 \in \pi^{-1}(x_0)$, and let $\alpha$ be a loop in $M$. Up to conjugation in $G$, the association $A \mapsto \mathrm{hol}_{A,p_0}(\alpha)$ is independent of the base point $x_0$, the choice of lift $p_0$, the gauge tranformation class of the connection $A$, and the homotopy class of $\alpha$. In other words, we have a well-defined map*

$$\mathrm{hol} : \mathcal{M}_P \to \mathrm{Hom}(\pi_1(M), G)/G,$$

*where $G$ acts on* $\mathrm{Hom}(\pi_1(M), G)$ *on the right by* $(\rho \cdot g)(\alpha) = g^{-1}\rho(\alpha)g$.

**Definition 2.20.** A *flat principal $G$-bundle* on a manifold $M$ is a pair $(P, A)$ consisting of a principal $G$-bundle $P \to M$ and a flat connection $A$ in $P$. Two flat principal $G$-bundles $(P, A)$ and $(P', A')$ are called isomorphic if there is a principal bundle homomorphism $\Phi : P \to P'$ such that $A = \Phi^*(A')$. The set $\mathcal{M}(M, G)$ of isomorphism classes is called the *moduli space of flat principal $G$-bundles on $M$*.

**Theorem 2.21.** *The map $\mathcal{M}(M, G) \to \mathrm{Hom}(\pi_1(M), G)/G$ mapping $[(P, A)]$ to $[\mathrm{hol}_A]$ is a bijection.*

### 2.5.2 The Chern–Simons line bundle

The goal of this section is to give a sketch of the construction that a certain subset of the moduli space is pre-quantizable.

From now on, assume always that $G$ is a simple, connected, simply connected, and compact Lie group. In this case, it is well-known that any principal $G$-bundle over $M$, where $\dim M \leq 3$, is trivializable. Let $Y$ be an oriented compact 3-manifold with boundary $\partial Y = \Sigma$, and let $P \to Y$ be a principal $G$-bundle. Trivializing the bundle $P \cong Y \times G$ by a trivialization $p \mapsto (\pi(p), g_p)$ is equivalent to a choice of a section $s : Y \to P$ through the identification $p \cdot g_p = s(\pi(p))$. Using such a section, the pullback of connections determines an identification $\mathcal{A}_P \cong \Omega^1(Y; \mathfrak{g})$, and likewise we can identify $\mathcal{G}_P \cong C^\infty(Y, G)$.

Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be an Ad-invariant inner product on $\mathfrak{g}$, and define for a connection $A \in \mathcal{A}_P$ with curvature $F_A \in \Omega^2(P; \mathfrak{g})$, the *Chern–Simons* form $\alpha(A) \in \Omega^3(P)$ by

$$\alpha(A) = \langle A \wedge F_A \rangle - \tfrac{1}{6} \langle A \wedge [A \wedge A] \rangle.$$

**Definition 2.22.** Let $s : Y \to P$ be a trivialization of $P \to Y$. The *Chern–Simons functional* or *Chern–Simons action* is given by

$$\mathrm{CS}_s(A) = \int_Y s^* \alpha(A) \in \mathbb{R}$$

Let $\theta \in \Omega^1(G; \mathfrak{g})$ be the *Maurer–Cartan form* defined by $\theta(v) = (dl_{g^{-1}})v \in \mathfrak{g}$ for $v \in T_g G$. The next proposition describes the behaviour of the Chern–Simons functional under gauge transformation (see [Fre95, Prop. 2.10]).

**Proposition 2.23.** *Let $\Phi : P \to P$ be a gauge transformation with associated map $u : P \to G$ and let $\theta_u = (u \circ s)^* \theta$. Then for $A \in \Omega^1(Y; \mathfrak{g})$,*

$$\begin{aligned} \mathrm{CS}_{\Phi \circ s}(A) &= \mathrm{CS}_s(\Phi^* A) \\ &= \mathrm{CS}_s(A) + \int_{\partial Y} \langle \mathrm{Ad}_{(u \circ s)^{-1}} A \wedge \theta_u \rangle - \int_Y \tfrac{1}{6} \langle \theta_u \wedge [\theta_u \wedge \theta_u] \rangle. \end{aligned}$$

Assume from now on that $\langle \cdot, \cdot \rangle$ is normalized so that $-\int_G \frac{1}{6} \langle \theta \wedge [\theta \wedge \theta] \rangle = 1$. Then the last integral of Proposition 2.23 is an integer.

*Remark* 2.24. In the case where $Y$ is closed we obtain the Chern–Simons action

$$\mathrm{CS}_s : \mathcal{A}_P / \mathcal{G}_P \to \mathbb{R}/\mathbb{Z}.$$

Since any two sections are related by a gauge transformation, by Proposition 2.23 this function is independent of $s$ and will be denoted CS.

Our next goal is to associate in the non-closed case, $\partial Y = \Sigma$, $Q = P|_\Sigma$, a certain complex line bundle $\mathcal{L}_Q$ over $\mathcal{A}_Q$ and use the Chern–Simons action to define a lift of the action of $\mathcal{G}_Q$ to $\mathcal{L}_Q$, ultimately giving rise to a line bundle over a subset of the quotient $\mathcal{A}_Q / \mathcal{G}_Q$. We first need the following (see [Fre95, Lem. 2.12]).

**Lemma 2.25.** *For any gauge transformation $g : Y \to G$, the functional*

$$W_{\partial Y}(g) = \int_Y -\tfrac{1}{6} \langle g^* \theta \wedge [g^* \theta \wedge g^* \theta] \rangle \pmod 1$$

*depends only on the restriction of $g$ to $\partial Y$.*

Let $\mathcal{G}_Q$ denote the space of gauge transformations in $Q$. Fix again a trivialization $s$ of $P$, let $A \in \mathcal{A}_Q \cong \Omega^1(\Sigma; \mathfrak{g})$ and let $g : \Sigma \to G$ be a gauge transformation in $Q$. Choose extensions $\tilde{A}$ and $\tilde{g}$ of $A$ and $g$ to a connection respectively a gauge transformation in $P$. By Proposition 2.23 and Lemma 2.25, the function $\Theta : \mathcal{A}_Q \times \mathcal{G}_Q \to \mathrm{U}(1)$ given by

$$\Theta(A, g) = \exp(2\pi i (\mathrm{CS}_s(\tilde{g}^* \tilde{A}) - \mathrm{CS}_s(\tilde{A}))) \tag{2.3}$$

depends neither on the choice of extensions of $A$ and $g$, nor of the preliminary choice of trivialization. It turns out that $\Theta$ satisfies the cocycle condition

$$\Theta(g^*A, h)\Theta(A, g) = \Theta(A, gh).$$

We turn now to the question of defining a symplectic structure on (an appropriate subspace of) the moduli space. We do this through a quotient construction, considering a symplectic structure on $\mathcal{A}_Q$. Notice that $\mathcal{A}_Q$ is really an affine infinite-dimensional manifold, modelled on $\Omega^1(\Sigma; \mathfrak{g})$. We ignore all technical details necessary to deal with such objects and simply notice that for a given connection $A \in \mathcal{A}_Q$, there is an identification $T_A\mathcal{A}_Q \cong \Omega^1(\Sigma; \mathfrak{g})$. There then is a natural symplectic form $\omega$ on $\mathcal{A}_Q$, invariant under $\mathcal{G}_Q$, defined by

$$\omega(\eta_1, \eta_2) = -\int_\Sigma \langle \eta_1 \wedge \eta_2 \rangle$$

for $\eta_1, \eta_2 \in \Omega^1(\Sigma; \mathfrak{g})$. Using a similar identification $T_{\mathrm{Id}}\mathcal{G}_Q \cong C^\infty(\Sigma, \mathfrak{g})$, a moment map $\mu : \mathcal{A} \to C^\infty(\Sigma; \mathfrak{g})$ for the action of $\mathcal{G}_Q$ on $\mathcal{A}_Q$ is given by

$$\mu_\xi(A) = 2\int_\Sigma \langle F_A \wedge \xi \rangle,$$

for $\xi \in C^\infty(\Sigma, \mathfrak{g})$, and $A \in \mathcal{A}_Q$ with curvature $F_A \in \Omega^2(\Sigma; \mathfrak{g})$. Notice now that the (infinite-dimensional analogue of the) Marsden–Weinstein quotient

$$\mathcal{M}_Q = \mu^{-1}(\{0\})/\!\!/\mathcal{G}_Q$$

is exactly the moduli space $\mathcal{M}_Q$ of flat connections on $Q$ up to gauge equivalence.

Let $\mathcal{A}_Q^*$ be the subset of $\mathcal{A}_Q$ consisting of flat *irreducible* connections in $Q$, i.e. flat connections $A$ such that $\nabla^A$ is injective, and let $\mathcal{M}_Q^* = \mathcal{A}_Q^*/\mathcal{G}_Q$. This space turns out to be an open subset of $\mathcal{M}$ obtaining naturally the structure of a symplectic manifold through the quotient construction.

Now, let $\tilde{\mathcal{L}}_Q = \mathcal{A}_Q \times \mathbb{C}$ be the trivial line bundle over $\mathcal{A}_Q$ and lift the action of $\mathcal{G}_Q$ to $\tilde{\mathcal{L}}_Q$ using $\Theta$. There is then a connection $B$ on $\tilde{\mathcal{L}}_Q$ given in a trivialization $s : \Sigma \to Q$ by

$$(B_s)_A(\eta) = \int_\Sigma \langle A \wedge \eta \rangle,$$

for $A \in \mathcal{A}_Q \cong \Omega^1(\Sigma; \mathfrak{g})$, $\eta \in T_A\mathcal{A}_Q \cong \Omega^1(\Sigma; \mathfrak{g})$. One checks that this indeed defines a connection on $\tilde{\mathcal{L}}_Q$, constructed to satisfy $F_B = \frac{i}{2\pi}\omega$. What is less obvious is that this connection is preserved by the lifted action of $\mathcal{G}_Q$ and induces a connection $\bar{B}$ on the line bundle $\mathcal{L} \to \mathcal{M}_Q^*$ defined to be all equivalence classes of elements of $\mathcal{A}_Q^* \times \mathbb{C}$ under the relation

$$(A, z) \sim (g^*A, \Theta(A, g)z)$$

for all $g \in \mathcal{G}_Q$. The line bundle $\mathcal{L}$ carries a Hermitian structure since $\Theta$ is U(1)-valued, and $\bar{B}$ is compatible with this structure. Thus we finally obtain the following:

**Theorem 2.26.** *Let $\Sigma$ be a closed oriented surface and $Q \to \Sigma$ a principal $G$-bundle. Then the moduli space $\mathcal{M}_Q^*$ of flat irreducible connections is prequantizable.*

Freed furthermore discusses the case of a surface with boundary, which is slightly more involved: Let $\Sigma$ be a compact surface with $k$ boundary components, and let $Q \to \Sigma$ be a principal $G$-bundle. Fix $h_1, \dots, h_k \in G$. Let $\mathcal{M}^{h_1,\dots,h_k}$ be the moduli space of flat connections having holonomy $h_i$ around the $i$'th boundary component. As before, this space is not necessarily smooth, the problem once again being reducible connections, but Freed goes on to prove that the smooth part consisting of irreducible connections is pre-quantizable.

An important special case is the following: Assume that $\Sigma$ is a compact surface with genus $g \geq 2$ and a single boundary component. The fundamental group of $\Sigma$ is then freely generated by $\alpha_i, \beta_i$, $i = 1, \dots g$. Let $p \in \partial\Sigma$, and let $\gamma = \prod_{i=1}^{g}[\alpha_i, \beta_i] \in \pi_1(\Sigma, p)$ be the class of a loop going once around the boundary. Let $G = \mathrm{SU}(n)$, and let $d \in \mathbb{Z}_n$ be relatively prime to $n$, or let $(n, d) = (2, 0)$ if $g = 2$. Let $D = e^{2\pi i d/n} I \in \mathrm{SU}(n)$, and define

$$\mathrm{Hom}_d(\pi_1(\Sigma, p), \mathrm{SU}(n)) = \{\rho \in \mathrm{Hom}(\pi_1(\Sigma, p), \mathrm{SU}(n)) \mid \rho(\gamma) = D\}.$$

The conjugation action acts on this subspace of $\mathrm{Hom}(\pi_1(\Sigma, p), \mathrm{SU}(n))$, and one finds that it consists of irreducible representations. The resulting moduli space

$$\mathcal{M}^d_{\mathrm{SU}(n)} = \mathrm{Hom}_d(\pi_1(\Sigma, p), \mathrm{SU}(n))/\,\mathrm{SU}(n)$$

is a smooth compact manifold that does not depend on $p$, and Freed [Fre95] proves that it admits a prequantum line bundle.

### 2.5.3 Teichmüller space and Hitchin's connection

Let $\Sigma$ be a compact surface, and let $\mathcal{C}(\Sigma)$ be the space of conformal equivalence classes of Riemannian metrics on $\Sigma$. Recall that two metrics are called conformally equivalent if they are related by multiplication by a positive function. The group $\mathrm{Diff}(\Sigma)$ of orientation-preserving diffeomorphisms of $\Sigma$ acts on $\mathcal{C}(\Sigma)$ by pulling back metrics.

**Definition 2.27.** The *Teichmüller space* of $\Sigma$ is the quotient

$$\mathcal{T}(\Sigma) = \mathcal{C}(\Sigma)/\,\mathrm{Diff}_0(\Sigma).$$

It is well-known that there is a bijective correspondence between elements of $\mathcal{C}(\Sigma)$ and complex structures on $\Sigma$ – see e.g. [Jos02]. For that reason, Teichmüller space is often refered to as the space of complex structures on $\Sigma$, even if this is slightly misleading. It is worth noting the well-known fact that $\mathcal{T}(\Sigma)$ is a contractible space and carries a natural complex structure.

From now on, let $P \to \Sigma$ be a principal $G$-bundle, and let $G = \mathrm{SU}(n)$. As the notation suggests, we want $\mathcal{T}$ to parametrize complex structures on the space $\mathcal{M}^*$ of irreducible flat $\mathrm{SU}(n)$-connections on $\Sigma$. A Riemannian metric (or a complex structure) on $\Sigma$ gives rise to a Hodge star operator $* : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$. Extending this to an operator on $\mathrm{Ad}_P$-valued 1-forms and using the fact that $H^1(\Sigma, \mathrm{Ad}_P)$ identifies with the space $\ker(d_A) \cap \ker(*d_A*)$ of harmonic $\mathrm{Ad}_P$-valued 1-forms, the operator $*$ acts on $T_{[A]}\mathcal{M}^* \cong H^1(\Sigma, \mathrm{Ad}_P)$. Furthermore, $*$ satifies $*^2 = -1$, and defines an almost complex structure $I_\sigma = -*$ on $\mathcal{M}^*$. By work of Narasimhan and Seshadri [NS64], this almost complex structure is integrable. Finally, it can be seen to be compatible with the symplectic structure defined previously and so $\mathcal{M}^*$ obtains the structure of a Kähler manifold $(\mathcal{M}^*, \omega, I_\sigma)$.

This defines a map $I : \mathcal{C}(\Sigma) \to C^\infty(\mathcal{M}^*, \mathrm{End}(T\mathcal{M}^*))$. The group $\mathrm{Diff}(\Sigma)$ also acts on $\mathcal{M}^*$ via its action on $\pi_1(\Sigma)$ and gives an induced action of $\mathrm{Diff}(\Sigma)$ on $C^\infty(\mathcal{M}^*, \mathrm{End}(T\mathcal{M}^*))$. The map $I$ is equivariant with respect to this action, and $\mathrm{Diff}_0(\Sigma)$ acts trivially on $\mathcal{M}^*$, so we obtain a map

$$I : \mathcal{T}(\Sigma) \to C^\infty(\mathcal{M}^*, \mathrm{End}(TM^*))$$

parametrizing Kähler structures on $\mathcal{M}^*$. In this case, the bundle $\mathcal{V}$ over $\mathcal{T}$ with fibres $\mathcal{V}_\sigma = H^0(\mathcal{M}^*, \mathcal{L}_\sigma^{\otimes k})$ does form a finite rank vector bundle, and the existence of a projectively flat connection in this bundle – as outlined in Section 2.3 – was proved by Hitchin [Hit90], and Axelrod, della Pietra, and Witten [AdPW91] using techniques from algebraic geometry. We will refer to this as *the Hitchin connection.*

In the case where $\Sigma$ is a compact surface with genus $g \geq 2$ and one boundary component, the space $\mathcal{M} = \mathcal{M}^d_{\mathrm{SU}(n)}$ carries the structure of a compact Kähler manifold $\mathcal{M}_\sigma = (\mathcal{M}, \omega, I_\sigma)$ for every $\sigma \in \mathcal{T}(\Sigma)$. This moduli space satisfies the conditions of [And06b, Thm. 1], and we could appeal to Andersen's construction to obtain an explicit expression for the Hitchin connection in this case.

## 2.6 Remarks on group cohomology

Later, we will need a group cohomology description of the cohomology groups arising from the sequence (2.1) when $A$ is a flat connection and $\nabla = \nabla^A$, and we introduce here the most basic concepts we will need. One further reference on group cohomology is [Bro82].

Let $G$ be any group. A *$G$-module* is an abelian group $N$ with a left action of $G$. The elements of $N$ invariant under the action will be denoted $N^G$. A *cocycle on $G$ with values in $N$* is a map $u : G \to N$ satisfying the cocycle condition

$$u(gh) = u(g) + gu(h).$$

A *coboundary* is a cocycle of the form $g \mapsto \delta m(g) := m - gm$ for some $m \in N$. The set of cocycles is denoted $Z^1(G, N)$, and the set of coboundaries is denoted $B^1(G, N)$. We define the first cohomology group of $G$ with coefficients of $N$ as the quotient

$$H^1(G, N) = Z^1(G, N)/B^1(G, N).$$

Notice that an element of $N$ satisfies $\delta m \equiv 0$ exactly when $m \in N^G$. We are led to define

$$H^0(G, N) = N^G.$$

Now, let $P \to M$ be a principal $G$-bundle over a 3-manifold $M$, and let $[A]$ be the gauge equivalence class of a flat connection in $P$, represented by a representation $\rho \in \mathrm{Hom}(\pi_1(M), G)$ using Theorem 2.21. The representation $\rho$ defines a $\pi_1(M)$-module structure on $\mathfrak{g}$ through the composition $\mathrm{Ad} \circ \rho : \pi_1(M) \to \mathrm{Aut}(\mathfrak{g})$. Let $H^i(M, \mathrm{Ad}_P)$ denote the cohomology of the complex (2.1) with the induced connection $\nabla^A$. The following theorem is well-known.

**Theorem 2.28.** *If $M$ has contractible universal covering space, there are isomorphisms*

$$H^0(M, \mathrm{Ad}_P) \cong H^0(\pi_1(M), \mathfrak{g}), \quad H^1(M, \mathrm{Ad}_P) \cong H^1(\pi_1(M), \mathfrak{g}).$$

# Chapter 3

# Topological quantum field theory

## 3.1 Historical background

We begin this chapter with a brief discussion of how Chern–Simons theory heuristically gives rise to the notion of topological quantum field theory (TQFT). Recall that for a closed 3-manifold $M$ with a principal $G$-bundle $P \to M$, the Chern–Simons action defines a map

$$\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}.$$

Witten, in the late 80's, considered this as the Lagrangian of a so-called quantum field theory. One feature of these is the existence of a partition function, in this case given by the path integral

$$Z_k(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} \exp(2\pi i k\, \mathrm{CS}(A)) \mathcal{D}A,$$

for $k \in \mathbb{N}$. From a mathematical point of view, this is ill-defined, as there is no way to make sense of the integral over the infinite-dimensional space $\mathcal{A}_P/\mathcal{G}_P$, but Witten argues on the physical level of rigour that it defines a topological invariant of the 3-manifold $M$ called the quantum $G$-invariant of $M$ at level $k$. It is worth noting that using the path integral, this invariant can be extended formally to an invariant of pairs $(M, L)$, where $L$ is a link in $M$, as follows: To every component $L_i$ of $L$ we associate a finite-dimensional representation $R_i$ of $G$ – referred to as a *colouring* – and let

$$Z_k(M, L, R) = \int_{\mathcal{A}_P/\mathcal{G}_P} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) \exp(2\pi i k\, \mathrm{CS}(A)) \mathcal{D}A.$$

In the non-closed case, $\partial M = \Sigma \neq \emptyset$, the aim is to associate to the boundary a vector space $V(\Sigma)$, which from a physical point of view represents physical states on $\Sigma$, and to the 3-manifold $M$ a vector $Z_k(M) \in V(\Sigma)$ which physically represents the time evolution of states. Let $P \to M$ be a principal $G$-bundle with a trivialization $s : M \to P$. As the moduli space $\mathcal{M}$ of irreducible connections in $P|_\Sigma \to \Sigma$ is pre-quantizable, we can use the general construction in Section 2.2 to obtain the vector space $V(\Sigma) = H^0(\mathcal{M}, \mathcal{L}^{\otimes k})$ of holomorphic sections of the line bundle $\mathcal{L}^{\otimes k} \to \mathcal{M}$. Heuristically, we obtain a vector in this space in the following way: For each $[A] \in \mathcal{M}$, let $\mathcal{A}_A$ be the connections on $M$ restricting to

$A$ on the boundary, and let $\mathcal{G}' \subseteq \mathcal{G}_P$ denote the gauge transformations restricting to gauge transformations on $P|_\Sigma$. Now, let

$$Z_k(M)([A]) = \int_{\mathcal{A}_A/\mathcal{G}'} \exp(2\pi i k\, \mathrm{CS}_s(A'))\mathcal{D}A'.$$

By the construction of $\mathcal{L}$, it turns out that this formally gives rise to a holomorphic section of $\mathcal{L}^{\otimes k} \to \mathcal{M}$. Again however, this is of course ill-defined as the integral is. Trying to axiomatize the physical formalism, one arrives at the notion of a TQFT. In this chapter, we first describe TQFT from a general point of view and go on to consider two rigorous constructions of such theories.

For more details on the general picture, see e.g. [Oht02, App. F] or Witten's papers [Wit89], [Wit88].

## 3.2 The axiomatic point of view

A full mathematical axiomatization of Witten's notion of a TQFT was first put forward by Atiyah, [Ati88]. As we will see later, in the various realizations of the axioms some adjustments are necessary, but the main philosophy of Atiyah's TQFTs will survive. In Atiyah's picture, a $(d+1)$*-dimensional TQFT* $(Z, V)$ *over a field* $\Lambda$ consists of the following data: To every (possibly empty) closed oriented smooth $d$-manifold $\Sigma$, we associate a vector space $V(\Sigma)$ over $\Lambda$, and to every (possibly empty) compact oriented smooth $(d+1)$-manifold $M$, we associate an element $Z(M) \in V(\partial M)$. The associations satisfy the following axioms:

1. $(Z, V)$ is functorial: To every orientation-preserving diffeomorphism
$$f : \Sigma \to \Sigma'$$
of $n$-dimensional manifolds, we associate a linear isomorphism
$$V(f) : V(\Sigma) \to V(\Sigma'),$$
satisfying, for a composition of $f : \Sigma \to \Sigma'$, $g : \Sigma' \to \Sigma''$, that
$$V(g \circ f) = V(g) \circ V(f).$$
If $f$ extends to an orientation-preserving diffeomorphism $M \to M'$ with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then
$$V(f)(Z(M)) = Z(M').$$

2. $(Z, V)$ is involutive: Let $-\Sigma$ denote the $n$-manifold $\Sigma$ with the opposite orientation. Then $V(-\Sigma) = V(\Sigma)^*$.

3. $(Z, V)$ is multiplicative: For disjoint unions, $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$, and if $M = M_1 \cup_{\Sigma_3} M_2$ is obtained by gluing two $(d+1)$-manifolds $M_1, M_2$ with boundaries $\partial M_1 = \Sigma_1 \cup \Sigma_3$, $\partial M_2 = \Sigma_2 \cup -\Sigma_3$ along $\Sigma_3$ (see Figure 3.1), then $Z(M) = \langle Z(M_1), Z(M_2)\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the pairing
$$V(\Sigma_1) \otimes V(\Sigma_3) \otimes V(\Sigma_3)^* \otimes V(\Sigma_2) \to V(\Sigma_1) \otimes V(\Sigma_2).$$

The last axiom allows us to think of $(Z, V)$ as a functor from a cobordism category to the category of vector spaces. Namely, if $\partial M = \Sigma_0 \cup -\Sigma_1$ for (possibly empty) $n$-manifolds $\Sigma_0$ and $\Sigma_1$, then we will view $Z(M)$ as an element

$$Z(M) \in V(\Sigma_0)^* \otimes V(\Sigma_1) = \mathrm{Hom}(V(\Sigma_0), V(\Sigma_1)).$$

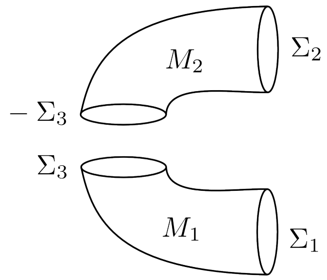


Figure 3.1: The gluing axiom.

The axiom also shows that $V(\emptyset) = V(\emptyset) \otimes V(\emptyset)$, so $V(\emptyset)$ is either trivial or isomorphic to $\Lambda$. We will explicitly require the latter. Similarly, $Z(\emptyset) \in \Lambda$ is either 0 or 1, and we explicitly require that $Z(\emptyset) = 1$. Finally, the axioms also imply that $Z(\Sigma \times I) \in \mathrm{End}(V(\Sigma))$ is idempotent, and we shall require that $Z(\Sigma \times I) = \mathrm{id}_{V(\Sigma)}$.

### 3.2.1 Mapping class group representations from TQFTs

The above axioms hint at how to obtain representations of mapping class groups of closed surfaces from a (2+1)-dimensional TQFT. Namely, let $\Sigma$ be a closed oriented surface, and let $f \in \Gamma(\Sigma)$ be the mapping class of an orientation-preserving diffeomorphism $\varphi : \Sigma \to \Sigma$. Put $\rho(f) = V(\varphi) : V(\Sigma) \to V(\Sigma)$, and let

$$M_\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$

be the mapping cylinder of $\varphi$ obtained by gluing together two copies of $\Sigma \times \{\frac{1}{2}\}$ using $\varphi$.

**Proposition 3.1.** *The map $\rho : \Gamma(\Sigma) \to \mathrm{End}(V(\Sigma))$ is a well-defined representation of $\Gamma(\Sigma)$. Furthermore, if $f$ is the mapping class of $\varphi$ as above, then $\rho(f) = Z(M_\varphi)$.*

*Proof.* Let $\varphi_t : \Sigma \to \Sigma$ be an isotopy between orientation-preserving diffeomorphisms $\varphi_0$ and $\varphi_1$. The map $\Sigma \times I \to \Sigma \times I$ given by

$$(x, t) \mapsto (\varphi_1 \varphi_t^{-1}(x), t)$$

extends the map $\varphi_1\varphi_0^{-1} \sqcup \mathrm{id} : \Sigma \sqcup -\Sigma \to \Sigma \sqcup -\Sigma$, and it follows from the axioms that $V(\varphi_1\varphi_0^{-1}) = \mathrm{id}$ and $V(\varphi_1) = V(\varphi_0)$.

To prove the last statement, notice that $\varphi \sqcup \mathrm{id} : \Sigma \sqcup -\Sigma \to \Sigma \sqcup -\Sigma$ extends to an orientation-preserving diffeomorphism $\Sigma \times I \to M_\varphi$. It follows that

$$Z(M_\varphi) = V(\varphi) = \rho(f).$$

□

Let $T_\varphi$ denote the mapping torus of a diffeomorphism $\varphi : \Sigma \to \Sigma$. That is, $T_\varphi$ is the result of identifying opposite ends of $\Sigma \times [0, 1]$ using $\varphi$. It then follows from the axioms that $Z(T_\varphi) = \mathrm{tr}\, V(\varphi)$. In particular one finds the dimensions of the TQFT vector spaces as

$$\dim V(\Sigma) = \mathrm{tr}\, V(\mathrm{id}\,|_{V(\Sigma)}) = Z(\Sigma \times S^1).$$

In general, the TQFT defines an invariant of *closed* oriented $(n+1)$-dimensional manifolds, often called a *quantum invariant*. The so-called *universal construction*, which we will discuss Section 3.4.4, gives a criterion for extending an invariant of closed manifolds to a TQFT functor. The main example in this report is the skein theoretical construction of [BHMV95]. Before going into details with this construction, we turn to the abstract category theoretical construction of [Tur94]. This construction summarizes the work by Reshetikhin and Turaev [RT90], [RT91], giving the historically first concrete realization of Witten's TQFT.

## 3.3 The Reshetikhin–Turaev TQFT

### 3.3.1 Ribbon categories and graphical calculus

We first set up the relevant categorical framework. The notion of a ribbon category will encompass the structures relevant for our constructions.

**Definition 3.2.** A *ribbon category* is a monoidal category $\mathcal{V}$ with a braiding $c$, a twist $\theta$, and a compatible duality $(*, b, d)$.

Let us discuss these concepts one at a time.

**Definition 3.3.** A *monoidal category* is a category $\mathcal{V}$ with a covariant functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ associating to two objects $V$ and $W$ an object $V \otimes W$ and to morphisms $f : V \to V'$ and $g : W \to W'$ a morphism $f \otimes g : V \otimes W \to V' \otimes W'$, such that the following properties are satisfied:

1. There exists a unit object $\mathbf{1}$ satisfying $V \otimes \mathbf{1} = V$, and $\mathbf{1} \otimes V = V$ for all objects $V$ in $\mathcal{V}$.
2. For triples $U, V, W$ of objects, we have $(U \otimes V) \otimes W = U \otimes (V \otimes W)$.
3. For morphisms $f$ in $\mathcal{V}$, $f \otimes \mathrm{id}_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} \otimes f = f$.
4. For triples $f, g, h$ of morphisms, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

*Remark* 3.4. The above definition is really that of a *strict* monoidal category. More generally one could consider categories, where the above equalities are replaced by fixed isomorphisms. By general abstract nonsense, nothing is lost by requiring equality, and we will avoid the more general case completely.

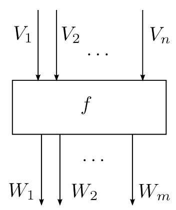


Figure 3.2: A morphism represented by a diagram.

We introduce now the notion of graphical calculus, and discuss it in parallel with the axioms of a ribbon category. A morphism

$$f : W_1 \otimes \cdots \otimes W_m \to V_1 \otimes \cdots \otimes V_n$$

in $\mathcal{V}$ is represented by a diagram with a box with several downward arrows as in Figure 3.2. The cases where either $m$ or $n$ is 0 (or both are) will be allowed as well; here, the corresponding tensor product will be **1** by convention. Composing morphisms is represented by stacking diagrams on top of each other, and tensoring morphims is represented by placing diagrams next to each other; an example is illustrated in Figure 3.3 where $g : U \to W'$ and $f : W \otimes W' \to V$ are two morphisms, and the diagram represents the morphism

$$f \circ (\mathrm{id}_W \otimes g) : W \otimes U \to V.$$

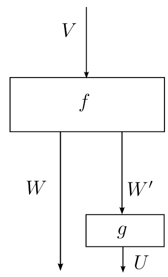


Figure 3.3: Composition and tensor products of morphisms.

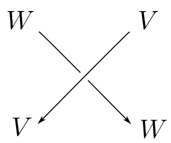


Figure 3.4: The braiding isomorphisms $c_{V,W}$.

**Definition 3.5.** A *braiding* in $\mathcal{V}$ is a family $c = \{c_{V,W} : V \otimes W \to W \otimes V\}$ of isomorphisms, represented in graphical calculus by the diagram in Figure 3.4, satisfying the following properties, where $U, V, W, V', W'$ are objects of $\mathcal{V}$, and $f : V \to V'$, $g : W \to W'$ are morphisms:

1. $c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W)$.
2. $c_{U\otimes V,W} = (c_{U,V} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W})$.
3. $(g \otimes f)c_{V,W} = c_{V',W'}(f \otimes g)$.

As an example, it follows from the definitions that

$$(\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = (c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W).$$

This isn't obvious from the algebraic statements above but is a natural consequence of the existence of graphical calculus and Theorem 3.9 below.

**Definition 3.6.** A *twist* in $(\mathcal{V}, c)$ is a family $\theta = \{\theta_V : V \to V\}$ of isomorphisms, represented diagrammatically as in Figure 3.5, satisfying the relations

$$\theta_{V\otimes W} = c_{W,V} c_{V,W}(\theta_V \otimes \theta_W),$$
$$\theta_V f = f\theta_U$$

for objects $V, U, W$ and morphisms $f : U \to V$ in $\mathcal{V}$.

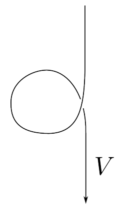


Figure 3.5: The twist isomorphisms $\theta_V$.

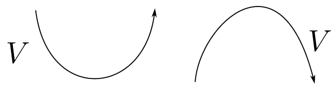


Figure 3.6: The duality morphisms $b_V$ and $d_V$ respectively.

**Definition 3.7.** A *duality* in $\mathcal{V}$ associates to every object $V$ of $\mathcal{V}$ an object $V^*$ of $\mathcal{V}$ and two morphisms $b_V : \mathbf{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbf{1}$, satisfying the following:

1. $(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V$.
2. $(d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$.

A duality $(*, b, d)$ in a category with braiding and twist, $(\mathcal{V}, c, \theta)$ is called *compatible* with the braiding and twist if furthermore

$$(\theta_V \otimes \mathrm{id}_{V^*})b_V = (\mathrm{id}_V \otimes \theta_{V^*})b_V.$$

In graphical calculus, a downward arrow coloured by an object $V^*$ will be used interchangeably with upward arrows coloured by $V$, and we will represent $b_V$ and $d_V$ by the diagrams in Figure 3.6.

### 3.3.2 The Reshetikhin–Turaev functor

The graphical calculus hints that we might be able to associate to an oriented graph in $\mathbb{R}^2 \times [0,1]$, where strings are coloured by objects of $\mathcal{V}$, and possibly containing a number of boxes, called coupons, a pair of objects in $\mathcal{V}$, and a morphism between them. The existence of the Reshetikhin–Turaev functor tells us that this is indeed the case. In this subsection, this is made somewhat more precise.

A *band* is a homeomorphic image in $\mathbb{R}^2 \times [0,1]$ of $[0,1] \times [0,1]$, an *annulus* is a homeomorphic image of $S^1 \times [0,1]$, and a *coupon* is a band with distinguished bases $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$.

**Definition 3.8.** A *ribbon* $(k,l)$*-graph* (or a *ribbon graph* for short) is an oriented surface embedded in $\mathbb{R}^2 \times [0,1]$, decomposed into a union of directed bands, directed annuli, and coupons, such that bases of bands meet either the planes $\mathbb{R}^2 \times \{0,1\}$ or the bases of coupons; see Figure 3.7 (see [Tur94] for a more precise statement). Here, $k$ bands meet $\mathbb{R}^2 \times \{0\}$, and $l$ bands meet $\mathbb{R}^2 \times \{1\}$.

To simplify drawings in what follows, we will often picture bands and annuli in ribbon graphs using the blackboard framing as in Figure 3.7; here, the 1-dimensional graph represents the ribbon graph obtained by letting bands and annuli be parallel to the plane of the picture. This is possible since ribbon graphs are always assumed to be orientable. Note that the theory of ribbon graphs therefore in particular contains the theory of banded oriented links in $S^3$.

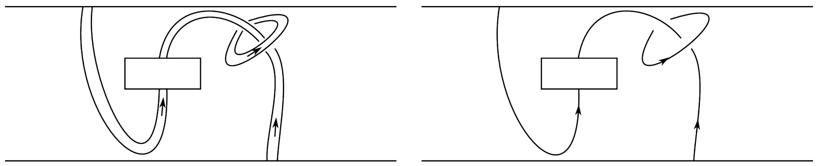

Figure 3.7: An example of a $(1,1)$-ribbon graph and how to represent it by a 1-dimensional graph using the blackboard framing.

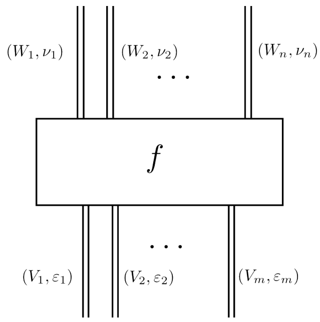


Figure 3.8: The colouring of a coupon.

Assume that $\mathcal{V}$ is a monoidal category with duality. A colouring[1] of a ribbon graph is an association of an object of $\mathcal{V}$ to every non-coupon band and every annulus, and an association of a morphism

$$f : V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n}$$

to every coupon meeting bands with colours $V_i, W_j$ as in Figure 3.8. Here, $\varepsilon_i, \nu_j \in \{-1, 1\}$, and $V^1 = V$, $V^{-1} = V^*$ for objects $V$. For every $V_i, W_j$, the appropriate sign is determined by the orientation of the bands in question, the sign being positive, if the band is directed downwards, and negative otherwise. This turns the set of coloured ribbon graphs into a monoidal category denoted $\mathrm{Rib}_{\mathcal{V}}$. Objects are sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$, $\varepsilon_i \in \{-1, 1\}$, and morphisms $\eta \to \eta'$ are *isotopy types* of coloured ribbon graphs meeting in $\mathbb{R}^2 \times \{0\}$ the sequence $\eta$, and similarly in $\mathbb{R}^2 \times \{1\}$ the sequence $\eta'$, where the numbers $\varepsilon_i$ determine the orientations of the bands. Here composition is given by stacking graphs and tensor product is given by placing graphs next to each other.

**Theorem 3.9.** *Let $\mathcal{V}$ be a ribbon category. There is a covariant functor*

$$F = F_{\mathcal{V}} : \mathrm{Rib}_{\mathcal{V}} \to \mathcal{V}$$

*preserving tensor product, transforming $(V, \varepsilon)$ to $V^\varepsilon$, and transforming morphisms to the natural corresponding morphisms in $\mathcal{V}$: Namely, consider the diagrams of graphical calculus as ribbon graphs, taking parallels as above. Then the functor transforms these graphs to the morphisms of $\mathcal{V}$ represented by the diagrams. Furthermore, imposing a few conditions outlined in [Tur94], the functor is unique.*

For a morphism $f : V \to V$ in a ribbon category, define $\mathrm{tr}(f) : \mathbf{1} \to \mathbf{1}$ by

$$\mathrm{tr}(f) = d_V c_{V,V^*}((\theta_V f) \otimes \mathrm{id}_{V^*}).$$

[1] In the language of [Tur94], this is a $v$-colouring.

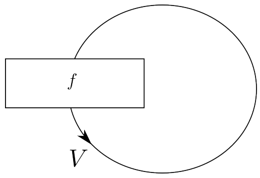


Figure 3.9: The coloured ribbon graph $\Omega_f$ corresponding under $F$ to $\mathrm{tr}(f)$.

Then $F(\Omega_f) = \mathrm{tr}(f)$, where $\Omega_f$ is the coloured ribbon graph shown in Figure 3.9. It follows immediately that $\mathrm{tr}(fg) = \mathrm{tr}(gf)$ and $\mathrm{tr}(f \otimes g) = \mathrm{tr}(g \otimes f)$. Define

$$\dim(V) = \mathrm{tr}(\mathrm{id}_V).$$

Then $\dim(V \otimes W) = \dim(V)\dim(W)$.

### 3.3.3 Modular categories

As already mentioned, our quest is to find invariants of 3-manifolds fitting into the framework of TQFT. The operator invariants of coloured ribbon graphs constructed above are too general to be useful for this purpose; rather, the ribbon categories used in the construction should satisfy several other conditions, summed up in the notion of a modular category.

**Definition 3.10.** An *Ab-category* is a monoidal category $\mathcal{V}$ such that the set $\mathrm{Hom}(V, W)$ of morphisms $V \to W$ has the structure of an additive abelian group and such that composition is bilinear. In particular, this holds for $\mathrm{End}(\mathbf{1})$, and this group obtains the structure of a (commutative) ring, with multiplication being given by composition; it is denoted $K$ and called the *ground ring* of $\mathcal{V}$.

**Definition 3.11.** An object $V$ of a ribbon Ab-category $\mathcal{V}$ is called *simple*, if the map $k \mapsto k \otimes \mathrm{id}_V$ defines a bijection $K \to \mathrm{Hom}(V, V)$.

**Definition 3.12.** Let $\{V_i\}_{i \in I}$ be a family of objects in a ribbon Ab-category $\mathcal{V}$. An object $V$ of $\mathcal{V}$ is *dominated* by $\{V_i\}$, if there exists a finite subfamily $\{V_{i(r)}\}_r$ and morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

**Definition 3.13.** A *modular category* is a ribbon Ab-category $\mathcal{V}$ with a finite set of simple objects $\{V_i\}_{i \in I}$ satisfying the following axioms:

1. $\mathbf{1} \in \{V_i\}_{i \in I}$.
2. For any $i \in I$, there exists $i^* \in I$ such that $V_{i^*}$ is isomorphic to $(V_i)^*$.
3. All objects of $\mathcal{V}$ are dominated by $\{V_i\}_{i \in I}$.
4. The matrix $S$ with entries $S_{i,j} = \mathrm{tr}(c_{V_j,V_i} c_{V_i,V_j}) \in K$ is invertible over $K$.

Let $(\mathcal{V}, \{V_i\}_{i \in I})$ be a modular category. We are now in a position to define the link invariant that will give rise to our 3-manifold invariant. Let $L$ be an oriented banded $m$-component link in $S^3$, viewed as a link in $\mathbb{R}^2 \times [0, 1]$. Let $\mathrm{col}(L)$ be the finite set of all possible colourings of $L$ by elements of $\{V_i\}_{i \in I}$. For $\lambda \in \mathrm{col}(L)$, let $\Gamma(L, \lambda)$ be the coloured ribbon $(0, 0)$-graph obtained from $L$ by colouring it according to $\lambda$. Now $F(\Gamma(L, \lambda))$ is an element of $K$, and we define

$$\{L\} = \sum_{\lambda \in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n)) F(\Gamma(L, \lambda)) \in K.$$

Note that this expression does not depend on the orientation of $L$.

### 3.3.4 Surgery and 3-manifold invariants

Let $L$ be an $m$-component banded link in $S^3$. Choose a regular closed neighbourhood $U$ of $L$, consisting of $m$ disjoint solid tori $U_1, \dots, U_m$. Each of these are homeomorphic to $S^1 \times D^2$ with boundary homeomorphic to $S^1 \times S^1$. Choose homeomorphisms $h_i : S^1 \times S^1 \to S^1 \times S^1$ and form the space

$$M_L = (S^3 \setminus U) \cup_{h_i} (\sqcup_{i=1}^m D^2 \times S^1)$$

which is the disjoint union of $S^3 \setminus U$ and $m$ copies of solid tori $D^2 \times S^1$, these two spaces being identified along their common boundary $\sqcup_{i=1}^m S^1 \times S^1$ using the homeomorphisms $h_i$. The resulting topological space $M_L$ is a closed orientable manifold. The space $M_L$ constructed as such depends of course on the homeomorphisms involved in the gluing. Using the banded structure of $L$, one canonically obtains particular homeomorphisms $h_i$ depending only on $L$, and the resulting surgery is referred to as *integral surgery.* Using this, we say that $M_L$ is *obtained by surgery on $S^3$ along $L$.*

**Theorem 3.14** (Lickorish, Wallace)**.** *Any closed connected oriented 3-manifold can be obtained by (integral) surgery on $S^3$ along a banded $m$-component link.*

Let $L$ be an oriented $m$-component banded link. Let $\sigma(L)$ be the signature of the linking matrix $A_{ij} = \mathrm{lk}(L_i, L_j)$ consisting of linking numbers of the components. Here, the linking number of a banded knot $L_i = K \times [0,1]$ with itself is defined to be the linking number of its boundary knots $K \times \{0\}$ and $K \times \{1\}$. Note that $\sigma(L)$ is independent of the orientations of the components.

Let as before $(\mathcal{V}, \{V_i\}_{i \in I})$ be a modular category. Assume that there is an element $\mathcal{D} \in K$ called a *rank* satisfying $\mathcal{D}^2 = \sum_{i \in I} (\dim(V_i))^2$. Since the $V_i$ are simple, each twist $\theta_{V_i}$ can be identified with an element $v_i \in K$, which is invertible, since $\theta_{V_i}$ is an isomorphism. Now, set

$$\Delta = \sum_{i \in I} v_i^{-1} (\dim(V_i))^2 \in K.$$

The dimensions $\dim(V_i)$ are invertible because of axiom (4) of a modular category, and it follows that both $\mathcal{D}$ and $\Delta$ are invertible in $K$.

We can now define the quantum invariant of 3-manifolds.

**Theorem 3.15** (Reshetikhin–Turaev)**.** *Let $M$ be a closed connected oriented 3-manifold obtained by surgery on $S^3$ along a banded link $L$. Then*

$$\tau(M) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \{L\} \in K$$

*is a topological invariant of $M$.*

Finally, we extend the invariant to an invariant of 3-manifolds containing coloured ribbon $(0,0)$-graphs. Let $M$ be a closed connected oriented 3-manifold containing a coloured ribbon graph $\Omega$. Assume that $M$ is the result of surgery along an $m$-component oriented link $L$ in $S^3$, and assume by applying isotopy that $\Omega$ does not meet the regular neighbourhood $U$ of $L$ used in the surgery, so we can view $\Omega$ as a ribbon graph in $S^3 \setminus U$. Now, let

$$\{L, \Omega\} = \sum_{\lambda \in \mathrm{col}(L)} \prod_{n=1}^{m} \dim(\lambda(L_n)) F(\Gamma(L, \lambda) \cup \Omega) \in K.$$

As before, this will not depend on the orientation of $L$ or the numbering of the components.

**Theorem 3.16.** *Let* $(M, \Omega)$ *be as above. Then*

$$\tau(M, \Omega) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \{L, \Omega\} \in K$$

*is a topological invariant of the pair* $(M, \Omega)$*.*

The proof of this theorem uses the theorem by Kirby stating that closed oriented manifolds obtained by surgery along banded links $L$ and $L'$ are homeomorphic (by orientation-preserving homeomorphism) if and only if $L$ and $L'$ are related by a certain sequence of moves on banded links called Kirby moves.

### 3.3.5 Modular categories from quantum groups and TQFT

The above approach defines a quantum invariant for any modular category. For completeness, we briefly include the main points of the construction by Reshetikhin and Turaev of the modular category, using quantum groups. A streamlined introduction giving all details is available in [Tur94].

For any given Hopf algebra $A$ over a commutative unital ring $K$, one might consider the category of representations of $A$, denoted $\mathrm{Rep}(A)$, with objects being finite rank $A$-modules and morphisms being $A$-homomorphisms. This category is a monoidal Ab-category. Furthermore, the category has a natural duality pairing. To provide this category with a braiding, the algebra should further have the structure of a *quasitriangular* Hopf algebra; this means that there is a distinct element $R \in A^{\otimes 2}$ (often refered to as an $R$-matrix) satisfying certain conditions. Finally, to get a twist in $\mathrm{Rep}(A)$, one fixes an element $v$ in the center of $A$ satisfying again particular conditions. With all of these in place, $\mathrm{Rep}(A)$ acquires the structure of a ribbon Ab-category. For the representation category to be a modular category, we furthermore need a finite collection of finite rank $A$-modules $\{V_i\}_{i \in I}$ that are *simple*, in the sense that their only endomorphisms are multiplications by scalars and which furthermore satisfy the following conditions:

1. For some element $0 \in I$, we have $V_0 = K$ (where $A$ acts by the Hopf algebra counit).

2. For every $i \in I$, there exists $i^* \in I$ so that $V_{i^*}$ and $(V_i)^*$ are isomorphic.

3. For every $k, l \in I$ the tensor product $V_k \otimes V_l$ splits as a finite direct sum of $V_i$, $i \in I$ and a module $V$ satisfying $\mathrm{tr}_q(f) = 0$ for any $f \in \mathrm{End}(V)$. Here, $\mathrm{tr}_q$ denotes the trace defined in Section 3.3.2.

4. Denoting by $S_{i,j}$ the quantum trace of $x \mapsto \mathrm{flip}_{A,A}(R)Rx$ on $V_i \otimes V_j$, we obtain an invertible matrix $[S_{i,j}]_{i,j \in I}$. Here, $\mathrm{flip}_{V,W} : V \otimes W \to W \otimes V$ is the homomorphism defined by $v \otimes w \mapsto w \otimes v$.

With all of these in place, we obtain exactly what we are looking for: If a collection of $A$-modules as above exist, then $\mathrm{Rep}(A)$ has a (non-full) modular subcategory.

It thus remains to construct Hopf algebras satisfying all of the above conditions. These arise in the language of quantum groups. While quantum groups can be defined for general simple Lie algebras, let us consider only $\mathfrak{sl}_2$ for which the construction is known to work out. The quantum group $U_q\mathfrak{sl}_2$ is defined to be the algebra over $\mathbb{C}$ generated by elements $K, K^{-1}, E, F$ with relations

$$K^{-1}K = KK^{-1} = 1,$$
$$KE = q^{-1}EK, \quad KF = qFK,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Assume for simplicity that $q$ is a primitive $l$'th root of unity with $l$ even; this turns out to be the most simple setting, and in general one cannot hope for the construction to work. The quantum group $U_q\mathfrak{sl}_2$ is not quite good enough for our purposes, and we consider instead the quotient $\tilde{U}_q\mathfrak{sl}_2$ of $U_q\mathfrak{sl}_2$ by the two-sided ideal generated by $E^{l/2}, F^{l/2}, K^l - 1$. Then $\tilde{U}_q\mathfrak{sl}_2$ can be endowed with an $R$-matrix and a twist satisfying the necessary conditions for its representation category to give a modular category. Here, the role of the simple modules described above will be played by certain irreducible $\tilde{U}_q\mathfrak{sl}_2$-modules. The construction for $\mathfrak{sl}_2$ was first carried out in [RT90], [RT91], and has since be generalized to other Lie algebras.

Having constructed now a collection of quantum invariants depending on a complex root of unity, we can appeal to the general arguments of the following sections to extend them to topological quantum field theories. Using the abstract framework of ribbon graphs it is also possible to describe the TQFT explicitly. We will not need this either but the idea of the construction is as follows: Given a modular category $\mathcal{V}$ with simple objects $\{V_i\}_{i\in I}$, we associate to a closed oriented surface $\Sigma$ of genus $g$ the module

$$V(\Sigma) = \bigoplus_{(i_1,\dots,i_g)\in I^g} \operatorname{Hom}\left(\mathbf{1}, \bigotimes_{r=1}^{g} (V_{i_r} \otimes V_{i_r}^*)\right)$$

To a compact oriented 3-manifold $M$ with boundary, $\partial M = \Sigma_1 \sqcup -\Sigma_2$, $M$ possibly containing a ribbon graph, we now associate a homomorphism $V(\Sigma_1) \to V(\Sigma_2)$ by gluing to $M$ handlebodies with boundaries $\Sigma_1$ and $\Sigma_2$ and adding to the handlebodies particular uncoloured ribbon graphs in such a way that a choice of colouring corresponds to a linear map $\tau(M, \Sigma_1, \Sigma_2) : V(\Sigma_1) \to V(\Sigma_2)$, using the quantum invariant $\tau$ on the closed manifold arising as the result of the gluing. The details of the construction are given in [Tur94, Ch. IV], and we simply record the following. For $l$ even, let $(\tau_l, V_l)$ denote the result of this construction using the representation theory of $U_q(\mathfrak{sl}_2)$.

**Theorem 3.17.** *$(\tau_l, V_l)$ defines a $(2 + 1)$-dimensional TQFT.*

## 3.4 Skein theory

We now switch gears again and turn to the skein theoretical construction of the quantum invariant which is in fact equivalent to the one by Reshetikhin–Turaev but more suited for our purposes. The version we describe was constructed in [BHMV92], [BHMV91] and proven to lead to a TQFT in [BHMV95].

### 3.4.1 Skein modules

**Definition 3.18.** Let $M$ be a compact oriented 3-manifold. The *Kauffman module* $K(M)$ of $M$ is the $\mathbb{Z}[A, A^{-1}]$-module generated by all isotopy classes of banded links in $M$ quotiented by the skein relations shown in Figure 3.10. Here, equivalences are assumed to take place in some small 3-ball in $M$.

It is well-known that

$$K(S^3) \cong \mathbb{Z}[A, A^{-1}].$$

The skein class of a banded link $L$ in $S^3$ is called its *Kauffman bracket* and is denoted $\langle L\rangle$. For a closed surface $\Sigma$, $K(\Sigma \times I)$ is a $\mathbb{Z}[A, A^{-1}]$-algebra with

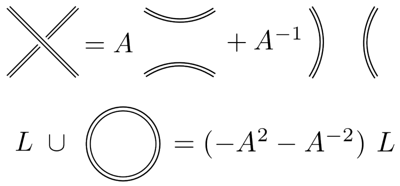


Figure 3.10: The skein relations.

multiplication given by stacking copies of $\Sigma \times I$. For the solid torus, the Kauffman module will be denoted $\mathcal{B}$ and is the algebra

$$\mathcal{B} = K(S^1 \times I \times I) \cong \mathbb{Z}[A, A^{-1}][z].$$

Under the latter isomorphism, $z^n$ corresponds to $n$ parallel unknotted longitudes in the solid torus[2]. The Kauffman module of a disjoint union of $n$ solid tori is $\mathcal{B}^{\otimes n}$.

**Definition 3.19.** Let $L$ be an $n$-component banded link in $S^3$ with ordered components $L_1, \ldots, L_n$, and let $z^{a_i}, \ldots, z^{a_n}$ be monomials in $\mathcal{B}$. Let $\langle z^{a_1}, \ldots, z^{a_n} \rangle_L$ be the result of replacing each $L_i$ by $a_i$ parallel copies and taking the Kauffman bracket of the resulting link in $S^3$. Extending linearly, we obtain the *meta-bracket*

$$\langle \cdot, \ldots, \cdot \rangle_L : \mathcal{B}^{\otimes n} \to \mathbb{Z}[A, A^{-1}].$$

Diagrammatically, this value will be pictured as in Figure 3.11. We say that $L_i$ is *coloured* by $a_i$.

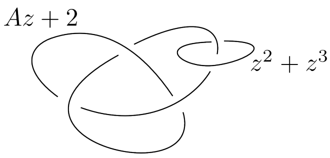


Figure 3.11: The metabracket of a 2-component link whose components are coloured by $Az + 2$ and $z^2 + z^3$.

## 3.4.2 The quantum invariant

Let $t : \mathcal{B} \to \mathcal{B}$ be the map induced on $\mathcal{B}$ by a twist about a meridian of the solid torus. We then obtain the following (see [BHMV92]).

**Lemma 3.20.** *There is a basis $e_i$, $i = 0, 1, \ldots,$ of $\mathcal{B}$ consisting of monic polynomials satisfying $e_0 = 1$, $e_1 = z$, $ze_j = e_{j+1} + e_{j-1}$. The $e_i$ are eigenvectors for $t$ with eigenvalues $\mu_i = (-1)^i A^{i^2+2i}$.*

This basis arises naturally from idempotents in the Temperley–Lieb algebra, as we will see in the next section.

From now on, we will assume that $A$ is a primitive root of unity of order $2p$ for some $p \geq 1$. In other words, we consider the above construction with coefficients in $\Lambda_p = \mathbb{Z}[A, A^{-1}]/\varphi_{2p}(A)$, where $\varphi_{2p}$ is the $2p$'th cyclotomic polynomial, and we let $\mathcal{B}_p = \mathcal{B} \otimes \Lambda_p$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on $\mathcal{B}_p$ defined by using the meta-bracket on the Hopf link, let $N_p$ denote the left-kernel of this form, and let $V_p = \mathcal{B}_p / N_p$.

[2]Here and in the following, we will not explicitly distinguish between banded links and isotopy classes of banded links.

**Theorem 3.21** ([BHMV92]). *The meta-bracket factors through* $V_p^{\otimes n}$*, and* $t$ *descends to a map* $t : V_p \to V_p$*. Furthermore,* $V_p$ *is a finite-dimensional algebra of rank* $p$ *for* $p = 1, 2$ *and of rank* $\lfloor (p-1)/2 \rfloor$ *for* $p \geq 3$*.*

We are now in a position to define the 3-manifold invariant. For a banded link in $S^3$, we denote by $b_+(L)$ and $b_-(L)$ the number of positive, respectively negative, eigenvalues of the linking matrix of $L$ as defined in Section 3.3.4. Define an element of $V_p$ (except for $p = 2$, where it is an element of $V_2 \otimes \mathbb{Z}[\frac{1}{2}]$) by $\Omega_1 = 1$, $\Omega_2 = 1 + \frac{z}{2}$, and

$$\Omega_p = \sum_{i=0}^{n-1} \langle e_i \rangle e_i$$

for $p \geq 3$, where $n = \lfloor (p-1)/2 \rfloor$ is the rank of $V_p$. It turns out that $\langle t^{\pm 1}(\Omega_p) \rangle$ are invertible elements of $\Lambda_p[\frac{1}{p}]$. Now, let $M_L$ be a closed oriented 3-manifold obtained by surgery along $L$, and define an element of $\Lambda_p[\frac{1}{p}]$ by

$$\theta_p(M_L) = \frac{\langle \Omega_p, \dots, \Omega_p \rangle_L}{\langle t(\Omega_p) \rangle^{b_+(L)} \langle t^{-1}(\Omega_p) \rangle^{b_-(L)}}. \tag{3.1}$$

**Theorem 3.22** ([BHMV92]). *The expression* $\theta_p$ *defines an invariant of closed oriented 3-manifolds.*

The proof once again relies on Kirby's theorem on how links giving rise to homeomorphic manifolds via surgery are related by certain moves. The right hand side of (3.1) is almost directly seen to be an invariant under Kirby moves, since in fact $\Omega_p$ satisfies $\langle t^{\pm 1}(\Omega_p), t^{\pm 1}(b) \rangle = \langle t^{\pm 1}(\Omega_p) \rangle \langle b \rangle$ for all $b \in \mathcal{B}$ (see also [BHMV92, Prop. 2.1]).

As before, we can extend the above invariant to an invariant of 3-manifolds with banded links. Let $K \subseteq M_L$ be a banded link viewed as a banded link in $S^3 \setminus L$.

**Theorem 3.23** ([BHMV91]). *The element of* $\Lambda_p[\frac{1}{p}]$ *defined by*

$$\theta_p(M_L, K) = \frac{\langle \Omega_p, \dots, \Omega_p, z, \dots, z \rangle_{L \cup K}}{\langle t(\Omega_p) \rangle^{b_+(L)} \langle t^{-1}(\Omega_p) \rangle^{b_-(L)}}$$

*is an invariant of closed oriented 3-manifolds containing banded links.*

### 3.4.3 The Temperley–Lieb algebra

**Definition 3.24.** The *n'th Temperley–Lieb algebra*, denoted $TL_n$, is the skein module $K(I \times I \times I, 2n)$ of the unit ball, where we fix $2n$ ordered small intervals on the boundary. That is, $TL_n$ consists of isotopy classes of banded tangles meeting the boundary in the $2n$ intervals. The algebra structure is given by using the ordering to the divide the $2n$ intervals into two sets of $n$ intervals, and gluing together copies of $I \times I \times I$, such that $n$ intervals are glued to $n$ intervals.

The Temperley–Lieb algebra $TL_n$ is generated by $n$ elements $1, a_1, \dots, a_{n-1}$ shown in Figure 3.12. Here, as before, arcs in the diagram will represent bands parallel to the plane of the diagram. By definition, an integer $i$ next to an arc corresponds to taking $i$ parallels of the arc in the plane.

Assume as before that $A$ is a primitive $2p$'th root of unity, $p > 2$, so that $A^2 - A^{-2}$ is invertible. Define $\Delta_i$ by

$$\Delta_i = (-1)^i \frac{A^{2(i+1)} - A^{-2(i+1)}}{A^2 - A^{-2}}.$$

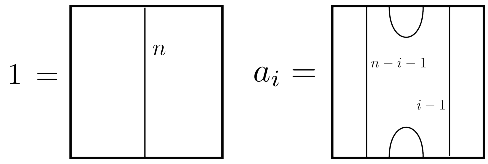


Figure 3.12: Generators of the Temperley–Lieb algebra $TL_n$.

If $A$ is chosen so that all $\Delta_0, \dots, \Delta_{n-1}$ are non-zero, then there exists a non-zero element $f^{(n)} \in TL_n$, called a *Jones–Wenzl idempotent*, satisfying

$$(f^{(n)})^2 = f^{(n)}, \quad f^{(n)}e_i = e_i f^{(n)} = 0,\ 1 \leq i \leq n-1.$$

In particular, these will exist if $A^4$ is not a $k$'th root of unity for any $k \leq n$. Diagrammatically, we denote $f^{(n)}$ as in Figure 3.13. The proof of the existence of the $f^{(n)}$ goes by inductively defining $f^{(n+1)}$ as in Figure 3.14. This recursive formula is due to Wenzl.

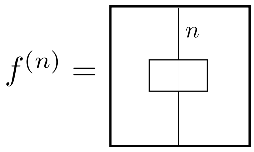


Figure 3.13: The diagram for a Jones–Wenzl idempotent.

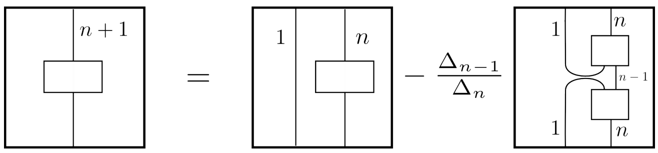


Figure 3.14: Wenzl's recursive formula for $f^{(n+1)}$.

We define a map $TL_n \to \mathcal{B}$ by placing $I \times I \times I$ in the solid torus and taking the closure, i.e. joining the $n$ intervals on the top to the $n$ intervals on the bottom by parallel arcs in the torus, encircling the torus. The image of $f^{(n)}$ is exactly the $e_n$ used to define $\Omega_p$, and $\langle e_n \rangle = \Delta_n$. The last equality follows by induction on the equality from Lemma 3.20 since $\langle z^k \rangle = (-A^2 - A^{-2})^k$.

### 3.4.4 The universal construction and TQFT

We turn now to the question of how to turn a 3-manifold invariant into a TQFT. The main focus here will be on a slightly modified version of the invariant above. We begin by generalizing the axioms presented in Section 3.2.

Consider as before a functor $(Z, V)$ from a cobordism category associating to a $d$-manifold $\Sigma$ this time a $k$-module for some commutative unital ring $k$ with a conjugation mapping $\lambda \mapsto \overline{\lambda}$. To an equivalence class of a cobordism $M$ between two $d$-manifolds, $\partial M = -\Sigma_1 \sqcup \Sigma_2$, we associate a linear map $Z(M) : \Sigma_1 \to \Sigma_2$. Here, cobordisms $M_1$ and $M_2$ between $\Sigma_1$ and $\Sigma_2$ are considered equivalent if there is an isomorphism $M_1 \to M_2$ acting identically on the boundary. The isomorphism will of course be required to preserve whatever structure the cobordisms are endowed with; typically they will simply be diffeomorphisms, but as we will see, cobordisms might contain certain extra structures.

Assume that $V(\emptyset) = k$. In this case, when $\partial M = \Sigma$, we write $Z(M)$ for $Z(M)(1) \in V(\Sigma)$, and for closed $M$, let $\langle M \rangle$ denote the corresponding element of $k$. We will call $(Z, V)$ a *quantization functor* if furthermore there is a non-degenerate sesquilinear form $\langle \cdot, \cdot \rangle_\Sigma$ on $V(\Sigma)$ such that for any $(d+1)$-manifolds $M_1, M_2$ with $\partial M_1 = \partial M_2 = \Sigma$, we have

$$\langle Z(M_1), Z(M_2) \rangle_\Sigma = \langle M_1 \cup_\Sigma (-M_2) \rangle.$$

If the set $\{Z(M) \mid \partial M = \Sigma\}$ generates $V(\Sigma)$, we say that the functor is *cobordism generated.*

On the other hand, a $k$-valued invariant $\langle \cdot \rangle$ of closed $(d+1)$-manifolds is called *multiplicative* if $\langle M_1 \sqcup M_2 \rangle = \langle M_1 \rangle \langle M_2 \rangle$ and it is called *involutive* if $\langle -M \rangle = \overline{\langle M \rangle}$.

Note that the invariant coming from a quantization functor is always both multiplicative and involutive. In fact, the converse is also true:

**Proposition 3.25** (The universal construction)**.** *Any multiplicative and involutive invariant $\langle \cdot \rangle$ of closed cobordisms extends to a unique cobordism generated quantization functor.*

*Proof.* Let $\Sigma$ be a $d$-manifold, and let $\mathcal{V}(\Sigma)$ be the $k$-module freely generated by all cobordisms from $\emptyset$ to $\Sigma$. For $M_1, M_2 \in \mathcal{V}(\Sigma)$, put

$$\langle M_1, M_2 \rangle_\Sigma = \langle M_1 \cup_\Sigma (-M_2) \rangle.$$

This extends to a form on $\mathcal{V}(\Sigma)$ which is sesquilinear by the involutivity of the invariant. Let $V(\Sigma)$ be the quotient of $\mathcal{V}(\Sigma)$ by the left kernel of this form. The form then descends to a non-degenerate sesquilinear form on $V(\Sigma)$. Finally, if $M$ is a cobordism from $\Sigma_1$ to $\Sigma_2$, define $Z(M) : V(\Sigma_1) \to V(\Sigma_2)$ by

$$Z(M)([M']) = M \cup_{\Sigma_1} M'.$$

Then $Z(M)$ is well-defined. Since furthermore multiplicativity of the invariant ensures that $V(\emptyset) = k$, the associations $(Z, V)$ give a unique cobordism generated quantization functor. □

Note that for any cobordism generated quantization functor we obtain a map

$$V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2). \tag{3.2}$$

We are now able to give the refined definition of a TQFT.

**Definition 3.26.** A cobordism generated quantization functor is called a *topological quantum field theory* if the map (3.2) is an isomorphism, if $V(\Sigma)$ is free of finite rank for all $\Sigma$, and if the pairing $\langle \cdot, \cdot \rangle_\Sigma$ determines an isomorphism $V(\Sigma) \to V(\Sigma)^*$.

It turns out that the invariants $\theta_p$ constructed in Section 3.4.2 need to be modified slightly to fit into the framework of TQFT. More precisely, the invariants have so-called *framing anomalies*, and the axioms hold only up to invertible scalar factors. The solution to this is to extend the cobordism category, requiring the manifolds in question to have certain extra structures. For our purpose, the relevant cobordism category is the category $C_2^{p_1}$ of smooth closed oriented 2-manifolds $(\Sigma, l)$ with $p_1$-structure (see [BHMV95, App. B] for details on these) and containing a set of banded intervals $l$, with bordisms being compact smooth oriented 3-manifolds $(M, L)$ extending the $p_1$-structures of the boundary components and containing banded links $L$ meeting the intervals specified in the boundary. In this category, cobordisms are equivalent if they are diffeomorphic through an orientation-preserving diffeomorphism, having isotopic links and $p_1$-structures homotopic relative to the boundary. We will not need these $p_1$-structures explicitly and rather than defining them precisely, we only observe the following:

**Proposition 3.27.** *For any closed 3-manifold $M$ there is a 1-1-correspondence, denoted $\sigma$, from (homotopy classes of) $p_1$-structures on $M$ to the integers. For 2-manifolds, $p_1$-structures are unique (up to homotopy).*

Let $k_p = \mathbb{Z}[A, A^{-1}, \kappa, \frac{1}{p}]/(\varphi_{2p}(A), \kappa^6 - u)$, where $u = A^{-6-p(p+1)/2}$. Then $k_p$ is a ring with involution $A \mapsto A^{-1}$, $\kappa \mapsto \kappa^{-1}$. Define $\eta \in k_p$ by $\eta = \kappa^3$ for $p = 1$, $\eta = (1 - A)\kappa^3/2$ for $p = 2$, and

$$\eta = \tfrac{1}{2}(A\kappa)^3(A^2 - A^{-2})p^{-1} \sum_{m=1}^{2p} (-1)^m A^{-m^2},$$

for $p = 3$. Let $b_i(M)$ denote the $i$'th Betti number of a manifold $M$. We then obtain the following main theorem of [BHMV95].

**Theorem 3.28.** *Let $M = (M, \alpha, L)$ be a closed 3-manifold with $p_1$-structure $\alpha$ and $L$ a banded link, and write $(M, L) = \bigsqcup_{i=1}^n (M_i, L_i)$, where the $M_i$ are connected components of $M$. Then the expression*

$$\langle M \rangle_p = \eta^{b_0(M)+b_1(M)} \kappa^{\sigma(\alpha)} \prod_{i=1}^n \theta_p(M_i, L_i),$$

*defines a multiplicative and involutive invariant on $C_2^{p_1}$ and thus gives rise to a quantization functor $(Z_p, V_p)$. Furthermore, if $p > 2$ is even, the functor is a TQFT.*

*Remark* 3.29. The resulting TQFT is known as the SU(2)-TQFT since it supposedly realizes Witten's Chern–Simons theory with gauge group $G = \mathrm{SU}(2)$. In the case where $p$ is odd, the above theorem fails to hold, as the map

$$V_p(\Sigma_1, l_1) \otimes V_p(\Sigma_2, l_2) \to V_p((\Sigma_1, l_1) \sqcup (\Sigma_2, l_2))$$

fails to be an isomorphism, unless one of the $l_i$ has an even number of components. Thus, in this case, we obtain a TQFT, called the SO(3)-TQFT by restricting to the cobordism category $C_2^{p_1}(\text{even})$ where objects are surfaces containing an even number of embedded intervals and morphisms are as before.

In fact, the SU(2)-theory is completely equivalent to the one constructed by Reshetikhin and Turaev for the quantum group $U_q(\mathfrak{sl}_2)$. Note also that the TQFT constructed in this section fits into the general framework of modular functors, even though we took a different approach. See [Tur94, Ch. XII] for details. More general SU($n$) theories have been constructed in the skein theoretical picture using the HOMFLY polynomial skein relations rather than the Kauffman bracket. See [Yok97], [Bla00].

In the following, we will largely ignore the $p_1$-structures in the definition of $\langle \cdot \rangle$ and consider it as an invariant defined only up to the framing anomalies. In general, anomalies could be dealt with in a variety of ways. For example, Turaev [Tur94, Ch. IV] describes the anomalies following an idea by Walker involving choices of Lagrangian subspaces in the homologies of the boundary surfaces, thus imposing extra structure on the cobordisms. Similarly, Roberts [Rob94] replaces cobordisms by framed cobordisms by changing the set of Kirby moves.

Either way, using the general construction of mapping class group representations described in Section 3.2.1, we can only expect the map $\rho$ to be a *projective* representation, i.e. it will satisfy $\rho(fg) = K\rho(f)\rho(g)$ for some invertible element $K$ in the ground ring, depending on $f$ and $g$.

### 3.4.5 Properties of the TQFT

We first note the following general fact about TQFTs which should be true by the general philosophy of Section 3.2.1. See [BHMV95, Thm. 1.2].

**Theorem 3.30.** *Let $(Z, V)$ be a TQFT in the sense of Definition 3.26, and let $M$ be a cobordism from $\Sigma$ to $\Sigma$. Let $M_\Sigma$ be the manifold obtained by identifying the two copies of $\Sigma$. Then $\langle M_\Sigma \rangle = \operatorname{tr} Z(M)$.*

In order to be able to do explicit calculations, we describe a basis of the $V(\Sigma, l)$ in terms of handlebodies containing particular banded links.

Throughout the rest of this chapter, consider the case where $p$ is an even number $p = 2n + 2$, and let $C_p = \{0, \ldots, n-1\}$. A triple $(a, b, c)$ of elements from $C_p$ is called *admissible*, if $a + b + c$ is even, $|a - b| \leq c \leq a + b$, and $a + b + c < 2n$. A *colouring* of a surface $(\Sigma, l)$ in $C_2^{p_1}$ is an assigment of a colour $c_j \in C_p$ to every component $l_j$ of $l$. A *banded trivalent graph* $G$ in a cobordism $M$ of $C_2^{p_1}$ is a graph contained in an oriented surface $SG \subseteq M$ such that

1. The graph $G$ meets $\partial M$ transversally in the set of monovalent vertices of $G$.

2. Every vertex of $G$ in the interior of $M$ has valency 2 or 3.

3. The surface $SG$ is a regular neighbourhood of $G$ in $SG$, and $SG \cap \partial M$ is a regular neighbourhood of $G \cap \partial M$ in $SG \cap \partial M$.

A *colouring* of a banded trivalent graph $G$ is a colouring of edges of $G$ by elements of $C_p$ such that colours of edges meeting in 2-valent vertices coincide, and such that colours of edges meeting a 3-valent vertex form an admissible triple.

We view a coloured graph $G$ of $M$ as a skein in the following way: The graph determines a collection of coloured embedded intervals $l$ in $\partial M$. Let $l_c$ be the *expansion* of $l$ in $\Sigma$ obtained by taking $c_j$ parallel copies of the component $l_j$, where $c_j$ is the colouring of $l_j$. The *expansion* of $(M, G)$ is the element of the module $K(M, l_c)$ (consisting of skeins in $M$ meeting $l_c$) obtained by splitting the graph $G$ into a union of I-shaped, O-shaped, and Y-shaped pieces and performing certain replacements: An I-shaped piece is a single edge coloured by $i \in C_p$ with two boundary vertices and gets replaced by the skein coloured by the idempotent $f^{(i)}$, viewed as a skein element in a ball in $M$. An O-shaped piece consists of a single $i$-coloured edge and a 2-valent vertex and similarly gets replaced by $e_i$, the closure of $f^{(i)}$. Finally, the Y-shaped pieces consist of 3 edges meeting in a trivalent vertex and here we make the expansion in Figure 3.15. The admissibility constraints ensures that this last assignment is possible, and piecing the various components back together we obtain an element of $K(M, l_c)$. Furthermore, this element does not depend on the decomposition of $G$ since the $f^{(i)}$ are idempotents. As always, the banded structures of the banded graphs are implicit in all diagrams.

We can define an invariant of closed oriented 3-manifolds with $p_1$-structures containing a banded trivalent graph by expanding the graph and using the invariant $\langle \cdot \rangle_p$. The invariant obtained this way will be multiplicative and involutive and thus define a quantization functor $(Z_p^c, V_p^c)$ on the extended category $C_2^{p_1,c}$, where surfaces and cobordisms have coloured structure. Since the invariants

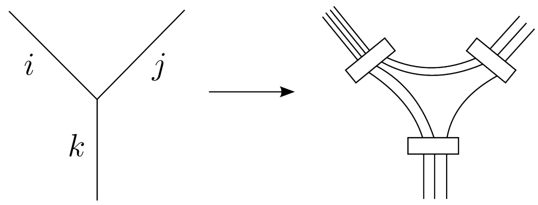


Figure 3.15: A Y-shaped piece with edges coloured by $i$, $j$, and $k$. In this example, $i = 4$, $j = 3$, $k = 3$.

coincide, for a surface with structure $(\Sigma, l)$ viewed as a surface with coloured structure by colouring the components of $l$ by $1 \in C_p$, the vector spaces $V_p^c(\Sigma)$ and $V_p(\Sigma)$ will coincide as well. Therefore, we simply write $Z_p = Z_p^c$, $V_p = V_p^c$, and $\langle \cdot, \cdot \rangle_\Sigma$ for the sesquilinear form. Furthermore, for a given cobordism with coloured structure, we will not distinguish between it and its expansion, when the meaning is clear from the context.

Let $(\Sigma, l, c)$ be a surface with coloured structure, and let $\gamma$ be a simple closed curve in $\Sigma$. Denote by ${}_i\Sigma(\gamma)_j$ the result of cutting $\Sigma$ along $\gamma$, capping off the two boundary components by disks containing 1-component banded intervals coloured by $i$ and $j$. Representing an element of $V({}_i\Sigma(\gamma)_i)$ by a manifold $M$, we obtain an element of $V(\Sigma)$ represented by the manifold $M'$ obtained by identifying the two disks, so $V({}_i\Sigma(\gamma)_i)$ embeds in $V(\Sigma)$.

**Theorem 3.31** (Coloured splitting theorem)**.** *Let $\gamma \subseteq \Sigma$ as above. Cutting along $\gamma$ gives an orthogonal decomposition*

$$V_p(\Sigma) = \bigoplus_{i=0}^{n-1} V_p({}_i\Sigma(\gamma)_i).$$

We end this chapter by describing explicitly a basis of the vector spaces $V_p(\Sigma, l, c)$.

**Theorem 3.32.** *Let $(\Sigma, l, c)$ be a connected closed surface with coloured structure. Let $H$ be a handlebody with boundary $\Sigma$, and let $G$ be a coloured banded graph with only 1-valent and 3-valent vertices, such that 1-valent vertices correspond to the intervals $l_i$, and such that $H$ is as a tubular neighborhood of $G$ (see Figure 3.16). Then $V_p(\Sigma, l, c)$ has an orthogonal basis consisting of colourings of $G$ compatible with the colouring of $l$ in the sense that colourings of edges incident to an interval in $\Sigma$ are coloured by the colour of the interval.*

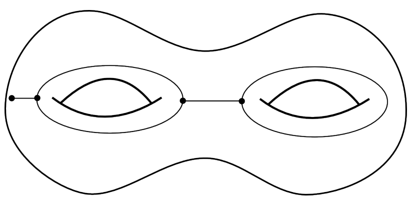

Figure 3.16: The standard basis graph of a surface $(\Sigma, l, c)$. Here $\Sigma$ has genus 2 and $l$ consists of a single component.

# Chapter 4

# Quantum representations

## 4.1 Skein theory revisited

We begin this chapter by describing two ways the skein theoretical construction of TQFTs gives rise to projective representations of the mapping class group of a surface. Let $p > 2$ be even, and let $(Z_p, V_p)$ be the corresponding TQFT.

Let $(\Sigma, l, c)$ be a surface with coloured structure, and let $(H, G)$ be one of the basis elements of Theorem 3.32. That is, $H$ is a handlebody with boundary $\Sigma$, and $G$ is a banded trivalent graph in $H$ endowed with a colouring compatible with that of $\Sigma$. By the general construction in Section 3.2.1, an action of the mapping class group is given by gluing to $H$ the mapping cylinder $M_\varphi$ for a homeomorphism $\varphi : \Sigma \to \Sigma$. In the case where $l = \emptyset$, this immediately defines a new element of $V_p(\Sigma, l, c)$, and in the case $l \neq \emptyset$, we extend the coloured structure in $H$ to $H \cup M_\varphi$ by extending the graph $G$ to a graph

$$G \cup (l \times [0, \tfrac{1}{2}] \cup_\varphi l \times [\tfrac{1}{2}, 1]) \subseteq H \cup M_\varphi,$$

and colouring the new edge compatibly. Here, mapping classes are assumed to preserve the coloured structure; equivalently, we consider mapping classes of $(\Sigma, l, c)$ viewed as a surface with boundary, each boundary component encircling a component of $l$.

Now, it is known that the mapping cylinder of a Dehn twist about a curve $\gamma$ in $\Sigma$ can be presented by surgery on $\Sigma \times [0, 1]$ along the curve $\tilde{\gamma} \times \{\frac{1}{2}\}$. Here, $\tilde{\gamma}$ is the banded link obtained by a full negative twist to the curve $\gamma$ viewed as a link with the blackboard framing with respect to the surface $\Sigma \times \{\frac{1}{2}\}$. In the language of surgery, $\tilde{\gamma}$ is the curve $\gamma$ with framing $-1$ with respect to $\Sigma \times \{\frac{1}{2}\}$. Thus, from the skein theoretical construction of TQFT, the action $\rho_p(t_\gamma)$ of a Dehn twist about $\gamma$ on $(H, G)$ is given by adding the banded link $\tilde{\gamma}$ coloured by $\Omega_p$ to the handlebody as in Figure 4.1. We could now simply define the projective action of a general mapping class $f$ by writing it as a product of Dehn twists, $f = t_{\alpha_1} \cdots t_{\alpha_n}$, and letting

$$\rho_p(f) = \rho_p(t_{\alpha_1}) \cdots \rho_p(t_{\alpha_n}).$$

Working in the skein module of the handlebody, it is now a feasible task to compute the action of a word of Dehn twists on $V_p$ by hand, at least for small enough values of $p$.

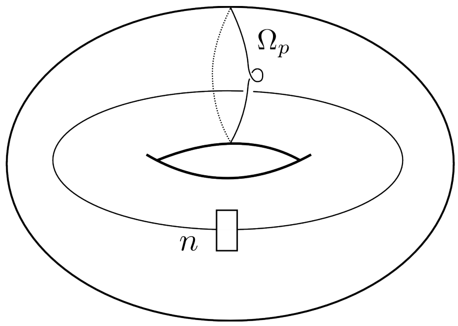


Figure 4.1: The action on $V_p$ of a meridian twist in the torus on the handlebody element containing the skein corresponding to $e_n$. We analyze this example further in Lemma 4.3.

### 4.1.1 On notation

Recall that we defined the TQFTs $(Z_p, V_p)$ only for $p > 2$ even, and that these arose from evaluating the Kauffman bracket at a primitive $2p$'th root of unity. Throughout the rest of this report, we will make a somewhat gross abuse of notation and write $(Z_k, V_k)$ for the TQFT $(Z_p, V_p)$, where $p = 2k + 4$. That is, for every $k = 0, 1, \dots,$ we let $(Z_k, V_k)$ denote the *$SU(2)$-TQFT at level $k$* obtained by evaluating the Kauffman bracket at a primitive $4k + 8$'th root of unity.

### 4.1.2 Roberts' construction

Roberts [Rob94] has given an alternative description of the mapping class group action which in some cases is more suited for direct calculations. Furthermore, its construction is significantly more elementary, lending itself not to the abstract setup of TQFT and surgeries but only to concrete manipulations with handlebodies.

For a compact oriented 3-manifold $M$, denote by $K_\xi(M)$ the complex vector space obtained from $K(M)$ by the homomorphism $A \mapsto \xi$ for a non-zero complex number $\xi$. Assume in the following that $\xi$ is a primitive root of unity of order $4k+8$. In this case, $K_\xi(S^3)$ is isomorphic to $\mathbb{C}$. Let $\Sigma$ be a closed oriented surface embedded into $S^3$, such that its complement is a union of two handlebodies $H$ and $H'$. We define a bilinear form

$$\langle \cdot, \cdot \rangle : K_\xi(H) \times K_\xi(H') \to K_\xi(S^3) = \mathbb{C}$$

on generators as follows: If $x \in K_\xi(H)$, $x' \in K_\xi(H')$ represent links $L, L'$ in $H, H'$ then $\langle x, x' \rangle$ is given by the value of $L \cup L'$ in $K(S^3)$, considering $H$ and $H'$ as subsets of $S^3$. Taking the quotient by the left kernel in $K_\xi(H)$ and right kernel in $K_\xi(H')$ we obtain a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : V_k(\Sigma) \times V_k'(\Sigma) \to \mathbb{C}.$$

It turns out that the $V_k(\Sigma)$ are finite-dimensional vector spaces, and by Proposition 1.9 of [BHMV95], they are isomorphic to the ones arising from TQFT.

We now proceed to describe the action of the Dehn twists on $V_k(\Sigma)$. Let $K$ denote the set of Dehn twists about curves in $\Sigma$ bounding discs in $H$, and let $K'$ be the set of those bounding discs in $H'$. Elements of $K$ extend in a unique way to homeomorphisms of $H$, giving rise to an action by such Dehn twists on $K_\xi(H)$ preserving the left kernel of the above form. Therefore, the group generated by the Dehn twists in $K$ act on $V_k(\Sigma)$. Denote this action by $\rho_k$. To describe the action of any element of the mapping class group, it now suffices to describe the action by elements of $K'$, since elements of $K \cup K'$ generate the mapping class group. For an element $f' \in K'$, define $\rho_k(f')$ by

$$\langle \rho_k(f')(x), y\rangle = \langle x, (f')^{-1}(y)\rangle,$$

for $x \in V_k(\Sigma)$, $y \in V_k'(\Sigma)$. Since the form is non-degenerate, this determines $\rho_k$ on the group generated by $K'$.

An element of the mapping class group could be written as a word in Dehn twists in more than one way, and for a mapping class $f = t_{\alpha_1} \cdots t_{\alpha_n}$, $t_{\alpha_i} \in K \cup K'$, one should verify that

$$\rho_k(f) := \rho_k(t_{\alpha_1}) \cdots \rho_k(t_{\alpha_n}),$$

is well-defined – at least up to a scalar factor, so that it gives rise once again to a projective representation of $\Gamma(\Sigma)$. This is Theorem 3.12 of [Rob94]. As projective representations $\rho_k : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}V_k)$, the $\rho_k$ agree with the ones arising from TQFT.

This construction extends immediately to the case where the surface and handlebodies have coloured structure, and we can use the basis for $V_k(\Sigma)$ of Theorem 3.32 to obtain an explicit expression for the projective representation in this case.

*Remark* 4.1. There is a general procedure to turn projective representations into honest ones. If $\rho : G \to PGL(V)$ is a projective representation of a group $G$ on a vector space $V$ with ground field $F$, there exists a central extension $0 \to A \to \tilde{G} \to G \to 0$ and a representation $\sigma : \tilde{G} \to GL(V)$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{G} & \xrightarrow{\pi} & G & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\rho} & & \\ 0 & \longrightarrow & F^* & \xrightarrow{\text{diag}} & GL(V) & \longrightarrow & PGL(V) & \longrightarrow & 0. \end{array}$$

Rather than considering the projective representations as homomorphisms to projective linear groups, it is some times more natural to consider them as lifts of homomorphisms determined on the generators. For example, in the case of the torus with mapping class group generators $t_a$ and $t_b$, the projective ambiguity turns out to lie completely in the relation $(t_a t_b)^6 = 1$, and one can lift the projective representation to the central extension $B_3$, once again generated by $t_a$ and $t_b$ but now with a single relation $t_a t_b t_a = t_b t_a t_b$. This generalizes to the higher genus case to some extent – see [MR95].

## 4.2 Connection with geometric quantization

We now return to the question of how to use geometric quantization to construct quantum representations of the mapping class group and compare the resulting representations with those arising from topological quantum field theory. Throughout this section, $G = \mathrm{SU}(n)$. One reference for the following is [And92].

Let $\Sigma$ be a closed surface with one boundary component, and let

$$\mathcal{M}_\sigma = (\mathcal{M}^*, \omega, I_\sigma)$$

be the Kähler manifolds with holomorphic line bundles $\mathcal{L}_\sigma^k \to \mathcal{M}_\sigma$ arising as in Section 2.5.3, with Kähler structures parametrized by Teichmüller space $\mathcal{T}$.

Here and in the following, we write $\mathcal{L}^k = \mathcal{L}^{\otimes k}$. Let $\mathcal{V}_k \to \mathcal{T}$ be the vector bundle over $\mathcal{T}$ with fiber $\mathcal{V}_{k,\sigma} = H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$. Let $\sigma \in \mathcal{T}$ be fixed.

Recall that $\mathrm{Diff}(\Sigma)$ acts on $\mathcal{M}^*$, and that the action of $\mathrm{Diff}_0(\Sigma) \subseteq \mathrm{Diff}(\Sigma)$ is trivial. It follows that the mapping class group $\Gamma(\Sigma)$ acts on $\mathcal{M}^*$, and this action lifts to an action on $\mathcal{L}^k$ in the following way: Let $\tilde{\mathcal{L}} = \mathcal{A} \times \mathbb{C}$ be the trivial bundle over $\mathcal{A}$, and define a map $\Psi : \mathcal{A} \times \mathrm{Diff}(\Sigma) \to \mathrm{U}(1)$ by

$$\Psi(A, f) = \exp(2\pi i(\mathrm{CS}(\widetilde{f^*A}) - \mathrm{CS}(\tilde{A})),$$

extending $A$ and $f^*A$ to connections $\tilde{A}$ and $\widetilde{f^*A}$ in a principal bundle over a 3-manifold with boundary $\Sigma$. This map satisfies

$$\Theta(f^*A, g)\Psi(A, f) = \Psi(g^*A, f)\Theta(A, g),$$

where $\Theta$ is the map (2.3) used to construct the prequantum line bundle. Furthermore, it can be proved that $\Psi(A, f) = 1$ for $f \in \mathrm{Diff}_0(\Sigma)$. Thus the map $\mathcal{L}_A \to \mathcal{L}_{f^*A}$ mapping $(A, z) \mapsto (f^*A, \Psi(A, f)z)$ projects to an action of $\Gamma(\Sigma)$ on the line bundle $\mathcal{L} \to \mathcal{M}^*$.

In fact, this action determines for a mapping class $f \in \Gamma(\Sigma)$ a map

$$f^* : H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma) \to H^0(\mathcal{M}_{f^*\sigma}, \mathcal{L}^k_{f^*\sigma}).$$

Now, choose a path $\gamma$ in $\mathcal{T}$ from $f^*\sigma$ to $\sigma$, and let

$$P_{f^*\sigma,\sigma} : H^0(\mathcal{M}_{f^*\sigma}, \mathcal{L}^k_{f^*\sigma}) \to H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma)$$

denote parallel transport in $\mathcal{V}_k$, determined by the Hitchin connection. Since the connection is projectively flat, this depends on the path chosen only up to scalar multiplication. Thus, the composition

$$\rho^n_k(f) = P_{f^*\sigma,\sigma} f^* \in \mathrm{Aut}(\mathbb{P}H^0(\mathcal{M}_\sigma, \mathcal{L}^k_\sigma))$$

determines a projective representation of $\Gamma(\Sigma)$. Using again the projective flatness of the connection, this is seen to depend on $\sigma$ only up to conjugation. We could also use the connection to canonically identify all fibers with the space $\mathbb{P}V_k$ of covariant constant sections in $\mathbb{P}\mathcal{V}_k$ and in this way obtain projective representations

$$\rho^n_k : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}V_k).$$

In the case where $\Sigma$ has a single boundary component and $d \in \mathbb{Z}_n$ satisfies $\gcd(n, d) = 1$ or $(n, d) = (2, 0)$, it is possible to lift the action of $\Gamma(\Sigma)$ to the line bundle over the moduli space $\mathcal{M}^d_{\mathrm{SU}(n)}$ defined in Section 2.5.2, using an argument similar to that of [Fre95] used to construct the line bundle in the non-closed case. Therefore, we obtain projective representations

$$\rho^{n,d}_k : \Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}V_k), \tag{4.1}$$

where $V_k$ is constructed as before. The projective representations arising from geometric quantization and topological quantum field theory have several similarities, a few of which are recorded in the next section. One important one is the fact that the dimensions of the representation spaces agree and are given by the *Verlinde formula*:

**Theorem 4.2.** *Let $\tilde{V}_k(\Sigma_g)$ be the representation spaces obtained by the above construction for $n = 2$ for a closed genus $g \geq 2$ surface, and let $V_k(\Sigma_g)$ denote the vector spaces constructed from the TQFT. Then*

$$\dim \tilde{V}_k(\Sigma_g) = \dim V_k(\Sigma_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g},$$

In work in progress, Andersen and Ueno ([AU07a], [AU07b], [AU06], [AU]) prove that the two constructions are in a sense completely equivalent for $G = \mathrm{SU}(2)$, when evaluating the skein theory representation at the root of unity $A = -e^{\frac{2\pi i}{4k+8}}$. See [And10, Thm. 8] for a precise statement of this result. Noting this, throughout the rest of this report, we stick to the notation of Section 4.1 and unless otherwise stated, we make no restriction on the primitive root of unity $A$ when formulating conjectures and results.

## 4.3 Kernels and images

One of our main goals is to understand the algebraic properties of the quantum representations and to analyze two conjectures involving their kernels and images. Note that for a surface $\Sigma$, we are interested in the kernel and image of $\rho_k$ as a map $\Gamma(\Sigma) \to \mathrm{Aut}(\mathbb{P}V_k(\Sigma))$. That is, a mapping class $f \in \Gamma(\Sigma)$ is said to be in the *kernel* of $\rho_k$, if $\rho_k(f)$ is a scalar multiple of the identity, and we can make sense of the *order* of an element in the usual way.

### 4.3.1 Dehn twists

It follows from Corollary 1.5 and the following result that none of the $\rho_k$ are faithful.

**Lemma 4.3.** *Dehn twists about non-separating curves in a closed oriented surface $\Sigma$ with empty coloured structure have order $4k+8$ in the projective representation $\rho_k$, $k \geq 2$.*

*Proof.* In the case $g = 0$ all curves are separating, and there is nothing to prove.

Consider the case $g = 1$. It is enough to prove the statement for a single non-separating curve $\gamma_0$ as for general non-separating curve $\gamma$ there exists a homeomorphism $\varphi$ taking $\gamma_0$ to $\gamma$ by the change of coordinates principle. Then, by Lemma 1.7,

$$\rho_k(\gamma) = \rho_k(\varphi)\rho_k(\gamma_0)\rho_k(\varphi)^{-1},$$

and the orders of $\rho_k(\gamma)$ and $\rho_k(\gamma_0)$ will coincide, since conjugation and taking powers commute. Let $\gamma_0$ be the meridian curve. It follows from Lemma 3.20 and skein theory considerations (or from Roberts' construction of $\rho_k$) that in the basis $e_0, \dots, e_k$ of $V_k$, the action of $t_{\gamma_0}$ is given by

$$\rho_k(t_{\gamma_0}) = \mathrm{diag}(\mu_0, \dots, \mu_k)^{-1}$$

up to a scalar. Clearly $\rho_k(t_{\gamma_0})^{4k+8}$ is the identity, and we only need to prove that the order of $\rho_k(t_{\gamma_0})$ is not less than $4k+8$. If some power $n$ of this matrix is a scalar times the identity, then this scalar is 1 since $\mu_0 = 1$. Now,

$$\mu_m = (-1)^m A^{m^2+2m} = A^{(2k+4)m+m^2+2m} = A^{(2k+6)m+m^2},$$

and we are done, if we can prove the following claim: If there exists $n$ such that $4k+8$ divides $((2k+6)m+m^2)n$ for all $m = 0, \dots, k$, then $n \in (4k+8)\mathbb{Z}$. Since $k \geq 2$, it suffices to prove that

$$1 = \gcd(4k+8, (2k+6)1+1^1, (2k+6)2+2^2) = \gcd(4k+8, 2k+7, 4k+16).$$

Now if a natural number $a$ divides $4k+8$ and $4k+16$, it divides 8 and is either 1 or even, and no even numbers divide $2k+7$.

The case $g \geq 2$ follows from the $g = 1$ case by Theorem 3.31, as one can find a separating curve in the surface, splitting the surface into a disjoint union of a genus $g-1$ surface and a torus. The corresponding TQFT vector space splits into a number of vector spaces associated to coloured surfaces, and the mapping class group action respects this splitting. Not it suffices to consider the action of the meridian curve in the torus with colour 0, which is exactly the one we considered before. □

*Remark* 4.4. In the case $k = 1$, the order in Lemma 4.3 is 4, since $\mu_1^4 = A^{12} = 1$.

For surfaces with coloured structure and for separating curves, pretty much anything goes, as the following examples illustrate.

Let $k \geq 6$ and consider the sphere with eight distinguished intervals with colourings given by $(1, 1, 1, 1, 1, 1, 1, 5)$ as in Figure 4.2. The corresponding graph in $B^3$ will have only 2 admissible colourings $(0, 1, 2, 3, 4)$ and $(2, 3, 4, 5, 6)$ (reading from left to right as illustrated on the figure), independent of the level. A Dehn twist about a curve bounding a disk in $B^3$ meeting only the edge coloured with 4 or 6 will be represented by the diagonal matrix $\operatorname{diag}(A^{24}, A^{48})$ in the basis given by the colourings, and at level $k$, the order of this matrix will be $(4k + 8)/\gcd(4k + 8, 24)$, which in general is less than $4k + 8$. Here, the pair $(4, 6)$ was simply chosen to ensure a large number of common divisors in the powers of the entries in the matrix representing the twist.

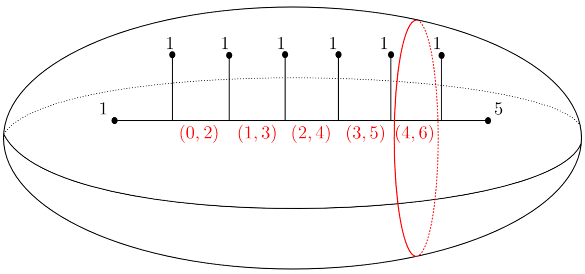


Figure 4.2: A sphere with 8 coloured intervals and a separating curve.

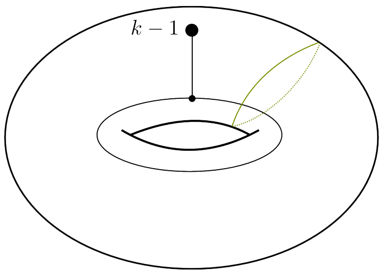


Figure 4.3: A torus with a single coloured interval.

Another example is given by the torus $\Sigma_1$ with coloured structure $(l, c)$, where $l$ consists of a single component. Let $G$ be the graph in the solid torus shown in Figure 4.3. If $k$ is even, and the component is coloured by $k$, there is only a single admissible colouring of $G$, and the corresponding projective representation is trivial. In general, when $l$ consists of a single component coloured by an even number $i$,

$$\dim V_k(\Sigma_1, l, c) = k - i + 1, \tag{4.2}$$

as is easily seen by admissibility constraints on $G$. Now, if $k$ is odd, and $l$ is coloured by $i = k-1$, we claim that the Dehn twist $t_a$ about the meridian curve shown in the image is represented by a matrix of order 4. In this case, the only admissible colourings of the non-coloured edge of $G$ are $i/2$ and $i/2+1$. Now $t_a$ acts diagonally on these as multiplication by $\mu_{i/2}$ and $\mu_{i/2+1}$ respectively. To prove that $t_a^4$ is in the kernel of $\rho_k^{k-1}$, we only have to realize that $\mu_{i/2}^4 = \mu_{i/2+1}^4$, which follows from the following simple calculation:

$$\begin{aligned}\mu_{i/2}^4\mu_{i/2+1}^{-4} &= (A^{i^2/4+i})^4(A^{i^2/4+i+1+i+2})^{-4} = A^{4i-8i-12}\\ &= A^{-(4i+12)} = A^{-(4k+8)} = 1.\end{aligned}$$

Now, the same holds true for any other non-separating curve by the argument of Lemma 4.3.

This argument appears to be hard to generalize to the higher genus case, as the number of admissible colourings grows rapidly. It is, however, not specific to the case where $l$ has one component, and a similar argument works for the torus containing the coloured structure shown in Figure 4.4; once again, the Dehn twists pictured all act with order 4.

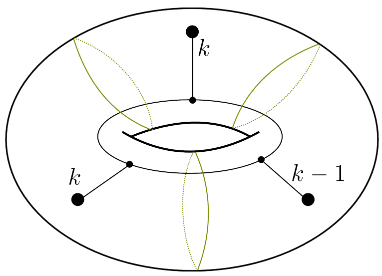


Figure 4.4: A torus with three intervals coloured $(k, k, k-1)$.

As the level increases, so does the dimensions of $V_k$ (when $g \geq 1$), and in the light of the Lemma 4.3, one might expect the representations to be increasingly faithful. The following result was first proven in the geometric case by Andersen [And06a], later in the skein theoretical setup (in the SU(2) case for all compact surfaces) by Freedman, Walker, and Wang [FWW02], and more recently by Marché and Narimannejad [MN08] using other methods. It is stated here in the geometric formulation.

**Theorem 4.5** (Asymptotic faithfulness). *Let $\Sigma$ be a compact surface of genus $g \geq 2$ with one boundary component, and let $\rho_k^{n,d}$ be the projective representation of $\Gamma(\Sigma)$ from (4.1).*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k^{n,d}) = \begin{cases} \{1, H\} & \text{if } g=2, n=2, d=0 \\ \{1\} & \text{otherwise} \end{cases},$$

*where $H$ is hyperelliptic involution.*

*Sketch of proof.* Let $\mathcal{M}_\sigma = (\mathcal{M}^d_{\mathrm{SU}(n)}, \omega, I_\sigma)$. A main point of the proof is that the only elements of $\Gamma(\Sigma)$ which act trivially on $\mathcal{M}$ are the elements specified on the right hand side in the theorem. Parallel transport in $\mathcal{V}_k$ induces a parallel transport in $\mathrm{End}(\mathcal{V}_k)$ (see [And06a]). Let $\varphi \in \Gamma(\Sigma)$, denote by $\varphi^*$ the action of $\varphi$ on $\mathcal{M}$, and let $f \in C^\infty(\mathcal{M})$ be any smooth function on $\mathcal{M}$. We have the following commutative diagram.

$$\begin{array}{ccccc}
H^0(\mathcal{M}_\sigma,\mathcal{L}^k_\sigma) & \xrightarrow{\varphi^*} & H^0(\mathcal{M}_{\varphi^*\sigma},\mathcal{L}^k_{\varphi^*\sigma}) & \xrightarrow{P_{\varphi^*\sigma,\sigma}} & H^0(\mathcal{M}_\sigma,\mathcal{L}^k_\sigma) \\
\downarrow{\scriptstyle T^k_{f,\sigma}} & & \downarrow{\scriptstyle T^k_{f\circ\varphi^*,\varphi^*\sigma}} & & \downarrow{\scriptstyle P_{\varphi^*\sigma,\sigma}T^k_{f\circ\varphi^*,\varphi^*\sigma}} \\
H^0(\mathcal{M}_\sigma,\mathcal{L}^k_\sigma) & \xrightarrow{\varphi^*} & H^0(\mathcal{M}_{\varphi^*\sigma},\mathcal{L}^k_{\varphi^*\sigma}) & \xrightarrow{P_{\varphi^*\sigma,\sigma}} & H^0(\mathcal{M}_\sigma,\mathcal{L}^k_\sigma)
\end{array}$$

Let $\varphi \in \cap_{k=1}^\infty \ker(\rho_k^{n,d})$, and let us see that $\varphi$ acts trivially on $\mathcal{M}$. By construction of $\rho_k^{n,d}$, we have that $P_{\varphi^*\sigma,\sigma}\circ\varphi^*$ is a scalar multiple of the identity, and by the above diagram, $T^k_{f,\sigma} = P_{\varphi^*\sigma,\sigma}T^k_{f\circ\varphi^*,\varphi^*\sigma}$. Andersen proves that for any two points $\sigma_0,\sigma_1\in\mathcal{T}$, the Toeplitz operators satisfy

$$\|P_{\sigma_0,\sigma_1}T^k_{f,\sigma_0} - T^k_{f,\sigma_1}\| = O(k^{-1}).$$

It follows from this that

$$\begin{aligned}
\lim_{k\to\infty}\|T^k_{f-f\circ\varphi^*,\sigma}\| &= \lim_{k\to\infty}\|T^k_{f,\sigma} - T^k_{f\circ\varphi^*,\sigma}\| \\
&= \lim_{k\to\infty}\|P_{\varphi^*\sigma,\sigma}T^k_{f\circ\varphi^*,\varphi^*\sigma} - T^k_{f\circ\varphi^*,\sigma}\| = 0.
\end{aligned}$$

Now, by Lemma 2.12, $f - f\circ\varphi^* = 0$, so $\varphi$ acts trivially on $\mathcal{M}$. □

### 4.3.2 Conjectures and experiments

In light of the previous section, we arrive at the following conjecture.

**Conjecture 4.6.** *Let $\Sigma$ be a surface, possibly with coloured structure. Then the kernel of the corresponding quantum representation $\rho_k$ at level $k$ will be generated by powers of Dehn twists of all possible curves together with hyper-elliptic involution in the cases $g = 1,2$. For non-separating curves in non-coloured surfaces, the powers are $4k+8$.*

This conjecture is hard to prove or disprove directly, since in general it is a non-trivial task to determine whether or not a general mapping class can be written as a word in the specific powers of Dehn twists. We can however see the following.

**Proposition 4.7.** *For a torus with empty structure, the conjecture fails to hold.*

*Proof.* For a group $G$ and a natural number $n$, let $G^n$ denote the normal subgroup of $G$ generated by all $n$'th powers of elements in $G$. In [New62], it is shown that the group $\mathrm{PSL}(2,\mathbb{Z})/\mathrm{PSL}(2,\mathbb{Z})^n$ has infinite order for $n = 6\cdot 72 = 432$. Since the surjective composition

$$\mathrm{SL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{PSL}(2,\mathbb{Z})/\mathrm{PSL}(2,\mathbb{Z})^n$$

factors over $\mathrm{SL}(2,\mathbb{Z})^n$, it follows that $\mathrm{SL}(2,\mathbb{Z})/\mathrm{SL}(2,\mathbb{Z})^n$ has infinite order.

The normal subgroup $\langle t_\alpha^n\rangle$ generated by $n$'th powers of all possible Dehn twists about curves in the torus is obviously contained in the group $\Gamma_1^n$ generated by *all* $n$'th powers. Since $\Gamma_1$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, the group $\Gamma_1/\langle t_\alpha^{432}\rangle$ has infinite order. The same will be true for $\Gamma_1/\langle t_\alpha^{432}, H_1\rangle$, where $H_1$ denotes the mapping class of elliptic involution.

In the torus case, the only curves giving rise to non-trivial Dehn twists are non-separating. Now, if the above conjecture were true, we would therefore obtain an isomorphism $\Gamma_1/\langle t_\alpha^{4k+8}, H_1\rangle \to \rho_k(\Gamma_1)$. It follows from a theorem by Gilmer, [Gil99], that the images $\rho_k(\Gamma_1)$ are finite for all levels, giving a contradiction at level $k = 106$, since $4\cdot 106 + 8 = 432$. □

The finiteness of the image used in the proof above turns out to occur only in the case of $g = 1$ and no coloured structure, as follows from the following result by Masbaum, [Mas99].

**Theorem 4.8.** *For a surface $\Sigma$ with genus $g \geq 2$, the image of $\rho_k$ is infinite for $k \neq 1, 2, 4, 8$.*

*Sketch of proof.* In [Mas99], Masbaum constructs explicitly a mapping class in the sphere containing four intervals coloured by the number 1 whose image under $\rho_k$ is infinite in the specified range. By Theorem 3.31, we can view this mapping class as a mapping class on $\Sigma$, still acting with infinite order, proving the theorem. We recall the construction of the mapping class. The vector space $V_k(S^2, l, (1,1,1,1))$ is two-dimensional for all levels $k \geq 2$ generated by the two handlebodies with coloured structure shown in Figure 4.5. Let $a$ and $b$ be the curves in $(S^2, l, (1,1,1,1))$ shown in Figure 4.6, and let $w = t_a^{-1} t_b$. Making a change of basis, Masbaum finds an explicit representation matrix $M_k$ for the action of $w$ and proves that, for a particular embedding $A \mapsto \xi$ of the ground ring into $\mathbb{C}$, the matrix has trace $|\mathrm{tr}(M_k)| > 2$ for $k \neq 1, 2, 4, 8$. Thus, since the vector spaces $V_k$ were two-dimensional, at least one eigenvalue $\lambda_k$ of $M_k$ satisfies $|\lambda_k| > 1$. Note that since the anomalies in an embedding will be complex roots of unity, and whereas the eigenvalues will change under scalar multiplication by these, their absolute values will not, and we conclude that $M_k$ has infinite order. □

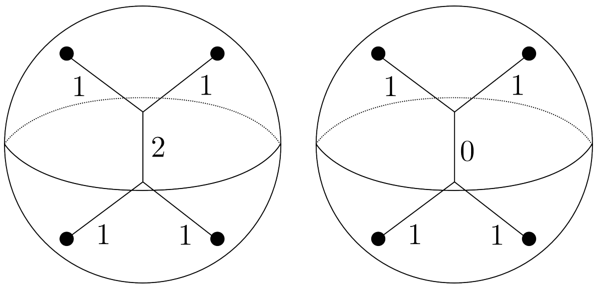


Figure 4.5: The generators of $V_k(S^2, l, (1,1,1,1))$.

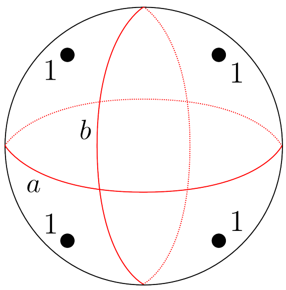


Figure 4.6: The curves $a$ and $b$.

The same example was carried out with the geometric construction of the quantum representations in the preprint [LPS10]. Here, the images of the mapping class group in $V_k(S^2, l, (1,1,1,1))$ are described explicity and all proven finite when $k = 1, 2, 4, 8$. Now, it does *not* follow, that the images of $\rho_k$ in

$V_k(\Sigma, l, c)$ are finite for general $\Sigma$ with genus $g \geq 2$ in these cases. We examine the case of $k = 8$ later.

Norbert A'Campo has written a PARI/GP tool[1] to calculate explicitly the matrices for the actions of certain Dehn twists in a particular handlebody basis of $V_k$ for low enough levels $k$. The software uses the fact that to evaluate general coloured link diagrams, it suffices to be able to evaluate coloured theta graphs and tetrahedral graphs, whose evaluations are computed in [MV94]. Using this, we can check some of the cases left out by the results above.

Consider the theory at level $k = 5$ in the case of the torus containing one interval coloured by 2. By (4.2), $\dim V_5(\Sigma_1, l, 2) = 4$. Letting as always $t_a$ and $t_b$ denote the Dehn twists generating the mapping class group, the evaluation of $\rho_5(t_a^{-1}t_b)$ at the primitive root of unity $A = e^{\frac{2\pi i \cdot 5}{4\cdot 5+8}}$ gives a matrix having an eigenvalue $\lambda$ of absolute value $|\lambda| > 1$. Thus, in this case as well, we can conclude that the image is infinite. In other words, the finiteness of the image in the genus $g = 1$ does not hold when the surface contains coloured structure.

For a surface with genus $g \geq 2$ and no coloured structure, the image turns out to be infinite at level $k = 8$; one of the cases not covered by Theorem 4.8. In this case, the TQFT vector space has dimension 165 and concrete calculations become rather messy. We can however apply the same procedure as before, using A'Campo's software: Let $t_{a_1}, \ldots, t_{a_5}$ be the twists in $\Sigma_2$ about the curves shown in Figure 1.6. Now, the evaluation of $\rho_8(t_{a_1}^{-1}t_{a_2}t_{a_3}^{-1}t_{a_4}t_{a_5}^{-1})$ at the root of unity $A = e^{\frac{2\pi i \cdot 3}{4\cdot 8+8}}$ once again gives an eigenvalue of absolute value greater than 1, and so the image of $\rho_8$ is infinite. Note that in the cases $k = 1, 2, 4$, the representations of this mapping class have orders 6, 6, and 36 respectively. It follows from the coloured splitting theorem that $\rho_8$ has infinite image for all surfaces of genus $g \geq 2$.

In the case of a surface containing a single coloured interval, we can also use the software to immediately describe the orders of elements as functions of the level and colouring. Let $\rho_k^i$ denote the quantum representation associated to a genus $g = 1$ surface containing an interval coloured $i$. Tables 4.1 and 4.2 describe the orders of two particular mapping classes. A dash represents an order greater than 100.

| $k \backslash i$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 1 | | | | | |
| 3 | 15 | 3 | | | | | |
| 4 | 12 | 3 | 1 | | | | |
| 5 | 12 | – | 3 | | | | |
| 6 | 12 | – | 3 | 1 | | | |
| 7 | 6 | 18 | – | 3 | | | |
| 8 | 30 | – | 30 | 3 | 1 | | |
| 9 | 15 | – | – | – | 3 | | |
| 10 | 12 | 12 | – | – | 3 | 1 | |
| 11 | 21 | – | – | – | – | 3 | |
| 12 | 24 | – | – | – | – | 3 | 1 |
| 13 | 60 | – | – | – | – | – | 3 |

Table 4.1: The orders of $\rho_k^i(t_a t_b^{-1})$.

For example we note again from this that we really do need the assumption of no coloured structure in the last part of Conjecture 4.6. Namely, we see that

[1] The software is currently available at http://www.geometrie.ch/TQFT/

| $k \backslash i$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| 2 | 8 | 1 | | | | | |
| 3 | 6 | 2 | | | | | |
| 4 | 12 | 12 | 1 | | | | |
| 5 | 4 | – | 2 | | | | |
| 6 | 16 | – | 16 | 1 | | | |
| 7 | 18 | – | – | 2 | | | |
| 8 | 12 | – | 60 | 20 | 1 | | |
| 9 | 10 | – | – | – | 2 | | |
| 10 | 24 | – | – | – | 24 | 1 | |
| 11 | 6 | – | – | – | – | 2 | |
| 12 | 8 | – | – | – | – | 28 | 1 |
| 13 | 6 | – | – | – | – | – | 2 |

Table 4.2: The orders of $\rho_k^i(t_a^2 t_b^{-1})$.

$(t_a^2 t_b^{-1})^2 \in \ker \rho_7^2$, but since the mapping class group in this case is

$$\Gamma_{1,1} \cong \langle t_a, t_b \mid t_a t_b t_a = t_b t_a t_b \rangle,$$

we have a well-defined homomorphism $l : \Gamma_{1,1} \to \mathbb{Z}$ mapping $t_a$ and $t_b$ to 1. Now, if Conjecture 4.6 were true in this case, we should expect that $l(\ker \rho_k^i) \subseteq \gcd(6, 4k+8)\mathbb{Z}$, since $l((t_a t_b)^3) = 6$, but $l((t_a^2 t_b^{-1})^2) = 2 \notin 6\mathbb{Z}$.

In view of these considerations, one might also hope for the last part of Conjecture 4.6 to hold in the coloured case when the image of the relevant quantum representation is infinite; no contradictions have been found in this case, neither by hand or by using computer calculations.

### 4.3.3 Pseudo-Anosov mapping classes

The mapping classes that were candidates for infinite order actions above were of course not chosen at random. They are exactly the simplest examples of pseudo-Anosov mapping classes that we obtain from Theorem 1.18. It is natural to examine exactly how the Nielsen–Thurston classification is reflected by the quantum representations. By Theorem 4.5 the collection of quantum representations detects the classification entirely, but one could wish for further criteria describing more precisely how the trichotomy becomes apparent in the representations. In [And08], Andersen describes an exclusion principle method for picking out pseudo-Anosov mapping classes using quantum representations. In [AMU06], the authors consider the mapping class group of a sphere with four coloured intervals and show – by relating the quantum representations to a well-known representation of the mapping class group – that pseudo-Anosov elements act with infinite order in the quantum representations at high enough levels, and that they furthermore determine the stretch factors. It is natural to conjecture that this happens in general.

**Conjecture 4.9** ([AMU06]). *Let $\Sigma$ be a hyperbolic surface and let $\varphi$ be a pseudo-Anosov mapping class. Then there exists $k_0$ such that $\rho_k(\varphi)$ has infinite order for $k > k_0$. Furthermore, the $\rho_k$ determine the stretch factor of $\varphi$.*

*Remark* 4.10. For surfaces with coloured structure, the coloured intervals are viewed as boundary components, so that e.g. a torus with one coloured interval is considered hyperbolic.

The $k_0$ of the conjecture can become arbitrarily large, as for every $k$ the element $t_{a_1}^{4k+8} t_{a_2}^{-4k-8} t_{a_3}^{4k+8} t_{a_4}^{-4k-8} t_{a_5}^{4k+8}$ in $\Gamma_2$ is in $\ker \rho_k$. The orders in Table 4.1

also show that for low levels, we can expect the elements to alternate between having finite and infinite order.

We return to this conjecture in Section 4.5.2.

## 4.4 Asymptotic expansion and growth rate

Throughout this section, quantum representations at level $k$ are evaluated at the primitive root of unity $A = -e^{2\pi i/(4k+8)}$. More information on the following conjectures can be found in [And02, Ch. 7.2] – much of it can be formulated for more general gauge groups, but to stick with our previous discussion, we restrict attention to $G = \mathrm{SU}(2)$. Let $M$ be a compact oriented 3-manifold. Since any principal bundle over $M$ is trivializable, we refer to the moduli space $\mathcal{M}$ of flat bundles on $M$ simply as the moduli space of flat connections on $M$.

It is known that $\mathcal{M}$ has only finitely many connected components. Here, we assume that $\pi_1(M)$ has $n$ generators and view $\mathcal{M}$ as a quotient of a subset of $\mathrm{SU}(2)^{\times n}$ using Theorem 2.21 with the natural topology. If furthermore $M$ is closed, the Chern–Simons action is known to be constant on connected components. In the case $G = \mathrm{SU}(2)$, the action is given by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(dA \wedge A + \tfrac{2}{3}A \wedge A \wedge A).$$

We can now formulate the following conjecture, which from a physical point of view is inspired by so-called stationary phase approximation of the path integral in the physical definition of $Z_k$ from Section 3.1.

**Conjecture 4.11** (The asymptotic expansion conjecture)**.** *Let $M$ be a closed oriented 3-manifold. Let $r = k + 2$. Let $\{c_0 = 0, \dots, c_m\}$ be the finitely many values of the Chern–Simons action on the moduli space of $M$. Then there exist $d_j \in \mathbb{Q}$, $b_j \in \mathbb{C}$, and $a_j^e \in \mathbb{C}$ for $j = 0, \dots, m$, $e = 1, 2, \dots$ such that*

$$Z_k(M) \sim_{k\to\infty} \sum_{j=0}^{m} e^{2\pi i r c_j} r^{d_j} b_j (1 + \sum_{e=1}^{\infty} a_j^e r^{-e})$$

*in the sense that*

$$\big|Z_k(M) - \sum_{j=0}^{m} e^{2\pi i r c_j} r^{d_j} b_j (1 + \sum_{e=1}^{E} a_j^e r^{-e})\big| = O(r^{d-E-1}),$$

*for $E = 0, 1, \dots,$ where $d = \max_j d_j$.*

*Remark* 4.12. If an asymptotic expansion as the above exists, it is well-known that the constants are more or less uniquely determined – see [And11].

Conjecturally, the constants appearing in this conjecture have various topological interpretations in terms of e.g. Reidemeister torsion and spectral flow. We will be particularly interested in the behaviour of the $d_j$.

Recall that for a flat connection $A$ in $P \to M$, we obtain a complex

$$\cdots \to \Omega^{k-1}(M, \mathrm{Ad}_P) \overset{\nabla^A}{\to} \Omega^k(M, \mathrm{Ad}_P) \overset{\nabla^A}{\to} \Omega^{k+1}(M, \mathrm{Ad}_P) \to \cdots.$$

Let $H^i(M, \mathrm{Ad}_P)$ denote the cohomology of this complex, and let

$$h_A^i = \dim H^i(M, \mathrm{Ad}_P).$$

**Conjecture 4.13** (The growth rate conjecture)**.** *Let $c_j, d_j$ as above, and let $\mathcal{M}_j$ denote the subspace of $\mathcal{M}$ consisting of connections with Chern–Simons action $c_j$. Then*

$$d_j = \frac{1}{2} \max_{[A]\in\mathcal{M}_j} (h_A^1 - h_A^0),$$

*where the* max *denotes the maximum over all Zariski open subsets of $\mathcal{M}_j$ with the property that $h_A^1 - h_A^0$ is constant on that subset.*

*Remark* 4.14. Consider the case where $M$ is the mapping torus of a homeomorphism of a surface of genus $g \geq 1$. Let $A$ be a flat connection in $P \to M$, and let $\rho$ be a representative of $[A]$ in $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))/\mathrm{SU}(2)$. The elements of $\mathfrak{su}(2)$ fixed by the action of $\pi_1(M)$ given by $\mathrm{Ad}\circ\rho$ are exactly those in the centralizer of the image $\rho(\pi_1(M))$, and so by Theorem 2.28,

$$h_A^0 = \dim \mathrm{Lie}(Z(\rho(\pi_1(M)))).$$

Similarly, Theorem 2.28 gives a description of $h_A^1$ using only the corresponding representation of $\pi_1(M)$.

Combining the conjectures, we can describe the growth rate of $Z_k(M)$ as follows.

**Conjecture 4.15.** *Let $d = \max_i\{d_i\}$ be the largest of the $d_i$ of Conjecture 4.13. Then*

$$|Z_k(M)| = O(k^d).$$

*Remark* 4.16. If in the above conjecture, $|Z_k(M)| = \Theta(k^d)$ for $d \geq 1$ (that is, there exist positive constants $c_1, c_2$ such that $c_1 k^d \leq |Z_k(M)| \leq c_2 k^d$ for all $k$), then the growth rate is determined as

$$d = \lim_{k\to\infty} \frac{\log|Z_k(M))|}{\log k}.$$

Conversely, if the limit exists and equals $d$, then $|Z_k(M)| = \Theta(k^d)$.

### 4.4.1 Mapping tori of torus homeomorphisms

Let $\Sigma$ be a closed surface, and let $T_\varphi$ be the mapping torus for a homeomorphism in the mapping class $\varphi \in \Gamma(\Sigma)$. Then by Theorem 3.30, $Z_k(T_\varphi) = \operatorname{tr} \rho_k(\varphi)$, and we can use the framework of quantum representations to study the conjectures above. We consider the case of the mapping tori of homeomorphisms of a torus and aim at proving the asymptotic expansion conjecture for this family of 3-manifolds. It is well-known that every orientable torus bundle over $S^1$ is homeomorphic to such a mapping torus, and thus these are covered by our analysis.

The calculations have been carried out for the homeomorphisms $U \in \Gamma_1$ satisfying $|\mathrm{tr}(U)| > 2$, for which the result is the following theorem; see [Jef92, Thm. 4.1].

**Theorem 4.17.** *Let*

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \cong \mathrm{SL}(2, \mathbb{Z})$$

*and assume that* $|\mathrm{tr}(U)| > 2$*. Then there exists a canonical choice of framing for* $T_U$*, and the quantum invariant is given by*

$$Z_k(T_U) = e^{2\pi i\psi(U)/(4r)}\,\mathrm{sgn}(d+a\mp 2)\sum_{\pm} \pm \frac{1}{2|c|\sqrt{|d+a\mp 2|}} \cdot \sum_{\beta=0}^{|c|-1}\sum_{\gamma=1}^{|d+a\mp 2|} \exp\left(2\pi i r\frac{-c\gamma^2+(a-d)\gamma\beta+b\beta^2}{d+a\mp 2}\right),$$

*where* $r = k+2$*, and* $\psi(U)\in\mathbb{Z}$ *depends only on* $U$ *and is given by [Jef92, (4.4)].*

*Remark* 4.18. Note that it follows from this that up to the framing correction – i.e. the number $e^{2\pi i\psi(U)/(4r)}$ – the sequence $\{Z_k(T_U)\}_k$ is periodic (with period a divisor of $(d+a-2)(d+a+2)$), so in particular it is bounded. Note also that the mapping classes $U\in\Gamma_1$ with $|\mathrm{tr}(U)|>2$ are exactly the pseudo-Anosov ones.

We should also note that Jeffrey as her definition of $Z_k(T_U)$ uses certain representations of $\mathrm{SL}(2,\mathbb{Z})$ arising from conformal field theory. These representations are known to coincide with the ones from geometric quantization, and by the announced theorem by Andersen and Ueno, they coincide with the combinatorial representations as well. In fact, for mapping tori over a torus, this is an older result, and we will use the notation $Z_k$ for all of the possible constructions.

### 4.4.2 Mapping tori of Dehn twists

Recall that for the torus, all mapping classes corresponding to non-trivial Dehn twists are conjugate and have trace 2. We consider now the conjectures for this class of mapping classes. Whereas we can not use Jeffrey's theorem directly, a very similar method of proof applies. We will need the following general quadratic reciprocity theorem.

**Theorem 4.19.** *Let* $a, b, c$ *be integers,* $a\neq 0$*,* $c\neq 0$*, and assume that* $ac+b$ *is even. Then*

$$\sum_{n=0}^{|c|-1} e^{\pi i(an^2+bn)/c} = |c/a|^{1/2} e^{\pi i(|ac|-b^2)/(4ac)}\sum_{n=0}^{|a|-1} e^{-\pi i(cn^2+bn)/a}.$$

See Figures 4.7—4.11 for plots of the first 200 values of $Z_k(T_{t_\gamma^m})$ for powers $m = 1,\dots,5$.

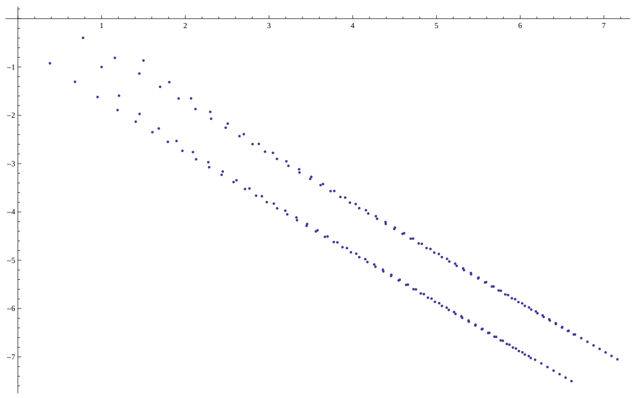


Figure 4.7: The first 200 values of $\mathrm{tr}(\rho_k(t_\gamma))$.

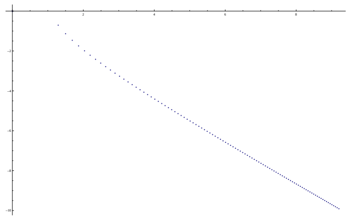

Figure 4.8: $m = 2$.

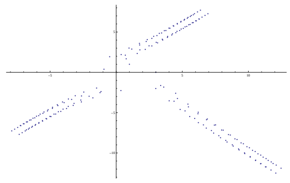

Figure 4.9: $m = 3$.

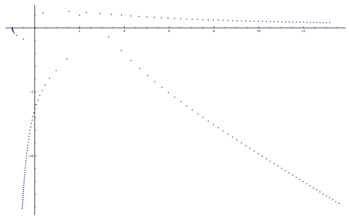

Figure 4.10: $m = 4$.

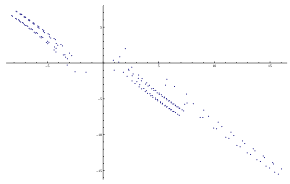

Figure 4.11: $m = 5$.

**Theorem 4.20.** *Let $\gamma$ be the isotopy class of an essential simple closed curve in $\Sigma_1$. Let $k > 0$ and $m \in \mathbb{Z}$, $m \neq 0$. Then*

$$Z_k(T_{t_\gamma^m}) = e^{\frac{\pi i m}{2r}} \left( \sqrt{\frac{r}{2|m|}} e^{-\operatorname{sgn}(m)\pi i/4} \sum_{n=0}^{|m|-1} e^{2\pi i r n^2/m} - \frac{e^{-\pi i r m/2}}{2} - \frac{1}{2} \right),$$

*where $r = k + 2$.*

*Proof.* By Lemma 1.7, it is enough to prove the Lemma for a single non-separating curve, so let $\gamma$ be the isotopy class of the meridian. With the notation $r = k+2$, the left hand side becomes

$$\begin{aligned} Z_k(T_{t_\gamma^m}) = \operatorname{tr}(\rho_k(t_\gamma^m)) &= \sum_{n=0}^{r-2} \mu_n^{-m} = \sum_{n=0}^{r-2} e^{-\frac{2\pi i}{4r}(n^2+2n)m} \\ &= e^{\frac{\pi i m}{2r}} \sum_{n=0}^{r-2} e^{-\frac{\pi i}{2r}(n+1)^2 m} = e^{\frac{\pi i m}{2r}} \sum_{n=1}^{r-1} e^{-\frac{\pi i}{2r} n^2 m}. \end{aligned}$$

Conjugating, it thus suffices to show that

$$\sum_{n=1}^{r-1} e^{\frac{\pi i}{2r} n^2 m} = \sqrt{\frac{r}{2|m|}} e^{\operatorname{sgn}(m)\pi i/4} \sum_{n=0}^{|m|-1} e^{-2\pi i r n^2/m} - \frac{e^{\pi i r m/2}}{2} - \frac{1}{2}.$$

An application of Theorem 4.19 with $a = m$, $b = 0$ and $c = 2r$ shows that

$$\sum_{n=0}^{2r-1} e^{\frac{\pi i}{2r} n^2 m} = \sqrt{\frac{2r}{|m|}} e^{\operatorname{sgn}(m)\pi i/4} \sum_{n=0}^{|m|-1} e^{-2\pi i r n^2/m},$$

and it remains to prove that

$$\begin{aligned} 2\sum_{n=1}^{r-1} e^{\frac{\pi i}{2r} n^2 m} &= \sum_{n=0}^{2r-1} e^{\frac{\pi i}{2r} n^2 m} - e^{\pi i r m/2} - 1 \\ &= \sum_{n=1}^{2r-1} e^{\frac{\pi i}{2r} n^2 m} - e^{\pi i r m/2}. \end{aligned}$$

Subtracting the first $r-1$ terms from the sum on the right hand side, we see that this is equivalent to

$$\sum_{n=1}^{r-1} e^{\frac{\pi i}{2r}n^2 m} = \sum_{n=r}^{2r-1} e^{\frac{\pi i}{2r}n^2 m} - e^{\pi i r m/2}$$
$$= \sum_{n=r+1}^{2r-1} e^{\frac{\pi i}{2r}n^2 m} = \sum_{n=1}^{r-1} e^{\frac{\pi i}{2r}(n+r)^2 m}.$$

Finally, this equality is indeed true, since the terms of the sums coincide. More precisely,

$$e^{\frac{\pi i}{2r}((r-1)-(n-1)+r)^2 m} = e^{\frac{\pi i}{2r}n^2 m}$$

for $n = 0, \dots, r-1$. This on the other hand follows from the observation that

$$(2r-n)^2 m = 4r(r-4n)m + n^2 m \equiv n^2 m \pmod{4r}.$$

□

The aim of the rest of this section is to prove the asymptotic expansion conjecture and the growth rate conjecture for $T_{t_\gamma^m}$.

Note that so far, we have more or less silently ignored the framing corrections in all calculations. It is in fact known, that if a manifold has a given asymptotic expansion, the framing corrections only affect the lower order terms of the expansion. Future work will make precise the dependence on the framing.

**Proposition 4.21.** *Let $\gamma$ be the isotopy class of an essential simple closed curve in $\Sigma_1$, and let $m \in \mathbb{Z}$, $m \neq 0$. The moduli space $\mathcal{M}$ of flat* SU(2)*-connections on $T_{t_\gamma^m}$ looks as follows:*

*For $m$ odd, it consists of a copy of $S^2$, $\frac{|m|-1}{2}$ copies of the 2-torus $T^2$, as well as a component containing a single point.*

*For $m$ even, it consists of 2 copies of $S^2$ and $\frac{|m|}{2} - 1$ copies of $T^2$.*

*The only irreducible connection is the one in the single point component for $m$ odd. On the various components, the Chern–Simons action takes the following (not necessarily distinct) values:*

$$\mathrm{CS}(\mathcal{M}) = \begin{cases} \{\frac{j^2}{m} \mid j = 0, \dots, \frac{|m|-1}{2}\} \cup \{1 - \frac{m}{4}\} & \textit{if } m \textit{ is odd,} \\ \{\frac{j^2}{m} \mid j = 0, \dots, \frac{|m|}{2}\} & \textit{if } m \textit{ is even.} \end{cases}$$

*Proof.* By Theorem 2.21, we can describe $\mathcal{M}$ by describing the representations of $\pi_1(T_{t_\gamma^m})$. It is well-known (see e.g. [Jef92]) that for a mapping torus $T_\varphi$, $\varphi : \Sigma \to \Sigma$, the fundamental group is given by the twisted product

$$\pi_1(T_\varphi) = \mathbb{Z} \tilde{\times} \pi_1(\Sigma),$$

where $\mathbb{Z}$ acts on $\pi_1(\Sigma)$ via $\varphi$. In our special case, the fundamental group therefore has the presentation

$$\pi_1(T_{t_\gamma^m}) = \langle \alpha, \beta, \delta \mid \alpha\beta = \beta\alpha, \delta\alpha\delta^{-1} = \alpha, \delta\beta\delta^{-1} = \alpha^m \beta \rangle.$$

Here, we simply note that two essential closed curves are homotopic if and only if they are isotopic (see e.g. [FM11, Prop. 1.10]), and we have simply let $\alpha$ be the homotopy class of any curve representing $\gamma$ and choose $\beta$ so that $i(\alpha, \beta) = 1$.

The moduli space of flat connections is identified with a quotient of a subset of $\mathrm{SU}(2)^{\times 3}$ as

$$\mathcal{M} \cong \{(A,B,C) \in \mathrm{SU}(2)^{\times 3} \mid AB = BA, CAC^{-1} = A, CBC^{-1} = A^m B\}/\sim,$$

where $\sim$ denotes simultaneous conjugation. Since $A$ and $B$ commute for any $[(A,B,C)] \in \mathcal{M}$, they both lie in the same maximal torus in $\mathrm{SU}(2)$, and by conjugating them simultaneously we may assume that they are both diagonal. In other words, they are both elements of $T := \mathrm{U}(1) \subseteq \mathrm{SU}(2)$. Here, for $a \in \mathrm{U}(1)$, we simply write $a$ for the matrix $\mathrm{diag}(a, \bar{a})$ in $\mathrm{SU}(2)$. We now consider three cases.

*Case 1.* Assume that $A, B \in Z(\mathrm{SU}(2))$. In this case, $B = A^m B$, so $A^m = 1$, and so $A$ must be the identity if $m$ is odd.

*Case 2.* Assume that $A \notin Z(\mathrm{SU}(2))$. Then $C \in N(T)$, where $N(T)$ is the normalizer of $T$, which is given by $N(T) = T \cup L$, where

$$L = \left\{ \begin{pmatrix} 0 & e^{2\pi i t} \\ -e^{-2\pi i t} & 0 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

If $C \in T$ then $B = BA^m$, and $A^m = 1$ is the only restriction. If $C \in N(T) \setminus T = L$, conjugation by $C$ corresponds to inversion for elements of $T$. Thus, for $C \in L$, we have $A^{-1} = A$ contradicting that $A \notin Z(\mathrm{SU}(2))$.

*Case 3.* Assume that $A \in Z(\mathrm{SU}(2)), B \notin Z(\mathrm{SU}(2))$. Again, $C \in N(T)$. If $C \in T$ we find again that $A^m = 1$, so $A = 1$ if $m$ is odd. If $C \in L$, then $B^{-1} = BA^m$, and $B^2 = A^{-m} = A^m$, which is impossible for $m$ even when $B \notin Z(\mathrm{SU}(2))$, but for $m$ odd, and $A = -1$, we get a contribution for $B = \pm i$.

In conclusion, when $m$ is odd,

$$\begin{aligned}\mathcal{M} \cong{}& ((\{1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup (\{e^{2\pi i j/m} \mid j = 0, \dots, |m|-1\} \setminus \{1\} \times T \times T) \\ & \cup (\{1\} \times T \setminus \{\pm 1\} \times T) \cup (\{-1\} \times \{\pm i\} \times L))/\sim,\end{aligned}$$

and when $m$ is even,

$$\begin{aligned}\mathcal{M} \cong{}& ((\{\pm 1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup (\{\, e^{2\pi i j/m} \mid j = 0, \dots, |m|-1\} \setminus \{\pm 1\} \times T \times T) \\ & \cup (\{\pm 1\} \times T \setminus \{\pm 1\} \times T))/\sim .\end{aligned}$$

In the case where $m$ odd, the last component is a union of two copies of $T$ where all points are identified under conjugation since

$$\begin{aligned}&\begin{pmatrix} 0 & e^{2\pi i s} \\ -e^{-2\pi i s} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{2\pi i t} \\ -e^{-2\pi i t} & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{2\pi i s} \\ -e^{-2\pi i s} & 0 \end{pmatrix}^{-1} \\ &\qquad = \begin{pmatrix} 0 & e^{-2\pi i s + 4\pi i t} \\ -e^{2\pi i s - 4\pi i t} & 0 \end{pmatrix}.\end{aligned}$$

This is the single point component of $\mathcal{M}$. If $m$ is odd or even, for the quotients of the first and third component in the above description, it suffices to consider the quotient of $\{1\} \times T \times T$ or $\{\pm 1\} \times T \times T$ respectively, since we may first identify any element of $\mathrm{SU}(2)$ with its diagonalization. Now $T \times T$ is identified with the torus $\mathbb{R}^2/\mathbb{Z}^2$, and the only conjugation action left is the action by the Weyl group $W \cong \mathbb{Z}_2$ which acts on $\mathbb{R}^2/\mathbb{Z}^2$ by $(t,s) \mapsto (-t,-s)$. The quotient of $\mathbb{R}^2/\mathbb{Z}^2$ under this action is homeomorphic to $S^2$, of which $\mathcal{M}$ therefore contains one or two in the cases $m$ odd or even respectively.

Finally, let $j \in \{0, \dots, |m|-1\}$, and assume that $j/m \notin \{0, \frac{1}{2}\}$. Arguing as above, the only conjugation action left on $\{e^{2\pi i j/m}\} \times T \times T$ is that of the Weyl group. Now, in this case, it acts non-trivially on the first factor, mapping

$e^{2\pi ij/m}$ to $e^{-2\pi ij/m}$, and the resulting quotient becomes a number of copies of $T \times T$ as claimed.

Finding the values of the Chern–Simons action is a well-studied problem, and in our case, the values on the components can be found using [Jef92, Thm. 5.11].

The claim about reducibility follows from the fact that an SU(2)-connection is reducible if and only if the corresponding representation has image contained in a maximal torus, which is the case for all representations above but the one mapping $C$ into $L$. □

**Corollary 4.22.** *The asymptotic expansion conjecture holds for $T_{t_\gamma^m}$ for all powers $m \in \mathbb{Z}$, $m \neq 0$, and all isotopy classes $\gamma$.*

*Proof.* As in the proof of Theorem 4.20, we note that

$$e^{2\pi ir((m-1)-(j-1))^2/m} = e^{2\pi irj^2/m}, \tag{4.3}$$

for $j = 0, \ldots, |m|-1$. To prove the corollary, it is now a matter of rearranging the terms in the formula for $Z_k(T_{t_\gamma^m})$.

Assume first that $m$ is even. In this case,

$$e^{-\pi irm/2} = e^{\pi irm/2} = e^{2\pi ir(\frac{|m|}{2})^2/m},$$

and it follows from (4.3) that

$$\begin{aligned}
Z_k(T_{t_\gamma^m}) = e^{\frac{\pi im}{2r}} \Bigg( & \sqrt{\frac{r}{2|m|}} e^{-\operatorname{sgn}(m)\pi i/4} \Bigg( 2 \sum_{n=1}^{|m|/2-1} e^{2\pi irn^2/m} + 1 + e^{\pi imr/2} \Bigg) \\
& - \frac{e^{-\pi irm/2}}{2} - \frac{1}{2} \Bigg) \\
= e^{\frac{\pi im}{2r}} \Bigg( & \sum_{n=1}^{|m|/2-1} e^{2\pi irn^2/m} \left[ \sqrt{\frac{2r}{|m|}} e^{-\operatorname{sgn}(m)\pi i/4} \right] \\
& + e^{2\pi ir\cdot 0/m} \left[ \sqrt{\frac{r}{2|m|}} e^{-\operatorname{sgn}(m)\pi i/4} - \frac{1}{2} \right] \\
& + e^{2\pi ir(\frac{|m|}{2})^2/m} \left[ \sqrt{\frac{r}{2|m|}} e^{-\operatorname{sgn}(m)\pi i/4} - \frac{1}{2} \right] \Bigg).
\end{aligned}$$

Now, one obtains the full asymptotic expansion of $Z_k(T_{t_\gamma^m})$ by introducing the Taylor series for $e^{\pi im/(2r)}$ and $1/\sqrt{r}$. For $m$ odd, the exact same argument shows that

$$\begin{aligned}
Z_k(T_{t_\gamma^m}) = e^{\frac{\pi im}{2r}} \Bigg( & \sum_{n=1}^{(|m|-1)/2} e^{2\pi irn^2/m} \left[ \sqrt{\frac{2r}{|m|}} e^{-\operatorname{sgn}(m)\pi i/4} \right] \\
& + e^{2\pi ir0/m} \left[ \sqrt{\frac{r}{2|m|}} e^{-\operatorname{sgn}(m)\pi i/4} - \frac{1}{2} \right] - e^{-\pi irm/2}\frac{1}{2} \Bigg).
\end{aligned}$$

□

Note that the proof of Corollary 4.22 gives us explicitly the leading order term of $Z_k(T_{t_\gamma^m})$, and in particular we are now able to turn to Conjecture 4.13 for $T_{t_\gamma^m}$.

**Theorem 4.23.** *Let $\mathcal{M}_{j/m}$, $j = 0, \dots, \lceil \frac{|m|+1}{2} \rceil$, and $\mathcal{M}_{-m/4}$ be the components of the moduli space of $T_{t_\gamma^m}$ arising from Proposition 4.21, and let*

$$d_i' = \frac{1}{2} \max_{[A]\in\mathcal{M}_i} (h_A^1 - h_A^0),$$

*where the* max *is as in Conjecture 4.13. Then $d_{j/m}' = \frac{1}{2}$ and $d_{-m/4}' = 0$. In particular, the growth rate conjecture holds true in this case.*

*Proof.* Abusing notation slightly, we write $\rho \in \mathcal{M}$ for the (conjugacy class of a) representation corresponding to a (gauge class of a) flat connection in $\mathcal{M}$. Let $A, B, C$ denote the images of generators $\alpha, \beta, \delta$ of $\pi_1(T_{t_\gamma^m})$ under $\rho$. Using the remark following Conjecture 4.13, we find that if $\rho \in \mathcal{M}_{j/m}$, then $h_\rho^0 = 1$ except in four or eight points in the cases where $m$ is odd or even respectively, those points corresponding to $A, B, C = \pm 1$. When $A, B, C = \pm 1$, we have $h_\rho^0 = 3$. For the representation $\rho \in \mathcal{M}_{-m/4}$, we have $h_\rho^0 = 0$.

We now describe $h_\rho^1$. The cocycles $Z^1(\pi_1(T_{t_\gamma^m}), \mathfrak{su}(2))$ embed in $\mathfrak{su}(2)^3$ under the map

$$u \mapsto (u(\alpha), u(\beta), u(\delta)).$$

The image can be determined since cocycles map the three relators

$$R_1 = \alpha\beta\alpha^{-1}\beta^{-1},\ R_2 = \alpha\delta\alpha^{-1}\delta^{-1},\ R_3 = \delta\beta\delta^{-1}\alpha^{-m}\beta^{-1}$$

of our presentation of $\pi_1(T_{t_\gamma^m})$ to $0 \in \mathfrak{su}(2)^3$. One finds that $Z^1(\pi_1(T_{t_\gamma^m}), \mathfrak{su}(2))$ can be identified with the kernel of the map $R = (\tilde{R}_1, \tilde{R}_2, \tilde{R}_3) : \mathfrak{su}(2)^3 \to \mathfrak{su}(2)^3$ determined by $R_1, R_2, R_3$ by the requirement that

$$\tilde{R}_i(u(\alpha), u(\beta), u(\delta)) = u(R_i).$$

Assume for simplicity that $m > 0$. Noting that in general,

$$u(g^{-1}) = -\operatorname{Ad}(\rho(g^{-1}))u(g),$$

the cocycle condition gives

$$\begin{aligned}
u(R_1) &= u(\alpha) - \operatorname{Ad}(B)u(\alpha) - u(\beta) + \operatorname{Ad}(A)u(\beta),\\
u(R_2) &= u(\alpha) - \operatorname{Ad}(C)u(\alpha) - u(\delta) + \operatorname{Ad}(A)u(\delta),\\
u(R_3) &= -\operatorname{Ad}(B)(\sum_{n=0}^{m} \operatorname{Ad}(A^n))u(\alpha) - u(\beta)\\
&\qquad + \operatorname{Ad}(C)u(\beta) + u(\delta) - \operatorname{Ad}(A^m B)u(\delta).
\end{aligned}$$

Here, the first two equalities are immediate, and the last one follows from

$$\begin{aligned}
u(R_3) &= u(\delta) + \operatorname{Ad}(C) + u(\beta\delta^{-1}\alpha^{-m}\beta^{-1})\\
&= u(\delta) + \operatorname{Ad}(C)(u(\beta) + \operatorname{Ad}(B)u(\delta^{-1}\alpha^{-m}\beta^{-1}))\\
&= u(\delta) + \operatorname{Ad}(C)u(\beta) + \operatorname{Ad}(CB)(u(\delta^{-1}) + \operatorname{Ad}(C^{-1})u(\alpha^{-m}\beta^{-1}))\\
&= u(\delta) + \operatorname{Ad}(C)u(\beta) - \operatorname{Ad}(CBC^{-1})u(\delta)\\
&\quad + \operatorname{Ad}(CBC^{-1})(u(\alpha^{-m}) + \operatorname{Ad}(A^{-m})u(\beta^{-1}))\\
&= u(\delta) + \operatorname{Ad}(C)u(\beta) - \operatorname{Ad}(A^m B)u(\delta)\\
&\quad - \operatorname{Ad}(CBC^{-1}A^{-m})u(\alpha^m) - \operatorname{Ad}(CBC^{-1}A^{-m}B^{-1})u(\beta)\\
&= u(\delta) + \operatorname{Ad}(C)u(\beta) - \operatorname{Ad}(A^m B)u(\delta) - \operatorname{Ad}(B)u(\alpha^m) - u(\beta)
\end{aligned}$$

since, in general

$$u(g^m) = \sum_{n=0}^{m-1} \mathrm{Ad}(\rho(g)^n) u(g).$$

In other words, $R$ is given by

$$R(x_1, x_2, x_3) = \begin{pmatrix} x_1 - \mathrm{Ad}(B)x_1 - x_2 + \mathrm{Ad}(A)x_2 \\ x_1 - \mathrm{Ad}(C)x_1 - x_3 + \mathrm{Ad}(A)x_3 \\ -\mathrm{Ad}(B)(\sum_{n=0}^{m} \mathrm{Ad}(A^n))x_1 - x_2 + \mathrm{Ad}(C)x_2 + x_3 - \mathrm{Ad}(A^m B)x_3 \end{pmatrix}.$$

Under this identification, the coboundaries $B^1(\pi_1(T_{t_\gamma^m}), \mathfrak{su}(2))$ become

$$\{(x - \mathrm{Ad}(A)x, x - \mathrm{Ad}(B)x, x - \mathrm{Ad}(C)x) \mid x \in \mathfrak{su}(2)\} \subseteq \ker R \subseteq \mathfrak{su}(2)^3.$$

Recall that $\mathfrak{su}(2)$ has a basis given by

$$\left\{ e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\}.$$

Consider first the case $\rho \in \mathcal{M}_{j/m}$. Write

$$A = \begin{pmatrix} e^{2\pi i j/m} & 0 \\ 0 & e^{-2\pi i j/m} \end{pmatrix}, \quad B = \begin{pmatrix} e^{2\pi i s} & 0 \\ 0 & e^{-2\pi i s} \end{pmatrix}, \quad C = \begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix}$$

for $j \in 0, \dots, \lceil \frac{m+1}{2} \rceil$, and $s, t \in [0, 1)$. A direct computation shows that the matrix representation of $R$ in the basis given above is

$$R = \begin{pmatrix} P - S(s) & -P + S(j/m) & 0 \\ P - S(t) & 0 & -P + S(j/m) \\ T(m, s) & -P + S(t) & P - S(s), \end{pmatrix},$$

where $P$, $S$, and $T$ are given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S(r) = \begin{pmatrix} \cos(4\pi r) & -\sin(4\pi r) & 0 \\ \sin(4\pi r) & \cos(4\pi r) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T(m, s) = \begin{pmatrix} -\eta\cos(4\pi s) & \eta\sin(4\pi s) & 0 \\ -\eta\sin(4\pi s) & -\eta\cos(4\pi s) & 0 \\ 0 & 0 & -m \end{pmatrix},$$

$$\eta = \sum_{n=0}^{m-1} e^{4\pi i j n/m} = \begin{cases} m, & \text{if } j/m \in \{0, \frac{1}{2}\} \\ 0, & \text{otherwise} \end{cases}.$$

One finds that $\dim(\ker R) = 6$ when $\frac{j}{m}, s, t \in \{0, \frac{1}{2}\}$ and that $\dim(\ker R) = 4$ otherwise. A similar computation shows that

$$B^1(\pi_1(T_{t_\gamma^m}), \mathfrak{g}) \cong \operatorname{span} \left\{ \begin{pmatrix} 1 - \cos(4\pi j/m) \\ -\sin(4\pi j/m) \\ 0 \\ 1 - \cos(4\pi s) \\ -\sin(4\pi s) \\ 0 \\ 1 - \cos(4\pi t) \\ -\sin(4\pi t) \\ 0 \end{pmatrix}, \begin{pmatrix} \sin(4\pi j/m) \\ 1 - \cos(4\pi j/m) \\ 0 \\ \sin(4\pi s) \\ 1 - \cos(4\pi s) \\ 0 \\ \sin(4\pi t) \\ 1 - \cos(4\pi t) \\ 0 \end{pmatrix}, 0 \right\},$$

so the subspace of coboundaries has dimension 0 when $\frac{j}{m}, s, t \in \{0, \frac{1}{2}\}$ and dimension 2 otherwise. Notice that by definition of the max of the Theorem, these finitely many special cases have no influence on $d_i'$.

Now, consider the case of $\rho \in \mathcal{M}_{-m/4}$, and write

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In this case, $R$ is given by

$$R = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ m & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -m & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

Now $\dim(\ker R) = 3$, and here we find that

$$B^1(\pi_1(T_{t_\gamma}), \mathfrak{g}) \cong \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\},$$

so all cocycles are coboundaries. It follows that

$$d'_{j/m} = \tfrac{1}{2}((4-2)-1) = \tfrac{1}{2},$$
$$d'_{-m/4} = \tfrac{1}{2}((3-3)-0) = 0.$$

□

## 4.5 Future plans

The considerations made in the previous sections leave a lot of questions unanswered, and there are several ways to go from here.

### 4.5.1 Generalizing asymptotic expansion of mapping tori

Understanding the powers of Dehn twists, we more or less understand all mapping tori over a torus. Jeffrey [Jef92] deals with all mapping classes $A \in \Gamma_1$ with trace $|\mathrm{tr}(A)| > 2$ in a slightly different setup. These are exactly the pseudo-Anosov mapping classes on the torus. Finite order homeomorphisms can be handled by the methods of [Jef92] as well, so by the Nielsen–Thurston classification, we only need to consider reducible mapping classes. If $A$ is reducible, then there exists $n > 0$ such that $A^n(\gamma) = \gamma$ for some (isotopy class of an) essential simple closed curve on the torus. By the change of coordinates principle, we find a $\varphi \in \mathrm{SL}(2,\mathbb{Z})$ so that $\varphi A^n \varphi^{-1}(\gamma_{\mathrm{m}}) = \gamma_{\mathrm{m}}$, where $\gamma_{\mathrm{m}}$ is the meridian curve in the torus. In other words, as we identify $\gamma_{\mathrm{m}} = e_1 \in \mathbb{R}^2$,

$$(\varphi A \varphi^{-1})^n = \varphi A^n \varphi^{-1} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

for some $r \in \mathbb{Z}$. Since $A$ is not finite order, $r \neq 0$. Writing

$$\varphi A \varphi^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we find that

$$\begin{pmatrix} a+cr & b+dr \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ar+b \\ c & cr+d \end{pmatrix}.$$

This implies that $c = 0$, so $a, d = 1$ or $a, d = -1$. Therefore there is an $s \in \mathbb{Z}$ such that $\varphi A \varphi^{-1} = t_{\gamma_{\mathrm{m}}}^s$ or $\varphi A \varphi^{-1} = H t_{\gamma_{\mathrm{m}}}^s$, where $H$ is hyper-elliptic involution. Since $H$ is central, we must have either $A = t_\gamma^s$ or $A = H t_\gamma^s$. Finally, we recall that $H$ is in the kernel of all quantum representations, and in particular that $\rho_k(t_\gamma^s) = \rho_k(H t_\gamma^s)$. To understand the asymptotic expansion conjecture for all reducible elements, we should simply check it for powers of Dehn twists and check that the Chern–Simons action takes the same values on $T_{t_\gamma^s}$ and $T_{H t_\gamma^s}$.

The moduli spaces of $T_{t_\gamma^m}$ and $T_{H t_\gamma^m}$ are identified through the map acting on $T \times T \times N(T)$ as the identity on the first two factors and swapping the components $T$ and $L$ of the third. More precisely the moduli space of $T_{H t_\gamma^m}$ is given by

$$\begin{aligned}\mathcal{M} = (&(\{\pm 1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup ((\{e^{2\pi i j/m} \mid j = 0, \dots, m-1\} \setminus \{\pm 1\}) \times T \times L) \\ &\cup (\{\pm 1\} \times T \setminus \{\pm 1\} \times L))/\sim\end{aligned}$$

if $m$ is even, and

$$\begin{aligned}\mathcal{M} = (&(\{1\} \times \{\pm 1\} \times \mathrm{SU}(2)) \cup ((\{e^{2\pi i j/m} \mid j = 0, \dots, m-1\} \setminus \{1\}) \times T \times L) \\ &\cup (\{1\} \times T \setminus \{\pm 1\} \times L) \cup (\{-1\} \times \{\pm i\} \times T))/\sim\end{aligned}$$

if $m$ is odd. Using once again [Jef92, Thm. 5.11], one obtains the exact same Chern–Simons values as in Proposition 4.21.

Besides this natural generalization which with a bit of work leads to a complete description of mapping tori over the torus, we mention a few other possible directions.

1. It would be obvious to try to carry out the same calculation for mapping tori over a higher genus surface. Andersen [And11] proves the asymptotic expansion conjecture for finite order homeomorphisms of surfaces of genus at least 2, and it would be natural to extend this to the case of Dehn twists. If $\gamma$ is the isotopy class of a meridian in $\Gamma_g$, we can use the coloured splitting theorem to obtain a concrete expression for $Z_k(T_{t_\gamma})$. The representation $\rho_k(t_\gamma)$ will once again have eigenvalues $\mu_i^{-1}$, this time with multiplicity given by an analogue of the Verlinde formula for non-closed surfaces.

   The manifolds $T_{t_\gamma^m}$ we have been considering are all Seifert fibered spaces, and it might be possible to work out calculations of Chern–Simons values and verifying the growth rate conjecture for this large family of spaces. This of course requires another framework than that of quantum representations – some work has been carried out in this direction in [Han99].

2. In understanding Conjecture 4.11, we have focused on the interpretations of $c_j$ and $d_j$. As mentioned, the constants $b_j$ conjecturally have similar topological interpretations in terms of well-known invariants (see [And02]), which it would be natural to try to understand.
3. Throughout the report, we have specialized to the case $G = \mathrm{SU}(2)$. Much is known about the $\mathrm{SU}(n)$ case, and – from the combinatorial viewpoint – about more general simple Lie groups, and it could be interesting to examine the behaviour of mapping tori in these cases as well.

### 4.5.2 Asymptotic expansion and stretch factors

Since the collection of all quantum representations determine the mapping class group (up to central elements) by Theorem 4.5, if Conjecture 4.11 holds for the mapping torus $T_\varphi$ of a pseudo-Anosov homeomorphism $\varphi$ on a surface $\Sigma$, we might expect to be able to read off the stretch factors directly from the expansion, thus testing the last part of Conjecture 4.9. In the case of the mapping torus of a torus, this is particularly easy (but recall that the first part of Conjecture 4.9 can not be true for the torus, since $\rho_k$ has finite image for all $k$ in this case). Let as always $r = k + 2$.

**Proposition 4.24.** *Let $\varphi : \Sigma_1 \to \Sigma_1$ be a pseudo-Anosov mapping class of the closed torus, given by the $\mathrm{SL}(2,\mathbb{Z})$ matrix*

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*and assume that $\mathrm{tr}(\varphi) > 2$. Then the stretch factor $\lambda$ of $\varphi$ is given by*

$$\lambda = \left(\lim_{n\to\infty} Z_{n(a^2+2ad+d^2-6)}(T_\varphi)\right)^{-2}$$

*Proof.* In the torus case, the stretch factor $\lambda$ is nothing but the largest eigenvalue. This is a fundamental fact but can also be seen as a consequence of the construction of incidence matrices in Section 1.3.1. In other words,

$$\lambda = \frac{(a+d) + \sqrt{(a+d)^2 - 4}}{2}.$$

By Theorem 4.17,

$$\begin{aligned} Z_{n(a^2+2ad+d^2-6)}(T_\varphi) &= Z_{n((a+d+2)(a+d-2)-2)}(T_\varphi) \\ &= e^{2\pi i\psi(U)/(4n(a+d+2)(a+d-2))} \sum_{\pm} \pm \frac{1}{2|c|\sqrt{d+a\mp 2}} \sum_{\beta=0}^{|c|-1} \sum_{\gamma=1}^{d+a\mp 2} 1 \\ &= e^{2\pi i\psi(U)/(4n(a+d+2)(a+d-2))} \sum_{\pm} \pm \frac{1}{2\sqrt{d+a\mp 2}} (d+a\mp 2) \\ &= e^{2\pi i\psi(U)/(4n(a+d+2)(a+d-2))} \left(\frac{\sqrt{d+a-2} - \sqrt{d+a+2}}{2}\right) \\ &\to \frac{\sqrt{d+a-2} - \sqrt{d+a+2}}{2} \end{aligned}$$

as $n \to \infty$. Thus,

$$\left(\lim_{n\to\infty} Z_{n(a^2+2ad-6)}(T_\varphi)\right)^2 = \frac{1}{2}\left(a + d - \sqrt{(a+d)^2 - 4}\right) = \lambda^{-1}.$$

□

We will try to generalize this argument to other surfaces.

### 4.5.3 Curve operators and Toeplitz operators

Recall that one way of defining quantum representations was by surgery along a knot $L$ in a cylinder over a surface, the knot being coloured by the special element $\Omega_p$. We might be able to gain an understanding of the quantum representations by splitting up the $\Omega_p$-coloured knot as a linear combination of knots coloured by the elements $e_i$ of $K(S^1 \times I \times I)$. This gives rise to so-called curve operators which are closely related to geometric quantization of the moduli space.

Consider the BHMV TQFT at level $k$. Let $\gamma$ be a simple closed curve in a surface $\Sigma$ of genus $g \geq 2$. Let $i \leq k$ be a colour, and view $\gamma$ as the coloured graph in $\Sigma \times I$ obtained by embedding $\gamma$ in $\Sigma \times \{\frac{1}{2}\}$, endowing it with the blackboard framing with respect to $\Sigma \times \{\frac{1}{2}\}$ and colouring it by $i$. This way we obtain an operator

$$Z_k(\gamma, i) \in \mathrm{End}(V_k(\Sigma))$$

called the *curve operator*. Let $\pi_i$ be an $(i+1)$-dimensional irreducible representation of SU(2), and let $\mathcal{M}$ denote the smooth part of the moduli space of flat SU(2) connections on $\Sigma$. Let $h_{\gamma,i} \in C^\infty(\mathcal{M})$ be the *holonomy function* given by

$$h_{\gamma,i}([A]) = \mathrm{tr}(\pi_i(\mathrm{hol}_A(\gamma))).$$

Andersen in [And10] claims the following result (in the general SU($n$) setup), providing a close relation between curve operators and holonomy functions using Toeplitz operators.

**Theorem 4.25.** *For any curve $\gamma$ in $\Sigma$, and any colouring $i$,*

$$\lim_{k \to \infty} \|Z_k(\gamma, i) - T_{h_{\gamma,i}}\| = 0.$$

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