SOLUTIONS TO EXERCISES

Set #1

1.1 Note that $x - x = 0$, that $x - y = -(y - x)$, and that $x - z = (x - y) + (y - z)$. Every real number is equivalent to an element in $[0, 1)$, but no elements in $[0, 1)$ are equivalent to each other. Thus the set of equivalence classes is in bijection with $[0, 1)$.

1.2 As always, the trivial and discrete topologies are topologies. Let us build all other topologies systematically. Clearly, $\{\emptyset, \{a\}, X\}$ is a topology, and similarly one gets two topologies by replacing a with b or c. If the topology contains two 1-element sets, say $\{a\}$ and $\{b\}$, then also their union $\{a, b\}$ is in the topology. Again, one could replace a, b with b, c or a, c. If all of $\{a\}$,$\{b\}$, and $\{c\}$ are in the topology, then it is discrete. Going to two-element subsets, we see that $\{\emptyset, \{a, b\}, X\}$ is a topology, and that $\{\emptyset, \{a, b\}, \{x\}, X\}$ is as well for any x, as are all permutations. If two two-element subsets in the topology overlap, say $\{a, b\}$ and $\{b, c\}$, then their intersection $\{b\}$ is in the topology as well, so it will consist of at least $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$, and this is a topology (as are permutations once more). If three two-element subsets are in the topology, then it is discrete. This covers all possible combinations of 1- and 2-element subsets.

1.3 Suppose that $U \in \mathcal{T}_2$. If id is continuous, then $U = \mathrm{id}^{-1}(U) \in \mathcal{T}_1$, so $\mathcal{T}_2 \subset \mathcal{T}_1$.

1.4 Let $\mathcal{T}$ be any topology on Y such that ι is continuous. Let $U \in \mathcal{T}_Y$. Then there is a set $V \subset X$, open in X, so that $U = Y \cap V$. We see that $U = Y \cap V = \iota^{-1}(V) \in \mathcal{T}$, since ι is continuous with respect $\mathcal{T}$, so $\mathcal{T}_Y \subset \mathcal{T}$.

1.5 Suppose that Y is open. If $U \subset Y$ is open in X, then $U = U \cap Y$ is open in Y. If U is open in Y, let U' be so that $U = Y \cap U'$ with U' open in X. Then U is open in X as the intersection of two open sets is open. Suppose that Y is closed; we will use Proposition 3.4 a few times. If $F \subset Y$ is closed in X, then $F = Y \cap F$ is closed in Y. On the other hand, if F is closed in Y there is a set $G \subset X$ which is closed in X, and $F = Y \cap G$, which is closed in X as the intersection of two closed sets is closed.

1.6 If f is continuous, then $\iota \circ f$ is continuous, since ι is, and since compositions of continuous are continuous. Suppose that $\iota \circ f$ is continuous, and let $U \in \mathcal{T}_Y$. Then there is an open set $U' \in \mathcal{T}_X$ with $U = Y \cap U'$. As in exercise 4, $U = \iota^{-1}(U')$, and so

$$f^{-1}(U) = f^{-1}(\iota^{-1}(U')) = (\iota \circ f)^{-1}(U')$$

which is open in Z.

1.7 (a): The basis elements are

$$\begin{aligned} P_a &= \{a, b, c, d\}, \\ P_b &= \{b, c\}, \\ P_c &= \{c\}, \\ P_d &= \{d\}. \end{aligned}$$

Recall that the topology generated by the basis can be described as all possible unions of basis elements, so it also contains $\emptyset$, $\{b, c, d\}$, and $\{c, d\}$. (b): The basis elements of $\mathcal{T}_{\leq}$ are of the form $P_x = \{y \mid x \leq y\} = [x, \infty)$, where x varies. We claim that $\mathcal{T}_{\leq} = \mathcal{T}_l$, the lower limit topology. Notice that if $x \in P_a$ for some a, then $x \in [a, x + 1) \subset P_a \in \mathcal{B}_l$, so $\mathcal{T}_l$ is finer than $\mathcal{T}_{\leq}$ by Lemma 2.15.

On the other hand, if $U \in \mathcal{B}_l$ is of the form $U = [a, b)$, then we can write $U = P_a \cap P_b$, so $U \in \mathcal{T}_{\leq}$; now, a general element of $\mathcal{T}_l$ is of the union of such U and thus also in $\mathcal{T}_{\leq}$.

1.8 We will use the lemma twice. First, let $(x, y) \in \mathbb{R}^2$ be contained in an open ball $B(z, r)$. Then clearly (draw!), by taking ε small enough, one finds that

$$\begin{aligned}\{(x', y') \mid x' \in (x-\varepsilon, x+\varepsilon), y' \in (y-\varepsilon, y+\varepsilon)\} &= (x-\varepsilon, x+\varepsilon) \times (y-\varepsilon, y+\varepsilon) \\ &\subset B(z, r).\end{aligned}$$

This set on the left hand side is $B(x, \varepsilon) \times B(y, \varepsilon)$ which is a basis element for the topology on $\mathbb{R} \times \mathbb{R}$. Therefore, the product topology is finer than the standard topology.

On the other hand, if $(x, y) \in B(z_1, r_1) \times B(z_2, r_2)$, one can find an r so small that

$$B((x, y), r) \subset B(z_1, r_1) \times B(z_2, r_2),$$

so the standard topology is finer than the product topology.

1.9 Let $r = d(x, y)/2$, $U_x = B(x, r)$, $U_y = B(y, r)$.

1.10 Let $x, y \in Y$, $x \neq y$. We then get U and V open disjoint neighbourhoods of x and y in X, and $Y \cap U$, $Y \cap V$ are open disjoint neighbourhoods of x and y in Y.

1.11 Suppose X_1 and X_2 are Hausdorff, and let $x = (x_1, x_2), y = (y_1, y_2) \in X_1 \times X_2$, $x \neq y$. Then either $x_1 \neq y_1$ or $x_2 \neq y_2$. Suppose $x_1 \neq y_1$. Then there are disjoint neighbourhoods U and V in X_1 of x_1 and y_1 respectively. It follows that $\pi_1^{-1}(U)$ and $\pi_1^{-1}(V)$ are disjoint neighbourhoods of x and y respecively. Similarly if $x_2 \neq y_2$.

1.12 Suppose X is Hausdorff and let $(x, y) \in \Delta^c$. Choose U, V disjoint neighbourhoods of x and y respectively. Then $U \times V \subset \Delta^c$ is an open neighbourhood of (x, y), so $(x, y) \in \mathrm{Int}\Delta^c$.

On the other hand, if Δ^c is open, then for any $(x, y) \in \Delta^c$ there are $U, V \subset X$ open so that $(x, y) \in U \times V \subset \Delta^c$, which means that U and V are disjoint open neighbourhoods of x and y.

1.13 Clearly, the topology induced by the new basis is coarser than the metric topology (since the basis is smaller). To see that it is also finer, let $x \in X$, and let $B_d(y, r)$ be a ball containing x. Then we have seen that $B_d(x, \varepsilon) \subset B_d(y, r)$ for all ε small enough. Now take n so $1/n < \varepsilon$.

1.14 (a): Suppose that $x \leq y$. If there is a z such that $x < z < y$, then let $U = (-\infty, z)$ and $V = (z, \infty)$. Else let $U = (-\infty, y)$, $V = (x, \infty)$. (b): We will show that $A = \{x \mid f(x) > g(x)\}$ is open by showing that $\mathrm{Int}A = A$. Let $x \in A$ and choose U and V disjoint neighbourhoods of $f(x)$ and $g(x)$ respectively so that $u > v$ for all $u \in U$, $v \in V$. Now let $W = f^{-1}(U) \cap g^{-1}(V)$. Clearly $x \in W$, and W is open, so we are done if $W \subset A$. Let $z \in W$. Then $f(z) > g(z)$ by choice of U and W, so $z \in A$.

1.15 First note that d is actually a metric. Let $(x_1, x_2) \in X_1 \times X_2$, and let $B_d((y_1, y_2), r)$ be an open ball containing (x_1, x_2). We then claim that

$$(x_1, x_2) \in B_{d_1}(y_1, r) \times B_{d_2}(y_2, r) \subset B_d((y_1, y_2), r).$$

First, note that

$$d_i(x_i, y_i) \leq d((x_1, x_2), (y_1, y_2)) < r,$$

so (x_1, x_2) lies in the product of balls. The inclusion holds since if $(z_1, z_2) \in B_{d_1}(y_1, r) \times B_{d_2}(y_2, r)$, then

$$d((z_1, z_2), (y_1, y_2)) = \max(d_1(z_1, y_1), d_2(z_2, y_2)) < r.$$

Similarly, if $(x_1, x_2) \in B_{d_1}(y_1, r_1) \times B_{d_2}(y_2, r_2)$ for some y_i, r_i, let $r = \min(r_1 - d_1(x_1, y_1), r_2 - d_2(x_2, y_2)) > 0$ so that $r \leq r_i - d_i(x_i, y_i)$ for $i = 1, 2$. We then claim that

$$(x_1, x_2) \in B_d((x_1, x_2), r) \subset B_{d_1}(y_1, r_1) \times B_{d_2}(y_2, r_2).$$

This time, it's obvious that (x_1, x_2) belongs to the ball. To see the inclusion, let $(z_1, z_2) \in B_d((x_1, x_2), r)$. Then

$$\begin{aligned}d_i(z_i, y_i) &\leq d_i(z_i, x_i) + d_i(x_i, y_i) \leq d((z_1, z_2), (x_1, x_2)) + d_i(x_i, y_i) \\ &< r + d_i(x_i, y_i) \leq r_i.\end{aligned}$$

1.16 Let $i : X \to X \times Y$ be the map $i(x) = (x, y_0)$. Then ι is continuous since the identity map is, and since constant maps are. Now, $h = F \circ i$, which is continuous since it is a composition of continuous functions. Similarly for g. Notice that $F(0,0) = 0$, $F(x,y) = xy/(x^2+y^2)$ is continuous in each variable but not continuous.

1.17 (a): We claim that if there exists a, b with $a \preceq b$, $a \neq b$, then the poset is not T_1. Notice that any basis element P_c which contains a will also contain b by transitivity. This implies there is no neighbourhood of a which does not contain b. Thus there can be no relations between any elements in a T_1 poset.

(b): Clearly $x \preceq x$. Assume that $x \preceq y$ and $y \preceq x$. Since X is T_0, if $x \neq y$ assume WLOG that there exists an open U, $x \in U$ so that $y \notin U$ but this means that $x \preceq y$ is false. Finally, suppose that $x \preceq y$ and $y \preceq z$. Let U be any neighbourhood of x. Then U is a neighbourhood of y and thus of z.

We claim that the poset topology agrees with the original topology. If U is open, we claim that

$$U = \bigcup_{x \in U} P_x$$

To see this, let $y \in P_x$ for some $x \in U$. Then $x \preceq y$ which means that $y \in U$. On the other hand, we claim that P_x is open for all x. Indeed, P_x is the intersection of all open subsets containing x, which is open when X is finite.

Set #2

2.1 Assume that X is connected, and let $X = C \cup D$. Then if C and D were both non-empty, $X = C^c \cup D^c$ would be a separation of X.

2.2 $X = \{a, b\} \cup \{c\}$ is a separation of X, so X is not connected. If X were path-connected, then X would also be connected.

2.3 (a): Recall first that open sets are all possible unions of basis elements. In this case, they are the sets of the form (a, ∞) themselves. Suppose that $x \in \{x_0\}'$. This means that any neighbourhood of x intersects $\{x_0\}$ in a point that is not x. A neighbourhood of x is a set (a, ∞), $a < x$, and any such neighbourhood will contain points that are not x, so it suffices to find those that also intersect $\{x_0\}$. If $x \leq x_0$, then clearly any of the neighbourhoods will intersect $\{x_0\}$. If $x > x_0$, choose a with $x_0 < a < x$. Then (a, ∞) is a neighbourhood of x that does not intersect $\{x_0\}$. It follows that $\{x_0\}' = (-\infty, x_0]$.

(b): By a theorem in the notes, $\overline{\{x_0\}} = \{x_0\} \cup \{x_0\}'$ so $\overline{\{x_0\}} = (-\infty, x_0]$.

(c): We claim that $\mathbb{R}$ is not Hausdorff in this topology: if $x < y$ are two points, and U is a neighbourhood of x, then $y \in U$.

2.4 We already know that the intervals are connected. Suppose that A is a connected subset of $\mathbb{R}$, and let $x, y \in A$. We claim that $r \in A$ for all $x < r < y$. If this were not the case, we could split $A = ((-\infty, r) \cap A) \cup ((r, \infty) \cap A)$.

2.5 Let us show that $X = \mathbb{R}^n \setminus \{0\}$ is path-connected. Let $x, y \in X$. Take the straight line from x to y. If this does not intersect 0, then it is a path from x to y. If the line does intersect zero, then $x = ay$ for an $a \in \mathbb{R}$. In this case, take a third point z which is not on the line itself, which is always possible for $n \geq 2$. Now take the line from x to z and concatenate it with the line from z to y to get a path from x to y.

2.6 Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism. Then $\mathbb{R}^n \setminus \{0\}$ is connected but $\mathbb{R} \setminus \{f(0)\}$ is not, which is a contradiction.

2.7 Write $\gamma = \gamma_1 \star \gamma_2$. Notice first that $[0,1]$ is a metric space, so it is first countable and we can use the theorem. Let $x_n \to x$ be a sequence. If $x < \frac{1}{2}$, then $x_n < \frac{1}{2}$ for all large enough n. It follows that for large enough n, $\gamma(x_n) = \gamma_1(2x_n) \to \gamma_1(2x) = \gamma(x)$ since γ_1 and $x \mapsto 2x$ are continuous. Likewise, if $x > \frac{1}{2}$ one can use the same argument with γ_2 instead. Suppose now that $x_n \to \frac{1}{2}$ and that there are subsequences $y_i = x_{n_i}$ with $y_i \leq \frac{1}{2}$ and $z_j = x_{n_j}$ with $z_j > \frac{1}{2}$, and so that each x_n is either a y_i or a z_j. Now also the subsequences converge, $y_i \to \frac{1}{2}$ and $z_j \to \frac{1}{2}$. Then as before $\gamma(y_i) \to \gamma(\frac{1}{2})$ and $\gamma(z_j) \to \gamma(\frac{1}{2})$. Then a standard argument in analysis, $\gamma(x_n) \to \gamma(x)$ (take the maximum of the Ns coming from y_i and z_j).

2.8 Let $x, y \in S^n$ and suppose first that the line from y and x does not contain 0; by Exercise 2.5 this can only happen if $y = -x$. Define a path $\gamma : [0,1] \to S^n$ by

$$\gamma(t) = \frac{(1-t)x + ty}{\|(1-t)x + ty\|}.$$

Then $\gamma(t) \in S^n$, γ is continuous, and γ is a path from x to y. If the line from x to y contains zero, i.e. if $y = -x$, take a third point z and use a concatenation of two paths.

2.9 Let $y \in Y$, and let U be a neighbourhood of y, and let $C \subset U$ be the connected component of y in the subspace U. We want to show that C is open, so let $C' = p^{-1}(C)$. By definition of the quotient topology, it then suffices to show that C' is open. So, let $x \in C'$ and let us show that $x \in \operatorname{Int} C'$. Since $p^{-1}(U)$ is a neighbourhood of x, and since X is locally connected, there is a connected neighbourhood V of x with $x \in V \subset p^{-1}(U)$. Then $p(V) \subset U$ is also connected, and $p(V)$ contains points from C (since $x \in C'$), so $p(V) \subset C$. This tells us that, $V \subset p^{-1}(C) = C'$, so $x \in \operatorname{Int} C'$, and C' is open.

2.10 Let A be a connected component, and let $x \in A$. We will show that $x \in \operatorname{Int} A$. There is a connected neighbourhood U of x, and $U \subset A$ by one of the theorems. Therefore $x \in \operatorname{Int} A$.

2.11 Let $U_n = A_1 \cup A_2 \cup \cdots \cup A_n$. We claim that U_n is connected. Clearly U_1 is. Suppose U_{n-1} is; then U_n is by one of the theorems. Thus $\{U_n\}$ is a family of connected subspaces that all contain a common point (any point from A_1 will do). Thus their union (which is $\bigcup_{n \in \mathbb{N}} A_n$) is connected.

2.12 One direction is an example. A counterexample is the rationals.

2.13 Define $g : S^1 \to \mathbb{R}$ by $g(x) = f(x) - f(-x)$. Then g is continuous, and so $g(S^1)$ is an interval. Notice that $g(x) = -g(-x)$, so if $g(x) = r \neq 0$ for some x, then $-r \in g(S^1)$, and $(-|r|, |r|) \subset g(S^1)$. This implies that $0 \in g(S^1)$ so no matter what there is an x with $g(x) = 0$ or $f(x) = f(-x)$.

2.14 Consider the map $g : [0,1] \to [-1,1]$ given by $g(x) = f(x) - x$. We need to find an x with $g(x) = 0$. Suppose that no such x exists. As before, $g([0,1])$ is connected, and since $g(0) = f(0) > 0$ we see that $g([0,1]) \subset (0,1]$. But $g(1) = f(1) - 1 \leq 0$.

2.15 Let us show that $X = \prod_{i \in I} X_i$ is connected. Let $x = (x_i)_{i \in I} \in X$ be an arbitrary point. For any finite subset $J \subset I$, let $X_J \subset X$ be the set of points whose coordinates may only vary from the x_i if $i \in J$, i.e.

$$X_J = \{y = (y_i)_{i \in I} \mid y_j = x_j \text{ if } j \notin J\}$$

Now X_J is homeomorphic to $\prod_{i \in J} X_i$ which is connected since we know how to deal with finite products. Notice also that $x \in X_J$ for every J. This implies that

$$Y = \bigcup_{J \subset I \text{ finite}} X_J$$

is connected. We claim that $X = \overline{Y}$ from which it follows that X is connected.

So, let $y \in X$ be arbitrary, and let U be any neighbourhood of y, and let us show that $U \cap Y \neq \emptyset$, so that $y \in \overline{Y}$. By possibly taking U smaller, we may assume that U is one of the basis elements. That is, $U = \prod_{i \in I} U_i$ where $U_i = X_i$ for all i outside a finite subset $K \subset I$.

Now let $z_i = y_i$ for $i \in K$, let $z_i = x_i$ for $i \notin K$, and consider $z = (z_i)_{i \in I}$. Then $z \in U$, and $z \in X_K \subset Y$, so $U \cap Y \neq \emptyset$.

Set #3

3.1 (a): Let $\mathcal{U}$ be a covering of $K_1 \cup \cdots \cup K_n$. Then in particular $\{U \cap K_i \mid U \in \mathcal{U}\}$ is a covering of K_i for each $i = 1, \ldots, n$, so that $U_1^i \cap K_i, \ldots, U_{n_i}^i \cap K_i$ cover K_i for some finite collection of $U_j^i \in \mathcal{U}$. Now

$$\{U_j^i \mid i = 1, \ldots, n,\ j = 1, \ldots, n_i\} \subset \mathcal{U}$$

is a cover of $K_1 \cup \cdots \cup K_n$.

(b): First of all, K_i is closed in X for all i since X is Hausdorff. Therefore $\bigcap K_i$ is closed. Now for any fixed j, $\bigcap K_i \subset K_j$, so $\bigcap K_i$ is compact.

(c): We see that $\emptyset \in \hat{\mathcal{T}}$ since $\emptyset \in \mathcal{T}$. Since $\emptyset$ is compact, $\hat{X} = (X \setminus \emptyset) \cup \{\star\} \in \hat{\mathcal{T}}$.

Suppose that $U_i \in \hat{\mathcal{T}}$ for $i \in I$. If $U_i \in \mathcal{T}$ for all $i \in I$, their union is also in $\mathcal{T}$ and thus in $\hat{\mathcal{T}}$. Suppose, on the other extreme, that $U_i \in \hat{\mathcal{T}} \setminus \mathcal{T}$ for all $i \in I$, and let K_i be the compacts sets in X so that $U_i = (X \setminus K_i) \cup \{\star\}$. Then

$$\bigcup_{i \in I} U_i = X \setminus \left(\bigcap_{i \in I} K_i \right) \cup \{\star\}.$$

From (b), $\bigcap K_i$ is compact in X. This shows that $\bigcup U_i \in \hat{\mathcal{T}}$.

Let us show that if $U \in \mathcal{T}$, $V \in \hat{\mathcal{T}} \setminus \mathcal{T}$, then $U \cup V \in \hat{\mathcal{T}}$. Write $V = (X \setminus K) \cup \{\star\}$. Then

$$U \cup V = X \setminus (X \setminus U) \cup (X \setminus K) \cup \{\star\} = X \setminus ((X \setminus U) \cap K) \cup \{\star\},$$

and we claim that $(X \setminus U) \cap K$ is compact: as before, K is closed, $X \setminus U$ is closed, so $(X \setminus U) \cap K$ is closed and thus compact, since it is contained in K.

That intersections of opens are open is similar: let us show that $U_1 \cap U_2$ is open when U_1 and U_2 are. As before, there is nothing to show if $U_1, U_2 \in \mathcal{T}$. If $U_1, U_2 \in \hat{\mathcal{T}} \setminus \mathcal{T}$, write $U_i = (X \setminus K_i) \cup \{\star\}$. Then

$$U_1 \cap U_2 = X \setminus (K_1 \cup K_2) \cup \{\star\},$$

which is open by (a). Finally, if $U \in \mathcal{T}$, $V \in \hat{\mathcal{T}} \setminus \mathcal{T}$, $V = (X \setminus K) \cup \{\star\}$, then

$$U \cap V = U \cap (X \setminus K),$$

and K is closed in X since X is Hausdorff, so $U \cap (X \setminus K)$ is open.

3.2 We very closely mimic the proof that compact subsets of Hausdorff spaces are closed. Let F and G be closed subsets of a compact Hausdorff space X. Then both F and G are compact. Fix an $x \in F$. As in the proof mentioned above, we obtain disjoint open subsets U^x, V^x so that $x \in U^x \subset X \setminus G$ and $G \subset V^x$. Now repeat this procedure to obtain an open cover of F, $F \subset \bigcup_{x \in F} U^x$ which then has a finite subcover $F \subset U^{x_1} \cup \cdots \cup U^{x_n}$. Let $U = U^{x_1} \cup \cdots \cup U^{x_n}$ and $V = V^{x_1} \cap \cdots \cap V^{x_n}$. Then $F \subset U$, $G \subset V$, and $U \cap V = \emptyset$.

3.3 Clearly the condition implies that X is locally compact: Take any set U to obtain a neighbourhood V of x with compact closure.

For the converse, suppose that X is locally compact, let $x \in X$, and let $U \subset X$ be a neighbourhood of x. Consider the one-point compactification $\hat{X}$ of X and notice that $C = \hat{X} \setminus U$ is closed in $\hat{X}$ and thus compact in $\hat{X}$ (since closed subsets of compact spaces are compact). As in the proof that compact subspaces of Hausdorff spaces are closed, we can find disjoint open sets V and W so that $x \in V$ and $C \subset W$. Since $\hat{X}$ is Hausdorff, $\overline{V}$ is compact, and we claim that $\overline{V} \cap C = \emptyset$ so $\overline{V} \subset U$. To see this, let $x \in C \subset W = \operatorname{Int} W$. Then there is a neighbourhood of x entirely contained in W; that is, it does not intersect V, so $x \notin \overline{V}$.

3.4 Write $Y \setminus X = \{\star_Y\}$ and $Y' \setminus X = \{\star_{Y'}\}$. Define $f : Y \to Y'$ by $f(x) = x$ for $x \in X$ and $f(\star_Y) = \star_{Y'}$. We claim that f is a homeomorphism. Clearly, f is bijective, and we will show that $f(U)$ is open, when U is open; then the same result will follow for the inverse f^{-1} by symmetry, and so f is a homeomorphism. So, let U be open in Y.

If $\star_Y \notin U$ then $f(U) = U$ which is open in X. Now X is open in Y' (as it is the complement of a single point set, which are always closed in Hausdorff spaces), so therefore U is open in Y'.

Suppose that $\star_Y \in U$. Then since $Y \setminus U$ is closed in Y, we get that $Y \setminus U$ is compact in Y, since Y is compact. Now $f(Y \setminus U)$ is compact in Y' since images of compact spaces are compact. Thus $f(Y \setminus U)$ is closed since Y' is Hausdorff, $Y' \setminus f(Y \setminus U) = f(U)$ is open in Y'.

3.5 Let $p : X \to X/C$ denote the projection, and let $[x], [y] \in X/C$, $[x] \neq [y]$. Suppose that $x \notin C$ and $y \notin C$. Since X is Hausdorff, we can find disjoint neighbourhoods U and V of x and y in X. Take $U' = U \setminus C$, $V' = V \setminus C$, which are again open since C was closed. Then $p(U')$ and $p(V')$ are disjoint open neighbourhoods of $[x] = \{x\}$ and $[y] = \{y\}$. The sets are open since $p^{-1}(p(U)) = U$ and $p^{-1}(p(V)) = V$.

Suppose that $y \in C$, so that $p(y) = [y] = C$. Now take open disjoint sets U and V so that $x \in U$ and $C \subset V$. Then as before, $p(U)$ and $p(V)$ are disjoint open neighbourhoods of $[x]$ and $[y]$.

3.6 Let us convince ourselves that this is indeed a basis. Clearly the balls cover $\mathbb{R}^n$. If $x \in B(y_1, r_1) \cap B(y_2, r_2)$, let $r = \min(r_1 - \|y_1 - x\|, r_2 - \|y_2 - x\|)$. Take any rational r' with $0 < r' < r$. Then $B(x, r') \subset B(y_1, r_1) \cap B(y_2, r_2)$. Finally, choose $y \in \mathbb{Q}^n$ so that $\|x - y\| < r'/2$. Then

$$x \in B(y, r'/2) \subset B(y_1, r_1) \cap B(y_2, r_2).$$

Let $\mathcal{T}$ be the topology generated by this basis. Since the basis is contained in the standard basis, it follows that $\mathcal{T}$ is coarser than the standard topology. We use Lemma 2.15 to see that it is also finer. That is, let $x \in \mathbb{R}^n$ and let $B(y, r)$ be any ball with $x \in B(y, r)$. Then we know that there is an r' with $B(x, r') \subset B(y, r)$. As before, take r'' rational with $0 < r'' < r'$ and choose $z \in \mathbb{Q}^n$ with $\|z - x\| < r''/2$. Then

$$x \in B(z, r''/2) \subset B(x, r'') \subset B(x, r') \subset B(y, r).$$

3.7 (a): Let $C \subset X \times Y$ be closed. We claim that $\pi(C)^c$ is open; to see this, let $x \in \pi(C)^c$ (assuming that the set is non-empty; if it's empty, we're done). That is, for each $y \in Y$, we have that $(x, y) \notin C$. By definition of the product topology, we can find for every y open neighbourhoods U_y and V_y of x and y respectively, so that $U_y \times V_y \subset C^c$. The V_y cover Y so by compactness, we can find $y_1, \dots, y_n$ so that $V_{y_1} \cup \cdots \cup V_{y_n} = Y$. Let $U = U_{y_1} \cap \cdots \cap U_{y_n}$. Then U is an open neighbourhood of x, so $x \in \operatorname{Int}(\pi(C)^c)$ since $U \cap \pi(C) = \emptyset$.

Notice that this looks a lot like the proof of the tube lemma. Indeed, the tube lemma can be applied to give a short proof: Notice that $\{x\} \times Y \subset C^c$, and C^c is open, so by the tube lemma, we can find an open neighbourhood $U \subset X$ of x so that $U \times Y \subset C^c$. This means that $U \subset \pi(C)^c$, so that once more, $x \in \operatorname{Int}(\pi(C)^c)$.

(b): Suppose first that G_f is closed, and let $C \subset Y$ be any closed set. Then

$$f^{-1}(C) = \pi((X \times C) \cap G_f),$$

which is closed by (a), since $X \times C$ is, and since G_f was assumed to be, so f is continuous. Notice that we did not use here that Y is Hausdorff.

Suppose that f is continuous. We will show that G_f^c is open, so let $(x, y) \in G_f^c$. That is, $y \neq f(x)$. Since Y is Hausdorff, we can find disjoint open sets U and V in Y so that $y \in U$, $f(x) \in V$. Since f is continuous at x, there is a neighbourhood W of x so that $f(W) \subset V$. We claim that $W \times U$ is a neighbourhood of (x, y) with $W \times U \subset G_f^c$ so that $(x, y) \in \operatorname{Int}(G_f^c)$. And this is clearly the case: if $(z, f(z)) \in G_f \cap (W \times U)$, then on one hand $f(z) \in U$ and on the other $z \in W$, so $f(z) \in f(W) \subset V$, but $U \cap V = \emptyset$.

3.8 Let $A \subset X$ be a non-empty subset which is bounded from above by some element x. We will assume for contradiction that A has no least upper bound. Let $a \in A$ be any element of A.

Let $y \in [a, x]$. If y is an upper bound for A, choose a smaller upper bound z and let $U_y = (z, \infty)$. If y is not an upper bound, choose an element $z \in A$ with $y \preceq z$, $y \neq z$, and let $U_y = (-\infty, z)$. In either case, $z \in [a, x]$, and since in either case $y \in U_y$, it follows that the collection $\{U_y\}_{y \in [a,x]}$ covers $[a, x]$. Since intervals are assumed to be compact, we obtain $y_1, \dots, y_n$ so that $U_{y_1}, \dots, U_{y_n}$ cover $[a, x]$. Now, out of these finitely many open intervals, a must belong to a set of the form $(-\infty, z)$, and x must belong to a set of the form (z, ∞). By splitting this finite subcover into the open intervals of either type, this implies that there are $b \in A \cap [a, x]$, c an upper bound of A, with

$$[a, x] \subset (-\infty, b) \cup (c, \infty).$$

Since c is an upper bound, we have $b \preceq c$. Therefore $b \notin (c, \infty)$ and since $b \notin (-\infty, b)$ it follows that $b \notin [a, x]$ which is a contradiction.

3.9 (a): Notice that a dense subset is one which intersects any non-empty open set. Since U_1 is dense, it follows that $B_0 \cap U_1 \neq \emptyset$. Take any point x in this intersection so that $B_0 \cap U_1$ is a neighbourhood of x. By Theorem 7.38 the locally compactness of X implies that there is a neighbourhood B_1 of this point x so that $\overline{B_1}$ is compact and $\overline{B_1} \subset B_0 \cap U_1$. Now B_1 will intersect the dense set U_2, so it is clear how to proceed inductively.

(b): Assume for contradiction that $\bigcap K_n = \emptyset$ and let $V_n = X \setminus K_n$. Then $\bigcup V_n = X$ and in particular, $\{V_n\}$ is an open cover of the compact set K_1 which therefore has a finite subcover $V_{i_1}, \dots, V_{i_n}$. Since the K_n are decreasing (with respect to the order $\subset$), the V_n are increasing and so $V_{i_1} \cup \dots \cup V_{i_n} = V_m$ for some m. That is, $K_1 \subset V_m = X \setminus K_m$ which is a contradiction since $K_m \subset K_1$ for all m.

We can now prove the main statement: we have a decreasing sequence $\overline{B_n}$ of compact subsets, so (b) applies. Since also $\overline{B_n} \subset U_n$ and $\overline{B_n} \subset B_0$ for all n, we have that

$$\emptyset \neq \bigcap_{n \in \mathbb{N}} \overline{B_n} \subset \bigcap_{n \in \mathbb{N}} U_n \cap B_0.$$

That is, since B_0 was an arbitrary open set, $\bigcap_{n \in \mathbb{N}} U_n$ must be dense.

Note that we really need to assume that the intersection is countable: $\mathbb{R} \setminus \{x\}$ is open and dense in $\mathbb{R}$, yet $\bigcap_{x \in \mathbb{R}} \mathbb{R} \setminus \{x\} = \emptyset$ is certainly not.

3.10 If F_n is closed with empty interior if and only if F_n^c is open and dense. Thus the result follows from the previous exercise.

Set #4

4.1 Let X be an n-manifold and Y an m-manifold. We need to show that every point $(x, y) \in X \times Y$ has a neighbourhood homeomorphic to $\mathbb{R}^{m+n}$, that $X \times Y$ is second-countable, and that $X \times Y$ is Hausdorff.

We know that there exist neighbourhoods U of x, V of y so that $U \simeq \mathbb{R}^n$, $V \simeq \mathbb{R}^m$. Then $U \times V \simeq \mathbb{R}^n \times \mathbb{R}^m$ is a neighbourhood of (x, y), and it is not hard to see that $\mathbb{R}^n \times \mathbb{R}^m \simeq \mathbb{R}^{m+n}$ (the special case $n = m = 1$ was Exercise 1.8).

Let $\{U_n\}$ be the countable basis for the topology on X, and let $\{V_m\}$ be the countable basis for the one Y. Then $\{U_n \times V_m\}$ is a basis for the product topology $X \times Y$; this basis is countable, so $X \times Y$ is second-countable.

Finally, $X \times Y$ is Hausdorff by an exercise from Set #1.

4.2 Let X be an n-manifold and an m-manifold, and let $x \in X$. Then we can find neighbourhoods U_n and U_m of x with homeomorphisms $f_n : U_n \to \mathbb{R}^n$ and $f_m : U_m \simeq \mathbb{R}^m$. Let $V = U_n \cap U_m$. Then we have a homeomorphism between the non-empty open sets $f_n(V)$ and $f_m(V)$, namely $f_m \circ f_n^{-1}|_{f(V)}$, so it follows from Theorem 6.18 that $n = m$.

4.3 Clearly, any star-shaped set A is path-connected: given $x, y \in A$, the concatenation of the line segment from x to a, and the line segment from a to y, is a path from x to y. We need to show that A has trivial fundamental group.

Let $\gamma : [0, 1] \to X$ be a loop based at a; we claim that γ is homotopic to the constant loop. For any point $x \in A$, let $l_x : [0, 1] \to A$ denote the line segment from x to a, $l_x(t) = (1 - t)x + ta$. Now, define a map $F : [0, 1] \times [0, 1] \to A$ by

$$F(s, t) = l_{\gamma(s)}(t) = (1 - t)\gamma(s) + ta.$$

Then F is clearly continuous, and

$$\begin{aligned} F(s,0) &= \gamma(s), \\ F(s,1) &= a = e_a(s), \\ F(0,t) &= (1-t)a + ta = a, \\ F(1,t) &= (1-t)a + ta = a. \end{aligned}$$

This exactly says that F is a path homotopy from γ to the constant loop e_a, so $\pi_1(A,a) = \{[e_a]\}$.

4.4 Let $\gamma : [0,1] \to \mathbb{Q}$ be any loop based at x. Since $[0,1]$ is connected, so is $\gamma([0,1])$. The connected components of $\mathbb{Q}$ are single point sets, so $\gamma([0,1]) = \{x\}$, or in other words, $\gamma = e_x$, and $\pi_1(\mathbb{Q}, x) = \{[e_x]\}$. However $\mathbb{Q}$ is not simply-connected since it is not path-connected. Note that no homotopy arguments were involved here.

4.5 Let $[\gamma] \in \pi_1(X,x)$ be a homotopy class. Then noting that $(\alpha \star \beta)^{\text{rev}} = \beta^{\text{rev}} \star \alpha^{\text{rev}}$, we have

$$\begin{aligned} \widehat{\alpha \star \beta}([\gamma]) &= [(\alpha \star \beta)^{\text{rev}}] \star [\gamma] \star [\alpha \star \beta] \\ &= [\beta^{\text{rev}}] \star [\alpha^{\text{rev}}] \star [\gamma] \star [\alpha] \star [\beta] = \hat{\beta}([\alpha^{\text{rev}}] \star [\gamma] \star [\alpha]) \\ &= \hat{\beta}(\hat{\alpha}([\gamma])) = \hat{\beta} \circ \hat{\alpha}([\gamma]). \end{aligned}$$

4.6 Suppose that $[\alpha] = [\beta]$, and let $[\gamma] \in \pi_1(X,x)$. Write $[\alpha]^{-1} = [\alpha^{\text{rev}}]$. This makes sense since $[\alpha^{\text{rev}}]$ is the left and right inverse of $[\alpha]$ and, importantly, uniquely determined. It follows that

$$\hat{\alpha}([\gamma]) = [\alpha^{\text{rev}}] \star [\gamma] \star [\alpha] = [\alpha]^{-1} \star [\gamma] \star [\alpha] = [\beta]^{-1} \star [\gamma] \star [\beta] = \hat{\beta}([\gamma]).$$

Alternatively, it is easy to see that $[\alpha^{\text{rev}}] = [\beta^{\text{rev}}]$ by explicitly constructing a path homotopy between the two paths.

4.7 Let $x \in X$ be the point $x = \alpha(0)$, and let $[\gamma] \in \pi_1(X,x)$. Then, using that f_* is a homomorphism, we have

$$f_* \circ \hat{\alpha}([\gamma]) = f_*([\alpha^{\text{rev}}] \star [\gamma] \star [\alpha]) = [f \circ \alpha^{\text{rev}}] \star [f \circ \gamma] \star [f \circ \alpha].$$

On the other hand,

$$\widehat{f \circ \alpha} \circ f_*([\gamma]) = \widehat{f \circ \alpha}([f \circ \gamma]) = [(f \circ \alpha)^{\text{rev}}] \star [f \circ \gamma] \star [f \circ \alpha].$$

It thus suffices to notice that $[f \circ \alpha^{\text{rev}}] = [(f \circ \alpha)^{\text{rev}}]$ which follows from the easy-to-check fact that $f \circ \alpha^{\text{rev}} = (f \circ \alpha)^{\text{rev}}$.

4.8 Let $(x,y) \in X \times Y$. We define a map $\Phi : \pi_1(X,x) \to \pi_1(Y,y) \to \pi_1(X \times Y, (x,y))$ as follows: Let $[\gamma_1] \in \pi_1(X,x)$, $[\gamma_2] \in \pi_1(Y,y)$. Then we have a natural loop $\gamma_1 \times \gamma_2(t) = (\gamma_1(t), \gamma_2(t))$ based at (x,y), so $[\gamma_1 \times \gamma_2] \in \pi_1(X \times Y, (x,y))$. We claim that the map $\Phi([\gamma_1],[\gamma_2]) = [\gamma_1 \times \gamma_2]$ is an isomorphism.

First, let us note that the map is actually well-defined. If $[\gamma_1] = [\widetilde{\gamma_1}]$ and $[\gamma_2] = [\widetilde{\gamma_2}]$, take path homotopies $F_1 : [0,1] \times [0,1] \to X$ and $F_2 : [0,1] \times [0,1] \to Y$ between the various loops. Then $F = (F_1, F_2) : [0,1] \times [0,1] \to X \times Y$ is a path homotopy from $\gamma_1 \times \gamma_2$ to $\widetilde{\gamma_1} \times \widetilde{\gamma_2}$, so the map is well-defined.

The natural bijections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ give rise to maps

$$(\pi_X)_* : \pi_1(X \times Y, (x,y)) \to \pi_1(X,x), \quad (\pi_Y)_* : \pi_1(X \times Y, (x,y)) \to \pi_1(Y,y),$$

and by construction, $\Phi^{-1} = (\pi_X)_* \times (\pi_Y)_* : \pi_1(X \times Y, (x,y)) \to \pi_1(X,x) \times \pi_1(Y,y)$, given concretely by

$$[(\gamma_1, \gamma_2)] \mapsto ([\gamma_1], [\gamma_2]).$$

Therefore, Φ is a bijection. Since $(\pi_X)_*$ and $(\pi_Y)_*$ are homomorphisms, so is Φ^{-1}, so it follows that Φ^{-1} (and thus Φ) is an isomorphism.

4.9 We define a homeomorphism $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1} \times (0,\infty)$ explicitly by

$$f(x) = \left(\frac{x}{\|x\|}, \|x\| \right).$$

Then f is continuous and bijective with continuous inverse f^{-1} given by

$$f^{-1}(x, r) = rx.$$

By the previous exercise, we have an isomorphism (and in particular a bijection) from $\pi_1(\mathbb{R}^n \setminus \{0\})$ to $\pi_1(S^{n-1}) \times \pi_1((0, \infty))$. Now clearly, $(0, \infty)$ is simply-connected (since for instance it is homeomorphic to $\mathbb{R}$ which is simply-connected) so this says that $\mathbb{R}^n \setminus \{0\}$ is simply-connected if and only if S^{n-1} is. This, in turn, is the case if and only if $n > 2$. That is $\mathbb{R}^n \setminus \{0\}$ is simply-connected for all $n \geq 3$, and $\pi_1(\mathbb{R}^2 \setminus \{0\}) = \mathbb{Z}$.

4.10 Let $p : S^n \to \mathbb{R}\mathrm{P}^n$ denote the projection $p(x) = [x] = \{x, -x\}$, and notice that $\mathbb{R}\mathrm{P}^n$ is path-connected since S^n is, so that it makes sense to talk about $\pi_1(\mathbb{R}\mathrm{P}^n)$.

We claim that p is a covering map. Clearly, p is surjective and continuous (as are all quotient maps). Now, let $y \in S^n$ and take ε so small that $B(y, \varepsilon)$ and $B(-y, \varepsilon)$ are disjoint in $\mathbb{R}^{n+1}$, let

$$U_y^+ = B(y, \varepsilon) \cap S^n, \quad U_y^- = B(-y, \varepsilon) \cap S^n$$

be the corresponding disjoint open sets in S^n, and let

$$\widetilde{U_y} = U_y^+ \cup U_y^-$$

be their union. Note that $U_y^\pm = U_{-y}^\mp$ and that $\widetilde{U_y}$ is open in S^n for all y.

Now, let $U_y = p(\widetilde{U_y})$. Everything has been chosen so that $p^{-1}(U_y) = \widetilde{U_y} = U_y^+ \cup U_y^-$ so that U_y is open in $\mathbb{R}\mathrm{P}^n$ (by definition of the quotient topology), and $p|_{U_y^+} : U_y^+ \to U_y$ and $p|_{U_y^-} : U_y^- \to U_y$ are homeomorphisms. Since moreover the U_y cover $\mathbb{R}\mathrm{P}^n$, we have all the ingredients that make up a covering map.

Now, let $[y] \in \mathbb{R}\mathrm{P}^n$ be arbitrary. Note that S^n is simply-connected since $n \geq 2$, so it follows from the proposition on lifting correspondences that we have a bijection

$$\pi_1(\mathbb{R}\mathrm{P}^n, [y]) \to p^{-1}(\{[y]\}) = \{y, -y\}.$$

That is, the fundamental group consists of two elements which is what we set out to show.

4.11 Let $\iota : A \to X$ denote the inclusion map. Notice that the condition on r is that $r \circ \iota = \mathrm{id}_A$.

(a) Let $[\gamma] \in \pi_1(A, a)$. Then $[\iota \circ \gamma] \in \pi_1(X, a)$, and

$$r_*([\iota \circ \gamma]) = r_* \circ \iota_*([\gamma]) = (r \circ \iota)_*([\gamma]) = (\mathrm{id}_A)_*([\gamma]) = [\gamma],$$

so r_* is surjective.

(b) Let us show that r_* is injective. That is, assume that $r_*([\gamma]) = [e_a]$ for some $[\gamma] \in \pi_1(X, a)$ – i.e. assume that $r \circ \gamma \sim_p e_a$ – and let us show that $[\gamma] = [e_a]$.

Define a map $G : [0, 1] \times [0, 1] \to X$ by

$$G(s, t) = F(\gamma(s), t),$$

where F is the homotopy from id_X to r provided to us. Then clearly, G is continuous, and

$$\begin{aligned} G(0, t) &= F(\gamma(0), t) = F(a, t) = a, \\ G(1, t) &= F(\gamma(1), t) = F(a, t) = a, \\ G(s, 0) &= F(\gamma(s), 0) = \mathrm{id}_X(\gamma(s)) = \gamma(s), \\ G(s, 1) &= F(\gamma(s), 1) = r(\gamma(s)) = r \circ \gamma(s). \end{aligned}$$

That is, G is a path homotopy from γ to $r \circ \gamma$, so $\gamma \sim_p r \circ \gamma \sim_p e_a$.

(c) Suppose that h has no fixed point, i.e. that $h(x) \neq x$ for all $x \in D^2$. Define a continuous map $r : D^2 \to S^1$ by drawing a line starting at $h(x)$, passing through x, and intersecting S^1 in a point that we call $r(x)$. Then by construction, $r(x) = x$ for $x \in S^1$, which means that r is a retraction. It follows that we have a surjection $r_* : \pi_1(D^2) \to \pi_1(S^1)$. This is impossible, though, since $\pi_1(D^2)$ consists of a single class whereas $\pi_1(S^1) = \mathbb{Z}$.