Notes on Topology

Lecture notes for Basic Topology 2014

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Contents

1. Concepts and terminology from set theory

The purpose of this section is only to provide very brief reviews of a few basic notions, enough to be able to use them practically. For a more careful treatment I refer to suitable text books in set theory.

1.1. Set theory and logic – terminology

We adopt the pragmatic view that a *set* is a "container" of elements of some (any) kind. Henceforth, capital letters denote sets and small letters denote elements in sets.

notation	we read	meaning
$a \in A$	a is an element, or a is contained, in A	obvious
$a \notin A$	a is not an element, or a is not contained, in A	obvious
$A \subset B$	A is a subset of B	if $a \in A$ then also $a \in B$

In words, if $A \subset B$ then every element in A is also an element in B. This includes the possibility $A = B$. If $A \subset B$ but $A \neq B$ then we write $A \subsetneq B$ and say "A is a proper subset of B". We use the notion of a set without elements, *the empty set*, that is denoted $\emptyset$. Sets are often defined by stating, explicitly or implicitly, which elements they contain.

notation	we read
$A = \{a, b, c\}$	A is the set containing the elements a, b, and c
$A = \{x \vert x \text{ is an even integer}\}$	A is the set of all x such that x is an even integer

For instance, the *power set* $\mathscr{P}(X)$ of a set X is the set of all subsets of X, i.e.

$$\mathscr{P}(X) = \{U | U \subset X\}.$$

In particular we have $\emptyset, X \in \mathscr{P}(X)$. The union and intersection of two sets A and B are defined as

$$A \cup B = \{x | x \in A \text{ or } x \in B\} \qquad \text{the } \textit{union} \text{ of } A \text{ and } B \tag{1.1}$$

$$A \cap B = \{x | x \in A \text{ and } x \in B\} \qquad \text{the } \textit{intersection} \text{ of } A \text{ and } B \tag{1.2}$$

The *or* in (1.1) has the logical meaning, i.e. it is allowed that $x \in A$ *and* $a \in B$.

It is often useful and efficient to make use of terminology borrowed from logic:

notation	reads
$\forall$	for all/each/every
$\exists$	there exist/exists
$\exists!$	there exists a unique
$\nexists$	there does not exist
$\Rightarrow$	implies
$\Leftrightarrow$	equivalence

Let us generalise the union and intersection defined above, and to that end introduce first the notion of a family of sets. Let I be a set. A collection $\{U_i\}_{i \in I}$ of sets is called a family of sets, parameterized by I. For any family $\{U_i\}_{i \in I}$ we can defined the union and intersection as

$$\bigcup_{i \in I} U_i = \{x | \exists i \in I : x \in U_i\} \tag{1.3}$$

$$\bigcap_{i \in I} U_i = \{x | x \in U_i \ \forall i \in I\} \tag{1.4}$$

If A and B are sets, we can also consider the set of elements contained in A but not in B. We write $A \backslash B$, or sometimes (for instance in [M]) $A - B$, and call this the difference between A and B. When all sets in question are subsets of another set, we talk about the *complement* of sets. If $U \subset X$, its complement is simply $U^c = X \backslash U$.

There are laws obeyed by union, intersection, and difference, known as the distributive laws and de Morgan's laws.

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \tag{1.5}$$

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z) \tag{1.6}$$

$$X \backslash \bigcup_{i \in I} U_i = \bigcap_{i \in I} X \backslash U_i \qquad \text{if } U_i \subset X \ \forall i \in I \tag{1.7}$$

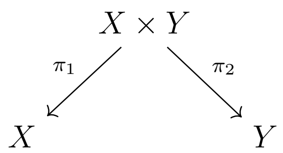
$$X \backslash \bigcap_{i \in I} U_i = \bigcup_{i \in I} X \backslash U_i \qquad \text{if } U_i \subset X \ \forall i \in I \tag{1.8}$$

1.2. Cartesian product

Let X and Y be sets, then their Cartesian product $X \times Y$ is defined as

$$X \times Y := \{(x, y)|x \in X \ \& \ y \in Y\}. \tag{1.9}$$

Note that the elements of $X \times Y$ are *ordered* pairs.There are two associated canonical surjections

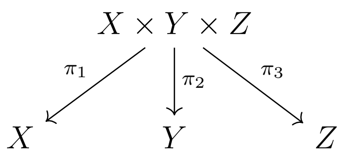


defined as $\pi_1 : (x, y) \mapsto x$, $\pi_2 : (x, y) \mapsto y$.

The Cartesian product can be iterated in various ways. For instance we can form both $X \times (Y \times Z)$, with elements $(x, (y, z))$, and $(X \times Y) \times Z$ with elements $((x, y), z)$. As it stands, these two sets are not strictly the same. They are, however, in bijection in an almost trivial way, namely by "re-bracketing". Henceforth we will tacitly, and without discussion, identify the different way to iterate the Cartesian product. There are situations where such an identification can lead to problems, but we will not encounter those. Furthermore, we will identify both iterations with the following construction

$$X \times Y \times Z := \{(x, y, z)|x \in X, \ y \in Y, \ z \in Z\}. \tag{1.10}$$

The latter product comes with three canonical surjections



defined in the obvious way.

The last construction may be generalized in a natural way to products of n sets. Instead we generalize to products of arbitrary families of sets. Let $\{X_i\}_{i \in I}$ be a family of sets parameterized by I. We defined their product as

$$\prod_{i \in I} X_i := \{f : I \to \cup_{i \in I} X_i | f(i) \in X_i \ \forall i \in I\}. \tag{1.11}$$

There is a canonical surjection $\pi_i : \prod_{i \in I} X_i \to X_i$ for $i \in I$, defined by $\pi_i : f \mapsto f(i)$. Restricting to the index set $\{1, 2, ..., n\}$ we get that an element in the product is nothing by an element $x_i \in X_i$ for each $i = 1, ..., n$, which contains precisely the same information as the n-tuple $(x_1, x_2, ..., x_n)$.

1.3. Relations

A relation C on a set X is simply a subset $C \subset X \times X$. We write xCy to mean that $(x, y) \in C$, or in words that x and y are in the relation C.

Definition 1.1. Let C be a relation on a set X. Then C is called

- *reflexive* if $xCx \; \forall x \in X$
- *symmetric* if $xCy \Rightarrow yCx$
- *anti-symmetric* if $xCy \;\&\; yCx \Rightarrow x = y$
- *transitive* if $xCy \;\&\; yCz \Rightarrow xCz$
- *total* if for every $x, y \in X$, xCy or yCx holds

The concept of a relation is quite general. We will only bother with a few kinds of relations, however.

Definition 1.2. Let X be a set. An *equivalence relation* on X is a relation C that is reflexive, symmetric, and transitive.

Equivalence relations are of course familiar to most of us; they occur frequently in mathematics and are usually denoted $\sim$.

Example 1.3. Let $X = \mathbb{Z}$, and let $C \subset \mathbb{Z} \times \mathbb{Z}$ be the subset consisting of the pairs (m, n) such that division with $p \in \mathbb{N}$ results in the same reminder. We thus have for example $(mp) \sim (np)$ for all $m, n \in \mathbb{Z}$, and more generally $(mp + r) \sim (np + r)$ for all $m, n \in \mathbb{Z}$ and every $r = 1, 2, ..., p - 1$

A, mayhap, less familiar example is that of a partial order.

Definition 1.4. Let X be a set. A relation $\preceq$ on X is called a *partial order* if it is reflexive, anti-symmetric, and transitive. If $\preceq$ is a partial order on X, the pair $(X, \preceq)$ is called a partially ordered set, or a *poset*.

Example 1.5.

(i) $(\mathbb{R}, \leq)$ is a poset; for every $x, y, z \in \mathbb{R}$ we have that $x \leq x$, that $x \leq y \;\&\; y \leq x \Rightarrow x = y$, and that $x \leq y \;\&\; y \leq z \Rightarrow x \leq z$.

(ii) Let X be a set, and $\mathscr{S} \subset \mathscr{P}(X)$ any collection of subsets of X. The pair $(\mathscr{S}, \subset)$ is a poset; for every set U it holds that $U \subset U$, if $U \subset V$ and $V \subset U$ then $U = V$, and finally if $U \subset V$ and $V \subset W$ then $U \subset W$.

Definition 1.6. Let $(X, \preceq)$ be a poset. The relation $\preceq$ is called a *total order* if it is in addition total, i.e. if for every $x, y \in X$ it holds that $x \preceq x$ or $y \preceq x$ (as usual, this includes the possibility that both relations hold). The pair $(X, \preceq)$ is then called a totally ordered set.

Example 1.7. $(X, \leq)$ is a totally ordered set; for every two real numbers x and y it is true that $x \leq y$ or $y \leq x$.

2. Topological spaces

This section is devoted to some basic definitions and examples.

2.1. Definition of topology

Definition 2.1. Let X be a set and $\mathscr{T} \subset \mathscr{P}(X)$ be a collection of subsets of X. The pair $(X, \mathscr{T})$ is called a *topological space* if

($T1$) $\emptyset, X \in \mathscr{T}$

($T2$) the union of any collection of elements in $\mathscr{T}$ is also an element in $\mathscr{T}$, i.e. if $U_i \in \mathscr{T}$ $\forall i \in I$, then $\cup_{i \in I} U_i \in \mathscr{T}$.

($T3$) the intersection of any *finite* collection of elements in $\mathscr{T}$ is an element in $\mathscr{T}$, i.e. if $U_1, U_2, ..., U_n \in \mathscr{T}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathscr{T}$.

The set $\mathscr{T}$ is called a *topology* on X, the elements of $\mathscr{T}$ are called the *open sets* of $(X, \mathscr{T})$, and a set $C \subset X$ is called *closed* if its complement is open, i.e. if $X \backslash C \in \mathscr{T}$.

■ We will often abuse notation and talk about the "topological space X", where we really mean $(X, \mathscr{T})$.

Example 2.2.

(i) For any set X the set $\mathscr{T} = \mathscr{P}(X)$ forms a topology, the topology where every subset is open. This is called the *discrete topology* on X.

(ii) For any set X the set $\mathscr{T} = \{\emptyset, X\}$ forms a topology. This is called the *trivial*, or *indiscrete*, topology on X.

Example 2.3. Let $X = \{a, b\}$ to be a two-element set. Which topologies can we define on X? We are looking for subset of $\mathscr{P}(X)$, so note first that

$$\mathscr{P}(X) = \{\emptyset, X, \{a\}, \{b\}\}.$$

For any subset $Y \subset X$ we have

$$\begin{aligned} \emptyset \cap Y &= \emptyset \\ \emptyset \cup Y &= Y \\ X \cup Y &= X \\ X \cap Y &= Y, \end{aligned}$$

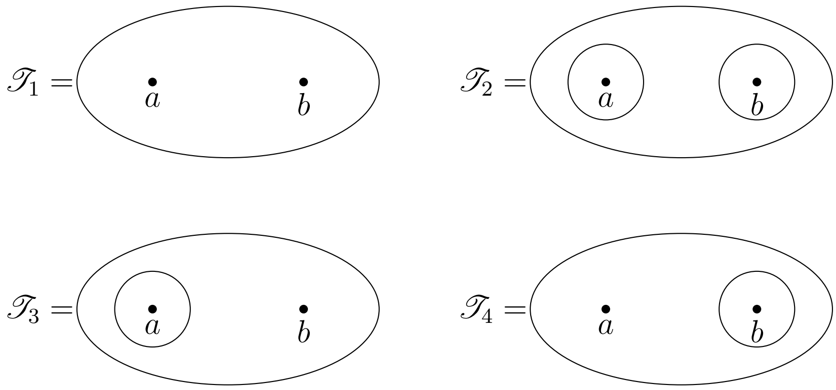
and furthermore

$$\begin{aligned} \{a\} \cap \{b\} &= \emptyset \\ \{a\} \cup \{b\} &= X. \end{aligned}$$

The possible subsets of $\mathscr{P}(X)$ satisfying conditions $(T1)-(T3)$ are thus

$$\begin{aligned}
\mathscr{T}_1 &= \{\emptyset, X\} \\
\mathscr{T}_2 &= \mathscr{P}(X) \\
\mathscr{T}_3 &= \{\emptyset, X, \{a\}\} \\
\mathscr{T}_4 &= \{\emptyset, X, \{b\}\}
\end{aligned}$$

These topologies may be illustrated as follows.



■ In Example 2.3, note that $\mathscr{T}_1 \subset \mathscr{T}_2$, $\mathscr{T}_1 \subset \mathscr{T}_3$, $\mathscr{T}_1 \subset \mathscr{T}_4$, $\mathscr{T}_3 \subset \mathscr{T}_2$, $\mathscr{T}_4 \subset \mathscr{T}_2$. Informally we can say that, for instance, $\mathscr{T}_2$ contains "smaller constituents" than $\mathscr{T}_3$, or that $\mathscr{T}_2$ is *finer* than $\mathscr{T}_3$.

Definition 2.4. Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on a set X. If $\mathscr{T} \subset \mathscr{T}'$ we say that $\mathscr{T}'$ is *finer* than $\mathscr{T}$, or equivalently that $\mathscr{T}$ is *coarser* than $\mathscr{T}'$. If $\mathscr{T} \subsetneq \mathscr{T}'$ we say that $\mathscr{T}'$ is *strictly finer* than $\mathscr{T}$, or equivalently that $\mathscr{T}$ is *strictly coarser* than $\mathscr{T}'$. We say that $\mathscr{T}$ and $\mathscr{T}'$ are *comparable* if either $\mathscr{T} \subset \mathscr{T}'$ or $\mathscr{T}' \subset \mathscr{T}$.

Proposition 2.5. *For a topological space X the following properties hold*

1. *$\emptyset$ and X are closed*
2. *If C_i is closed for every $i \in I$, then $\cap_{i\in I} C_i$ is closed*
3. *If $C_1, C_2, ..., C_n$ are closed, then $C_1 \cup C_2 \cup \cdots \cup C_n$ is closed*

Proof. Exercise. □

It is natural to ask if a topology can be generated by a subset of open sets. To this end we introduce

Definition 2.6. Let X be a set and $\mathscr{B} \subset \mathscr{P}(X)$. The collection $\mathscr{B}$ is called a *basis* for a topology on X if

$(B1)$ If $x \in X$, then there exists a $B \in \mathscr{B}$ such that $x \in B$

(B2) If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathscr{B}$, then there exists a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$

■ We can express $(B1)$ by saying that $\mathscr{B}$ *covers*, or $\mathscr{B}$ is a *cover* of, X.

■ For $\mathscr{B}$ satisfying $(B1)$ and $(B2)$, then we can define the topology $\mathscr{T}_{\mathscr{B}}$ *generated by* $\mathscr{B}$ on X by declaring $U \subset X$ to be open if for every $x \in U$ there exists a basis element $B \in \mathscr{B}$ such that $x \in B \subset U$.

Proof that the collection $\mathscr{T}_{\mathscr{B}}$ generated by a basis $\mathscr{B}$ indeed forms a topology. Let $\mathscr{T}_{\mathscr{B}}$ be the collection of subsets generated as above from a basis $\mathscr{B}$. The empty set $\emptyset$ is clearly in $\mathscr{T}_{\mathscr{B}}$ (check that there is nothing to check). Clearly $X \in \mathscr{T}_{\mathscr{B}}$ since by $(B1)$, for every $x \in X$ there exists a $B_x \in \mathscr{B}$ containing x, so $X = \cup_{x\in X} B_x$. Let $U_i \in \mathscr{T}_{\mathscr{B}}$ for every $i \in I$, and let $U = \cup_{i\in I} U_i$. If $x \in U$, then there exists an $i \in I$ such that $x \in U_i$, and therefore there exists a $B \in \mathscr{B}$ s.t. $x \in B \subset U_i \subset U$. It follows that U is open. Let $U_1, U_2 \in \mathscr{T}_{\mathscr{B}}$, and consider $x \in U_1 \cap U_2$, i.e. $x \in U_1$ and $x \in U_2$. There thus exist $B_1, B_2 \in \mathscr{B}$ such that $x \in B_i \subset U_i$, $i = 1, 2$, and by $(B2)$ there exists a $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. It follows that $U_1 \cap U_2 \in \mathscr{T}_{\mathscr{B}}$. We can now use induction to show that if $U_1, U_2, ..., U_n \in \mathscr{T}_{\mathscr{B}}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathscr{T}_{\mathscr{B}}$. Assume it is true that $U_1 \cap U_2 \cap \cdots \cap U_{n-1} \in \mathscr{T}_{\mathscr{B}}$, then we have just shown that $(U_1 \cap U_2 \cap \cdots \cap U_{n-1}) \cap U_n = U_1 \cap U_2 \cap \cdots \cap U_n \in \mathscr{T}_{\mathscr{B}}$. This finishes the proof. □

Lemma 2.7. *Let $\mathscr{T}_{\mathscr{B}}$ be the topology generated from the basis $\mathscr{B}$, then $\mathscr{T}_{\mathscr{B}}$ consists of all unions of basis elements, i.e. all unions of elements in $\mathscr{B}$.*

Proof. Take $U \in \mathscr{T}_{\mathscr{B}}$, then by construction there exists for every $x \in U$ a set $B_x \in \mathscr{B}$ satisfying $x \in B_x \subset U$. Now, note that $\cup_{x\in U} B_x = U$ (this follows since $B_x \subset U$ for every $x \in U$, and $y \in \cup_{x\in U} B_x$ for every $y \in U$). To see that every union of basis elements is an open set, note that in particular we have $\mathscr{B} \subset \mathscr{T}_{\mathscr{B}}$, so the statement follows from $(T2)$. □

Example 2.8.

(i) Let $X = \{a, b\}$. Then $\mathscr{B} = \{\{a\}, \{b\}\}$ is a basis for a topology on X. The topology generated by $\mathscr{B}$ is the discrete topology, denoted $\mathscr{T}_2$ in Example 2.3.

(ii) Let X be an arbitrary set, and let $\mathscr{B}$ be the set of *singletons*, i.e. one-element subsets, of X. Then $\mathscr{B}$ is a basis for a topology on X, and the topology generated by $\mathscr{B}$ is the discrete topology.

■ Let $B(x, r) = \{y \in \mathbb{R}^n | \|y - x\| < r\}$ be the open ball in $\mathbb{R}^n$ with center in x and radius r.

Proposition 2.9. *The collection*

$$\mathscr{B} = \{B(x, r) | x \in \mathbb{R}^n, r > 0\}$$

is a basis for a topology on $\mathbb{R}^n$.

The proof follows from a more general result further down.

Remark 2.10. The topology generated from open balls in $\mathbb{R}^n$ is called the *standard topology* on $\mathbb{R}^n$, where the open sets are exactly those familiar from calculus.

The following result is useful, see [M, Lemma 13.3] for the proof.

Lemma 2.11. *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies $\mathscr{T}$ and $\mathscr{T}'$ respectively on X. Then the following are equivalent*

1. *$\mathscr{T} \subset \mathscr{T}'$ ($\mathscr{T}'$ is finer than $\mathscr{T}$)*
2. *$\forall\, x \in X,\ B \in \mathscr{B}:\ x \in B, \exists B' \in \mathscr{B}' : x \in B' \subset B$ (in words: for every $x \in X$ and basis element B of $\mathscr{B}$ containing x, there is a basis element B' of $\mathscr{B}'$ containing x and being a subset of B)*

Example 2.12.

(i) Consider the set $\mathscr{B}'$ consisting of all half-open intervals of the form $[a, b) = \{x \in \mathbb{R} | a \leq x < b\}$. The set $\mathscr{B}'$ is a basis for a topology $\mathscr{T}_l$ called the *lower limit topology* on $\mathbb{R}$. Denote the topological space $(\mathbb{R}, \mathscr{T}_l)$ by $\mathbb{R}_l$.

(ii) Let $K \subset \mathbb{R}$ be the set of all elements of the form $1/n$ for $n \in \mathbb{N}$, and consider the collection $\mathscr{B}''$ consisting of all open intervals together with all sets of the type $(a, b)\backslash K$. The set $\mathscr{B}''$ is a basis for the so-called *K-topology* on $\mathbb{R}$. Denote $\mathbb{R}$ with this topology $\mathbb{R}_K$.

Lemma 2.13. *Let $(X, \mathscr{T})$ be a topological space, and let $\mathscr{C} \subset \mathscr{T}$ be such that for every $U \in \mathscr{T}$ and every $x \in U$, $\exists C \in \mathscr{C}$ s.t. $x \in C \subset U$. Then $\mathscr{C}$ is a basis for the topology $\mathscr{T}$.*

Proof. Note that $(B1)$ is automatically satisfied by $\mathscr{C}$. Let $C_1, C_2 \in \mathscr{C}$, then $C_1 \cap C_2 \in \mathscr{T}$. For every $x \in C_1 \cap C_2$ there exists, by assumption, $C_3 \in \mathscr{C}$ s.t. $x \in C_3 \subset C_1 \cap C_2$, so also $(B2)$ holds for $\mathscr{C}$, which is therefore a basis.

Denote by $\mathscr{T}'$ the topology generated by $\mathscr{C}$. If $U \in \mathscr{T}$ and $x \in U$, then by assumption there exists a $C \in \mathscr{C}$ s.t. $x \in C \subset U$. By definition we then have $U \in \mathscr{T}'$. Conversely, if $U \in \mathscr{T}'$ it follows from Lemma 2.7 that U is a union of elements of $\mathscr{C} \subset \mathscr{T}$, so is in particular an element of $\mathscr{T}$. □

We end this subsection by showing some properties of $\mathbb{R}_l$ and $\mathbb{R}_K$.

Lemma 2.14. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$. Furthermore, they are not comparable.*

Proof. Take $x \in \mathbb{R}$ and (a, b) containing x, i.e. $a < x < b$. Then $x \in [x, b) \subset (a, b)$. By Lemma 2.11 it follows that the topology of $\mathbb{R}_l$ is finer than the standard topology. Conversely, $[x, b)$ contains b, but contains no open interval (a, b) that contains x, and thus strictness follows.

Again, take $x \in (a,b)$, then the same interval (a,b) is also a basis element for the topology of $\mathbb{R}_K$, so this topology is finer than the standard topology. Take $0 \in (-1,1)\backslash K$. There is no open interval containing 0 and contained in that basis element, and strictness thus follows. □

2.2. Metric spaces

We next introduce a generalisation of the Euclidean space $\mathbb{R}^n$, i.e. the corresponding vector space together with the Euclidean distance function.

Definition 2.15. A *metric space* (X, d) is a set X together with a function $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying

$(M1)$ $d(x,y) = 0$ iff $x = y$

$(M2)$ $d(x,y) = d(y,x)$

$(M3)$ $d(x,y) \leq d(x,z) + d(z,y)$ (the triangle inequality)

The function d is called a *metric* on X.

■ If (X,d) is a metric space we can define for every element $x \in X$ and every $r \in \mathbb{R}_{>0}$ the *open ball*

$$B_d(x,r) = \{y \in X | d(x,y) < r\} \tag{2.1}$$

As in $\mathbb{R}^n$ we can define open sets, i.e. a topology, in terms of open balls.

Proposition 2.16. *If (X,d) is a metric space then the collection*

$$\mathscr{B} = \{B_d(x,r) | x \in X,\ r \in \mathbb{R}_{>0}\}$$

of open balls forms a basis for a topology. The topology generated by $\mathscr{B}$ is called the metric topology.

Proof. $(B1)$: trivial, choose any $r > 0$ and consider $B_d(x,r)$, this open set contains x and so is non-empty.

$(B2)$: Assume $x \in B_d(y_1,r_1) \cap B_d(y_2,r_2)$, and let $\epsilon = \min(r_1 - d(x,y_1), r_2 - d(x,y_2))$ (note that $\epsilon > 0$). Then $x \in B_d(x,\epsilon) \subset B_d(y_1,r_1) \cap B_d(y_2,r_2)$ since for any $y \in B_d(x,\epsilon)$ we have

$$d(y,y_i) \leq d(y,x) + d(x,y_i) < \epsilon + d(x,y_i) \leq r_i - d(x,y_i) = r_i.$$

□

Remark 2.17. Let (X,d) be a metric space, and consider the metric topology on X. By definition, a subset $U \subset X$ is then open if for every $x \in U$ there exists some open ball $B_d(y,r)$ containing x, and contained in U. This sounds like how we define an open set in the standard topology on $\mathbb{R}^n$; the only difference is that in the latter case we require that for some radius $\epsilon > 0$ the open ball $B_d(x,\epsilon)$ is contained in U. In $\mathbb{R}^n$ with the Euclidean metric it is easy to see that the two definitions are equivalent: If there exists some open ball containing x and contained in U it is easy to construct an open ball *centered in* x and contained in U, thus a set is open in one definition iff it is open in the other. What is the situation in a general metric space? Let $x \in B_d(y, r)$, then it follows using the triangle inequality (check this!) that $B_d(x, \epsilon) \subset B_d(y, r)$, where $\epsilon = r - d(x, y) > 0$. In other words, we can alternatively characterize the metric topology as the topology where a set is open if around every point in the set we can also fit an open ball centered in that point.

Example 2.18.

(*i*) In $\mathbb{R}^n$ we can define the Euclidean metric d as

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2} = \|x - y\|.$$

This is (obviously!) a metric space, and the metric topology is (as we have seen!) precisely the standard topology on $\mathbb{R}^n$.

(*ii*) Let X be any set, and define a function $d : X \times X \to \mathbb{R}_{\geq 0}$ as

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

It is very quickly verified that (X, d) is a metric space. The metric topology in this case is the discrete topology on X. To see this, let us first describe the open balls in this metric. The open ball $B_d(x, y)$ consists of all points in X at distance less than r from x. If $0 < r \leq 1$, then only the point x itself satisfies this condition, thus $B_d(x, r) = \{x\}$. If $r > 1$, then *all* points in X satisfies this condition, since their distance to x is either 1 or 0, both of which are smaller than r. In other words, if $r > 1$ it holds that $B_d(x, r) = X$. From an earlier example we know that $\{\{x\} | x \in X\}$ is a basis for the discrete topology on X. It is easy to see (check this!) that adding X to the basis does not change the topology generated by the basis, and the statement thus follows.

2.3. Continuous functions

The motivation for introducing topological spaces in the first place, is that they provide the minimal structure necessary to define continuity. Since continuous maps play a central role in topology, we introduce them already here. To check that our abstract definition of continuity is sensible, we verify that when applied to the standard topology on $\mathbb{R}^n$ our definition is equivalent to the classical $\epsilon - \delta$-definition.

Definition 2.19. Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. A function $f : X \to Y$ is called (*i*) continuous if $U \in \mathscr{T}_Y \Rightarrow f^{-1}(U) \in \mathscr{T}_X$ (in words: the pre-image of an open set is open)

(*ii*) continuous at $x \in X$ if for every $U \in \mathscr{T}_Y$ containing $f(x)$, $\exists V \in \mathscr{T}_X$ such that $x \in V$ and $f(V) \subset U$.

Example 2.20.

(*i*) Let $(X, \mathscr{T})$ be a topological space. Then the identity map $id : (X, \mathscr{T}) \to (X, \mathscr{T})$ is a continuous map.

(*ii*) Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, and $p \in Y$. Then the constant map

$$c_p : X \to Y,\ c_p : x \mapsto p,\ \forall x \in X$$

is continuous. To see this, note that for any $U \in \mathscr{T}_Y$ not containing p, $c_p^{-1}(U) = \emptyset$, while for any $U \in \mathscr{T}_Y$ containing p, $c_p^{-1}(U) = X$.

(*iii*) Let X have the discrete topology, and let Y be any topological space. Then any function $f : X \to Y$ is continuous, since the pre-image of any open set is a subset of X and thus open in the discrete topology.

(iv) Finally, let X be any topological space and Y be equipped with the trivial topology. Then any function $f : X \to Y$ is continuous, since the pre-image of $\emptyset$ is just $\emptyset$, and the pre-image of Y is X.

We summarize a few important properties of continuous maps in

Theorem 2.21.

(i) If $f : X \to Y$, $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

(ii) We have the equivalence

$$f : X \to Y \text{ continuous}$$
$$\Leftrightarrow$$
$$F \subset Y \text{ closed} \Rightarrow f^{-1}(F) \text{ closed}$$

(iii) $f : X \to Y$ is continuous iff f is continuous at all points $x \in X$

We now make contact with the classical definition of continuity.

Definition 2.22. Recall from calculus that a function $f : \mathbb{R}^n \to \mathbb{R}$ is called continuous at $x \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon \text{ for every } y \in \mathbb{R}^n \text{ such that } \|y - x\| < \delta.$$

If f is continuous at all points in $\mathbb{R}^n$ it is simply called continuous.

Starting from our abstract definition we can instead show

Theorem 2.23. *Let (X, d_X) and (Y, d_Y) be metric spaces with the respective metric topologies. A function $f : X \to Y$ is continuous iff*

$$\forall x \in X, \epsilon > 0,\ \exists \delta > 0 : d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon.$$

Before proving this theorem, let us disentangle the meaning of it. The last line of the theorem states that for every choice of $x \in X$ it holds that for every $\epsilon > 0$ there exists a $\delta > 0$ such that the distance between $f(x)$ and $f(y)$ is smaller than ϵ for any $y \in X$ at distance less than δ from x. Applied to the standard topology on $\mathbb{R}^n$, this is just the definition of a continuous function. The theorem thus states that our abstract definition of continuity is equivalent to the classical definition in the classical case.

As a step in the proof, we first prove the following

Lemma 2.24. *Let (X, d_X) and (Y, d_Y) be metric spaces with the metric topologies. Then a function $f : X \to Y$ is continuous at $x \in X$ iff*

$$\forall \epsilon > 0,\ \exists \delta > 0 : f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon).$$

Proof. $\Rightarrow$: Choose a $\epsilon > 0$. $f(x) \in B_{d_Y}(f(x), \epsilon) \Rightarrow \exists V$ open in X with the properties $x \in V$ and $f(V) \subset B_{d_Y}(f(x), \epsilon)$. However, since V is open it follows that there exists a $\delta > 0$ such that $B_{d_X}(x, \delta) \subset V$ (see Remark 2.17), and thus $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon)$.

$\Leftarrow$: Pick an open subset U containing $f(x)$. Then there exists $\epsilon > 0$ such that $B_{d_X}(f(x), \epsilon) \subset U$, and by assumption therefore a $\delta > 0$ such that $f(B_{d_X}(x, \delta)) \subset B_{d_Y}(f(x), \epsilon) \subset U$. From the definition it then follows that f is continuous at $x \in X$. □

Proof of Theorem 2.23. The proof now follows immediately by combining Lemma 2.24 with Theorem 2.21 (*iii*). □

Let us end with a result that is often useful in showing that a given function is continuous.

Lemma 2.25 (The pasting lemma). *Let X be a topological space, and let $U, V \subset X$ be two open (closed) subsets such that $X = U \cup V$. If $f : U \to Y$, and $g : V \to Y$ are continuous functions (here U and V are viewed as subspaces of X) with the property $f(x) = g(x)$ for every $x \in U \cap V$, then the function $h : X \to Y$ defined by $h|_U = f$ and $h|_V = g$ is continuous.*

Proof. First, note that h is well-defined since $f(x) = g(x)$ whenever $x \in U \cap V$. Next, let $W \subset Y$ be open (closed). Note that $h^{-1}(W) = f^{-1}(W) \cup g^{-1}(W)$. $f^{-1}(W) \subset U$ is open (closed), hence also open (closed) as a subset of X. Analogously, $g^{-1}(W) \subset V$ is open (closed) and hence open (closed) as a subset of X. The union of two open (closed) subsets is open (closed), so h is continuous. □

2.4. The subspace topology

There is a natural way to give a topology to any subset of a topological space.

Definition 2.26. Let $(X, \mathscr{T})$ be a topological space, and let $Y \subset X$ be any subset of X. We define a topology $\mathscr{T}_Y$, the *subspace topology* (or sometimes the *relative* topology), on Y by defining $U \subset Y$ to be open if $\exists \widetilde{U} \in \mathscr{T}$ such that $U = Y \cap \widetilde{U}$. The topological space $(Y, \mathscr{T}_Y)$ is then called a subspace of $(X, \mathscr{T})$.

Proof that the subspace topology is a topology.

$(T1)$: obviously $\emptyset, Y \in \mathscr{T}_Y$.

$(T2)$: Let $U_i \in \mathscr{T}_Y$ $\forall i \in I$, then for each $i \in I$ $\exists \widetilde{U}_i \in \mathscr{T}$ such that $U_i = Y \cap \widetilde{U}_i$. By de Morgan's law it follows that

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} Y \cap \widetilde{U}_i = Y \cap \bigcup_{i\in I} \widetilde{U}_i \in \mathscr{T}_Y.$$

$(T3)$: Let $U_1, U_2, ..., U_n \in \mathscr{T}_Y$, then $\exists$ $\widetilde{U}_1, \widetilde{U}_2, ..., \widetilde{U}_n \in \mathscr{T}$ such that $U_i = Y \cap \widetilde{U}_i$, $i = 1, ..., n$. Clearly

$$U_1 \cap U_2 \cap ... \cap U_n = (Y \cap \widetilde{U}_1) \cap (Y \cap \widetilde{U}_2) \cap ... \cap (Y \cap \widetilde{U}_n) = Y \cap (\widetilde{U}_1 \cap \widetilde{U}_2 \cap ... \cap \widetilde{U}_n) \in \mathscr{T}_Y,$$

so it follows that $\mathscr{T}_Y$ is a topology on Y. □

Proposition 2.27. *Let $(X, \mathscr{T})$ be a topological space, and $(Y, \mathscr{T}_Y)$ a subspace.*

(i) The inclusion map $\imath : Y \to X$, $\imath : y \mapsto y$, is continuous.

(ii) If $f : X \to Z$ is continuous, then the restriction $f|_Y : Y \to Z$ is also continuous.

(iii) $F \subset Y$ is closed iff $\exists \widetilde{F} \subset X$ such that $\widetilde{F}$ is closed and $F = \widetilde{F} \cap Y$.

Proof. (i): Trivial, note that $\imath^{-1}(U) = Y \cap U$.

(ii): Again, note that $f|_Y^{-1}(U) = Y \cap f^{-1}(U)$.

(iii): $\Leftarrow$: $X\backslash\widetilde{F} \in \mathscr{T} \Rightarrow Y \cap (X\backslash\widetilde{F}) \in \mathscr{T}_Y$, but

$$Y \cap (X\backslash\widetilde{F}) = Y\backslash(\widetilde{F} \cap Y) = Y\backslash F.$$

We have shown that $Y\backslash F$ is open, and thus F is closed.
$\Rightarrow$: F closed $\Rightarrow Y\backslash F$ open, and there exists $G \in \mathscr{T}$ such that $Y\backslash F = Y \cap G$.

$$F = Y\backslash(Y\backslash F) = Y\backslash(Y \cap G) = Y \cap (X\backslash G).$$

We have now shown that $\widetilde{F} := X\backslash G$, which is closed, satisfies $F = Y \cap \widetilde{F}$. □

Example 2.28. Let $\mathbb{R}$ have the standard topology, and consider two subsets with the corresponding subspace topologies.

(*i*) The subspace topology on $\mathbb{Z} \subset \mathbb{R}$ coincides with the discrete topology.

(*ii*) The subspace topology on $\mathbb{Q} \subset \mathbb{R}$ is, however, *not* the discrete topology. In fact, every open set has infinitely many points.

We end this subsection by stating two results without proofs.

Proposition 2.29. *Let $Z \subset Y \subset X$, and let $(X, \mathscr{T})$ be a topological space. Then $(Z, \mathscr{T}_Z) = (Z, (\mathscr{T}_Y)_Z)$.*

Proposition 2.30. *If (X, d) is a metric space, and $Y \subset X$ a subset, then $d_{Y\times Y} : Y \times Y \to \mathbb{R}_{\geq 0}$ is a metric on Y. The metric topology on $(Y, d_{Y\times Y})$ coincides with the subspace topology on Y given by the metric topology on X.*

Example 2.31. Let $\mathbb{R}^2$ be equipped with the standard topology. Then the subspace topology on the set $\mathbb{R} \equiv \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ coincides with the standard topology on $\mathbb{R}$.

2.5. The poset and order topologies

Posets are perhaps the most important machine to produce topologies.

Proposition 2.32. *Let $(X, \preceq)$ be a poset and define for every $a \in X$ the set $P_a = \{x \in X | a \preceq x\}$. Then $\mathscr{B} = \{P_a | a \in X\}$ is a basis for a topology $\mathscr{T}_{\mathscr{B}}$, called the* poset *topology, on X.*

Proof. $(B1)$: Since $x \preceq x$ we have $x \in P_x$ $\forall x \in X$.

$(B2)$: Let $x \in P_a \cap P_b$, thus $a \preceq x$ and $b \preceq x$. It holds that $x \in P_x \subset P_a \cap P_b$ since $x \preceq y \Rightarrow a \preceq y$ and $b \preceq y$ by transitivity of $\preceq$. □

Example 2.33. In the poset $(\mathbb{R}, \leq)$ we have for $a \in \mathbb{R}$ that $P_a = [a, \infty)$. Thus, in the poset topology on $\mathbb{R}$ a set $U \subset \mathbb{R}$ is open iff for every $x \in U$ there exists a "closed ray" $[a, \infty)$ containing x and contained in U. Note that this is true for every "open ray" (a, ∞), so the open rays are open in the poset topology.

■ Let $(X, \preceq)$ be a totally ordered set. Let $a, b \in X$, and define subsets of various types as

$$\begin{aligned}
(a, b) &:= \{x \in X | a \preceq x \preceq b,\ x \neq a,\ x \neq b\} \\
[a, b) &:= \{x \in X | a \preceq x \preceq b,\ x \neq b\} \\
(a, b] &:= \{x \in X | a \preceq x \preceq b,\ x \neq a\} \\
[a, b] &:= \{x \in X | a \preceq x \preceq b\} \\
(a, \infty) &:= \{x \in X | a \preceq x, x \neq a\} \\
[a, \infty) &:= \{x \in X | a \preceq x\} \\
(-\infty, b) &:= \{x \in X | x \preceq b, x \neq b\} \\
(-\infty, b] &:= \{x \in X | x \preceq b\}
\end{aligned}$$

In case X has a smallest element a_0, i.e. $a_0 \preceq x$ $\forall x \in X$, then there are some identifications among subsets i.e. $(-\infty, b) = [a_0, b)$. Likewise if there is a largest element b_0.

Proposition 2.34. *Let $(X, \preceq)$ be a totally ordered set, and denote the smallest and largest elements by a_0 respectively b_0 if they exist. The collection*

$$\mathscr{B} = \{(a,b)|a,b \in X,\ a \preceq b\} \cup \{[a_0,b)|b \in X\} \cup \{(a,b_0]|a \in X\},$$

where the latter two sets are included only if they exist, constitute a basis for a topology on X. The topology $\mathscr{T}_{\mathscr{B}}$ is called the order topology *on X.*

Example 2.35. It follows that the standard topology on $\mathbb{R}$ coincides with the order topology given by the total order $\leq$ on $\mathbb{R}$.

Remark 2.36. Consider the example of $(\mathbb{R}, \leq)$ to see that the poset topology on a totally ordered set is not the same as the order topology.

2.6. The product topology

Recall that we can take (Cartesian) products of arbitrary families of sets. It is natural to ask if we can do a similar construction with families of topological spaces, i.e. if there is a product of topological spaces. There is, as we shall soon see, but let us first introduce a new notion that will be of some help. A basis for a topology gives a collection of subsets that is more manageable than the topology itself, but there are still some conditions that need checking. In fact it is, in a rather trivial way, possible to consider even smaller collections than bases to generate topologies.

Definition 2.37. Let X be a set. A *subbasis* $\mathscr{C}$ for a topology on X is a collection of subsets that cover X. The topology $\mathscr{T}_{\mathscr{C}}$ generated by the subbasis $\mathscr{C}$ is the collection of all unions of finite intersections of elements of $\mathscr{C}$. The topology $\mathscr{T}_{\mathscr{C}}$ is the coarsest topology containing $\mathscr{C}$, and it is tautological that it satisfies $(T1)$, $(T2)$, and $(T3)$.

We can now define a topology on the product of a family of topological spaces, the product topology.

Definition 2.38. Let $\{X_i\}_{i \in I}$ be a family of topological spaces, and let $X = \prod_{i \in I} X_i$ be the product of the underlying sets. The *product topology* on X is defined to be the coarsest topology such that each canonical surjection π_i, $i \in I$, is continuous.

The definition is not very concrete, so let us see what we can say about it starting from the topologies on the individual factors. First, since we require the canonical projections to be continuous, the pre-images of open sets under these maps must be open in the product topology. If $U \subset X_i$ is open, then $\pi_i^{-1}(U)$ is U in the $i'th$ factor, and X_j in the $j'th$ factor for all $j \neq i$. The product topology contains all unions of finite intersections of such sets. All possible finite intersections of sets of this type are products where all but finitely many factors are $X_i's$, and finitely many factors are allowed to be smaller open sets in the respective topologies. If $|I| < \infty$ we are thus looking at all sets of the type $U_1 \times U_2 \times ... \times U_n$, where $U_i \subset X_i$ are open for $i = 1, ..., n$. The product topology consists of all unions of such sets, i.e. we have constructed a basis for the product topology.

Remark 2.39. Allowing arbitrary, not only finite, intersections in the description above also defines a topology on the product set, called the *box topology*. That topology is strictly finer than the product topology.

To clarify the construction we spell it out in some detail for the product $X \times Y$. The set $\mathscr{B} = \{U \times V | U \overset{open}{\subset} X,\ V \overset{open}{\subset} Y\}$ is a basis for the product topology on $X \times Y$, and an open set in the latter is thus a union of elements in $\mathscr{B}$.

Theorem 2.40. *Let X be a topological space and let $\{Y_i\}_{i\in I}$ be a family of topological spaces. A function $f : X \to \prod_{i\in I} Y_i$ consists of a family of functions $\{f_i\}_{i\in I}$ where $f_i : X \to Y_i$. The function f is continuous iff f_i is continuous for each $i \in I$.*

Proof. $\Rightarrow$: Assume f is continuous. We have for each $i \in I$ that $f_i = \pi_i \circ f$ so f_i is the composition of two continuous functions, and is therefore continuous.
$\Leftarrow$: Suppose now that f_i is continuous for each $i \in I$. To show that f is continuous it is enough to show that the pre-images of all elements of a subbasis are open. A typical subbasis element is of the form $\pi_i^{-1}(U_i)$ where $U_i \subset Y_i$ is open. Since $f_i = \pi_i \circ f$, we have that $f^{-1}(\pi_i^{-1}(U_i)) = f_i^{-1}(U_i)$,. The latter set is open since f_i is continuous by assumption. □

2.7. Interior, closure, boundary, and limit points

Definition 2.41. Let $(X, \mathscr{T})$ be a topological space, and $Y \subset X$ a subset.

(i) The *interior* of Y is
$$\mathring{Y} := \bigcup_{\substack{U\in\mathscr{T}\\ U\subset Y}} U$$

(ii) The *closure* of Y is
$$\overline{Y} := \bigcap_{\substack{U\in\mathscr{T}\\ Y\subset X\setminus U}} X\setminus U$$

(iii) Y is called *dense* (in X) if $\overline{Y} = X$.

Some immediate consequences of the definition follow

- $\mathring{Y}$ is open
- $\overline{Y}$ is closed
- $\mathring{Y} \subset Y \subset \overline{Y}$
- Y is open iff $\mathring{Y} = Y$, and we can characterize $\mathring{Y}$ as the largest open set contained in Y
- Y is closed iff $\overline{Y} = Y$, and we can characterize $\overline{Y}$ as the smallest closed set containing Y

- $\mathring{Y} = X\backslash\overline{(X\backslash Y)}$ since

$$X\backslash \bigcap_{\substack{U\in\mathscr{T}\\ X\backslash Y\subset X\backslash U}} X\backslash U = \bigcup_{\substack{U\in\mathscr{T}\\ U\subset Y}} X\backslash(X\backslash U) = \bigcup_{\substack{U\in\mathscr{T}\\ U\subset Y}} U.$$

  The next to last step follows from de Morgan's law together with $X\backslash Y \subset X\backslash U \Leftrightarrow U \subset Y$.

- $\overline{Y} = X\backslash(X\mathring{\backslash}Y)$ since

$$X\backslash \bigcup_{\substack{U\in\mathscr{T}\\ U\subset X\backslash Y}} = \bigcap_{\substack{U\in\mathscr{T}\\ U\subset X\backslash Y}} X\backslash U.$$

  To get to the right hand side we have again used de Morgan's law.

Proposition 2.42. *Let X be a topological space and $Y, Z \subset X$ subsets. Then*

(i) $\overline{Y \cup Z} = \overline{Y} \cup \overline{Z}$

(ii) $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$

(iii) $\mathring{Y} \cup \mathring{Z} \subset (Y \mathring{\cup} Z)$

(iii) $\mathring{Y} \cap \mathring{Z} = (Y \mathring{\cap} Z)$

Proof. (i) $\subset$: Note that since $Y \subset \overline{Y}$ and $Z \subset \overline{Z}$ we have that $Y \cup Z \subset \overline{Y} \cup \overline{Z}$. Since the latter is a closed subset containing $\overline{Y \cup Z}$ we must have $\overline{Y \cup Z} \subset \overline{Y} \cup \overline{Z}$.
$\supset$: Since $Y \subset \overline{Y \cup Z}$ and $Z \subset \overline{Y \cup Z}$, and the right hand side is closed, it follows that $\overline{Y} \subset \overline{Y \cup Z}$ and $\overline{Z} \subset \overline{Y \cup Z}$ (since the closure of a set is the smallest closed set containing the set). It follows that $\overline{Y} \cup \overline{Z} \subset \overline{Y \cup Z}$, and thus $\overline{Y} \cup \overline{Z} = \overline{Y \cup Z}$

(ii): Since $Y \cap Z \subset \overline{Y} \cap \overline{Z}$, and the latter is a closed set, it follows that $\overline{Y \cap Z} \subset \overline{Y} \cap \overline{Z}$.

(iii) & (iv): Exercise, follows from similar arguments.

□

■ For any $x \in X$ in a topological space, define a *neighbourhood* of x to denote an open set containing x.

Definition 2.43. Let X be a topological space, and $Y \subset X$ a subset.

(i) The *boundary* of Y is

$$\partial Y := \{x \in X | \forall \text{ neighbourhoods } U \text{ of } x : U \cap Y \neq \emptyset \neq U \cap (X\backslash Y)\}.$$

In words: $x \in \partial Y$ iff all neighbourhoods of x intersect both Y and $X\backslash Y$.

(ii) A *limit point* of Y is a point $x \in X$ such that all the neighbourhoods of x intersect Y in a point not equal to x. Let

$$Y' := \{x \in X | x \text{ is a limit point of } Y\}.$$

Example 2.44. $Y = (0,1) \cup \{2\} \subset \mathbb{R}$ with the standard topology. Then

- $\mathring{Y} = (0,1)$
- $\overline{Y} = [0,1] \cup \{2\}$
- $\partial Y = \{0,1,2\}$
- $Y' = [0,1]$

Theorem 2.45. *Let X be a topological space and $Y \subset X$ a subset.*

(i) $\partial Y = \overline{Y} \cap \overline{(X \backslash Y)}$

(ii) $\overline{Y} = Y \cup \partial Y$

(iii) $\overline{Y} = Y \cup Y'$

Proof. (i) Consider the complement of both sides of the equality, i.e.

$$X \backslash \partial Y = X \backslash \overline{Y} \cup X \backslash \overline{(X \backslash Y)} = (X \mathring{\backslash} Y) \cup \mathring{Y}.$$

$\subset$: Take $x \in X \backslash \partial Y$, then there exists a neighbourhood of x contained in Y or in $X \backslash Y$.
$\supset$: If $x \in (X \mathring{\backslash} Y)$, then there exists a neighbourhood of x not intersecting Y (since any interior is open) so $x \in X \backslash \partial Y$. Similarily, if $x \in \mathring{Y}$, then there exists a neighbourhood of x not intersecting $X \backslash Y$, and it follows again that $x \in X \backslash \partial Y$. We have shown $X \backslash \partial Y \subset (X \mathring{\backslash} Y) \cup \mathring{Y}$ and $X \backslash \partial Y \supset (X \mathring{\backslash} Y) \cup \mathring{Y}$, so the equality follows.

(ii): $Y \cup \partial Y = Y \cup (\overline{Y} \cap \overline{(X \backslash Y)}) = (Y \cup \overline{Y}) \cap (Y \cup \overline{(X \backslash Y)}) = \overline{Y} \cap X = \overline{Y}$

(iii) Use (ii), and consider the sets $\partial Y \backslash Y$ and $Y' \backslash Y$. If $x \in \partial Y \backslash Y$ then there exists some neighbourhood of x that intersects Y, and it follows that there is a point distinct from x in that intersection. Thus $\partial Y \backslash Y \subset Y' \backslash Y$. If $x \in Y' \backslash Y$ then every neighbourhood of x intersects both Y and $X \backslash Y$ (since x is contained in that intersections), hence $Y' \backslash Y \subset \partial Y \backslash Y$. It follows that $\partial Y \backslash Y = Y' \backslash Y$. Taking the union with Y we get $Y \cup Y' = Y \cup \partial Y = \overline{Y}$. □

Proposition 2.46. *Let $\{X_i\}_{i \in I}$ be a family of topological spaces, and let $A_i \subset X_i$ be a subset for each $i \in I$. Then $\prod_{i \in I} \overline{A}_i = \overline{\prod_{i \in I} A_i}$.*

Proof. Let $x = \{x_i\}_{i \in I} \in \prod i \in I \overline{A}_i$, and let $U = \prod_{i \in I} U_i$ be a basis element containing x. Since $x_i \in U_i$ for each $i \in I$ there is a point $y_i \in U_i \cap A_i$ for each $i \in I$. It follows that $y = \{y_i\} \in U$ and $y \in \prod_{i \in I} A_i$, i.e. the intersection of any neighbourhood of x with $\prod_{i \in I} A_i$ is non-empty, so $x \in \overline{\prod_{i \in I} A_i}$.

Conversely, suppose that $x = \{x_i\}_{i \in I} \in \overline{\prod_{i \in I} A_i}$, and let V_i be an open set in X_i containing x_i. The canonical projection π_i is continuous, so $\pi_i^{-1}(V_i) \subset \prod_{i \in I} X_i$ is open and contains a point y of $\prod_{i \in I} A_i$ (since x lies in the closure of the latter). In other words, $y_i = V_i \cap A_i$, which is therefore non-empty, and it follows that $x \in \prod_{i \in I} \overline{A}_i$. □

Example 2.47. We end with a few (counter)-examples concerning dense sets.

(*i*) X is dense in X

(*ii*) The rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$ (with the standard topology). To see this, use $\overline{\mathbb{Q}} = \mathbb{Q} \cup \partial\mathbb{Q}$. Let $x \in \mathbb{R}$ be arbitrary. Any open interval containing x intersects both $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ in infinitely many points. Thus $\partial\mathbb{Q} = \mathbb{R}$, and $\overline{\mathbb{Q}} = \mathbb{R}$.

2.8. Separation axioms – part 1

Definition 2.48. A topological space $(X, \mathscr{T})$ is called

(*i*) T_0 if for every pair $x, y \in X$, $x \neq y$, there is a neighbourhood of x not containing y, or there is a neighbourhood of y not containing x. Both may be true, of course.

(*ii*) T_1 if for any pair $x, y \in X$, $x \neq y$, x has a neighbourhood not containing y, and y has a neighbourhood not containing x.

(*iii*) T_2, or *Hausdorff*, if any two distinct points, x, y, have disjoint neighbourhoods, i.e. there exist $U_x, U_y \in \mathscr{T}$ such that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

■ It is immediate that $T_2 \Rightarrow T_1 \Rightarrow T_0$.

Proposition 2.49. *A topological space X is T_1 iff $\{x\}$ is closed $\forall x \in X$.*

Proof. Let $x \in X$ and take $y \in X \backslash \{x\}$. If X is T_1 then there exists a neighbourhood U_y of y not intersecting x, i.e. $U_y \subset X \backslash \{x\}$. It follows that $X \backslash \{x\}$ is open, and $\{x\}$ is therefore closed. Conversely, assume $\{x\}$ is closed, i.e. $X \backslash \{x\}$ is open. Then for every $y \in X \backslash \{x\}$ there is a neighbourhood U_y of y contained in $X \backslash \{x\}$. In other words, y has a neighbourhood not containing x. In the same way it follows that x has a neighbourhood not containing y, and thus X is T_1. □

Example 2.50.

(*i*) Let X be a topological space with at least two points, and with the trivial topology. Then X is not T_0 (and therefore also not T_1 or Hausdorff) since every point in X has only the neighbourhood X.

(*ii*) A poset $(X, \preceq)$ with the poset topology is T_0. Take two points $x, y \in X$. If $x \preceq y$ then $y \in P_y$, but $x \notin P_y$, thus P_y is a neighbourhood of y not containing x. We argue analogously if $y \preceq x$. If there is no relation between y and x, then $P_x \cap P_y = \emptyset$, and thus x has a neighbourhood (P_x) not containing y.

(*iii*) Consider the poset topology on $X = \{a, b, c\}$ with the partial order $a \preceq b$, $a \preceq c$. We have $P_a = \{a, b, c\} = X$, $P_b = \{b\}$, and $P_c = \{c\}$. The poset topology is then

$$\{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}.$$

It follows that the closed sets are

$$\{\emptyset, X, \{a, c\}, \{a, b\}, \{a\}\}.$$

The set $\{b\}$ is not closed, and the topology is therefore not T_1. It follows that there are topological spaces that are T_0 but not T_1.

(*iv*) If (X, d) is a metric space with the metric topology, then X is T_1. The proof of this statement is an exercise.

(*v*) Let $(X, \preceq)$ be a totally ordered set with the order topology, then X is T_1. Let $x \in X$. If there are no smallest or largest elements in X then

$$X \backslash \{x\} = (-\infty, x) \cup (x, \infty),$$

i.e. $X \backslash \{x\}$ is open, and thus $\{x\}$ is closed. If there is a smallest element a_0 in X, then

$$X \backslash \{x\} = [a_0, x) \cup (x, \infty).$$

The set $[a_0, x)$ is a basis element, and therefore open, so $\{x\}$ is again closed. The remaining cases (with a largest element) follow analogously.

(*vi*) Let X be an infinite set equipped with the *cofinite* topology, i.e. a set $F \subset X$ is closed iff F is finite, $|F| < \infty$. It is easily checked that this indeed defines a topology on X. Two sets U_1, U_2 are open iff they are complements of closed sets F_1, F_2. We see

$$U_1 \cap U_2 = (X \backslash F_1) \cap (X \backslash F_2) = X \backslash (F_1 \cup F_2) \neq \emptyset,$$

since $F_1 \cup F_2$ is again a finite set. It follows that X is not Hausdorff because no two proper open sets are disjoint. However, the set $\{x\}$ is finite and therefore closed for every $x \in X$, so X is T_1. We have shown that there are sets that are T_1 but not T_2.

2.9. Sequences and convergence

Definition 2.51. Let X be a topological space. A *sequence* in X is a family $\{x_n\}_{n \in \mathbb{N}}$ of points in X. The sequence $\{x_n\}$ *converges* to $x \in X$ if for every neighbourhood U of x, $\exists N > 0 : n > N \Rightarrow x_n \in U$. We write $x_n \to x$. A *subsequence* of $\{x_n\}$ is a sequence $\{y_n\}$ such that $y_i = x_{n_i}$ for some $n_1 < n_2 < ....$

Proposition 2.52. *If $x_n \to x$ then $x_{n_i} \to x$ for every subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

Proof. Trivial. □

Example 2.53.

(*i*) In the trivial topology, every sequence converges to every point.

(*ii*) The constant sequence $x_n = x \ \forall n \in \mathbb{N}$ converges to x in any topology

(*iii*) In the discrete topology, a sequence $\{x_n\}$ converges iff there exist a $N \in \mathbb{N}$ such that the subsequence $\{y_n\}$, $y_n := x_{n+N}$, is constant.

Proposition 2.54. *In a metric space (X, d), a sequence $\{x_n\}$ converges to x iff $\forall \epsilon > 0$, $\exists N > 0$: $n > N \Rightarrow d(x_n, x) < \epsilon$*

Proof. $\Rightarrow$: For every $\epsilon > 0$, $B_d(x, \epsilon)$ is a neighbourhood of x, thus there exists an $N > 0$ such that $n > N \Rightarrow x_n \in B_d(x, \epsilon)$.

$\Leftarrow$: Let U be a neighbourhood of x. Since U is open, there exists an $\epsilon > 0$: $B_d(x, \epsilon) \subset U$. By assumption there exists an $N > 0$ such that $n > N \Rightarrow x_n \in B_d(x, \epsilon) \subset U$. □

Remark 2.55. Note that $x_n \to x$ in a metric space (X, d) iff $d(x_n, x) \to 0$ in the standard topology on $\mathbb{R}$.

Proposition 2.56. *Let $\{x_n\}$ be a sequence in a Hausdorff space X such that $x_n \to x$ and $x_n \to y$. Then $x = y$.*

Proof. Assume $x \neq y$, then there exist disjoint neighbourhoods of x and y, say U_x and U_y. Therefore there exist $N_x > 0$ and $N_y > 0$ such that $n > N_x \Rightarrow x_n \in U_x$ and $n > N_y \Rightarrow x_n \in U_y$. Thus, for $n > \max(N_x, N_y)$ it follows that $x_n \in U_x \cap U_y = \emptyset$, which is a contradiction. We conclude that $x = y$. □

Proposition 2.57. *A metric space (X, d) is Hausdorff.*

Proof. Take $x, y \in X$, $x \neq y$, and define $r := d(x, y)$, and take $z \in B_d(x, r/2) \cap B_d(y, r/2)$. Then

$$d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2 = r,$$

which is a contradiction and it follows that $B_d(x, r/2) \cap B_d(y, r/2) = \emptyset$. □

We want to generalize the familiar characterisations of closed sets and continuous functions in terms of sequences. To do so in a maximally efficient way it is necessary to introduce the notion of countable basis and first countability.

Definition 2.58. A space X has a *countable basis* at $x \in X$ it there is a collection of neighbourhoods $\{B_n\}_{n\in\mathbb{N}}$ of x such that for any neighbourhood U of x, there is some $n \in \mathbb{N}$ satisfying $B_n \subset U$. A space X is called *first countable* if every point has a countable basis.

Proposition 2.59. *A metric space (X, d) is first countable, and*

$$\{B_d(x, 1/n)\}_{n\in\mathbb{N}}$$

is a countable basis at x.

Proof. Every neighbourhood of x contains some open ball $B_d(x, \epsilon)$, so take n so that $\epsilon > 1/n$ □

Now, the following result is a characterization of closed sets.

Lemma 2.60 (The sequence lemma). *Let X be a topological space and $Y \subset X$ a subset. We have*

$$\exists \{x_n\}_{n\in\mathbb{N}} \text{ in } Y : x_n \to x \Rightarrow x \in \overline{Y}$$

In words: every limit of a sequence in Y is contained in the closure $\overline{Y}$. Furthermore, the converse is true if X is first countable.

Proof. $\Rightarrow$: Let $\{x_n\}$ be a sequence in Y such that $x_n \to x$. By the definition of convergence, every neighbourhood U of x contains x_n for infinitely many n, and it follows that the intersection $U \cap Y$ is non-empty for every neighbourhood U of x. Therefore $x \in \overline{Y}$, since $x \in Y$ or $x \in \partial Y$ (if every neighbourhood also intersects $X \backslash Y$ non-trivially).

$\Leftarrow$: Let $x \in \overline{Y}$, and let $\{B_n\}$ be a countable basis at x. For each $n \in \mathbb{N}$, $\bigcap_{i=1}^n B_i$ is a neighbourhood of x, so it follows that $Y \cap \bigcap_{i=1}^n B_i \neq \emptyset$. For every $n \in \mathbb{N}$, choose $x_n \in \bigcap_{i=1}^n B_i$. We will now show that $\{x_n\}$ converges to x. Let V be a neighbourhood of x, then $B_N \subset V$ for some $N \in \mathbb{N}$. Therefore also $\bigcap_{i=1}^n B_i \subset V$ for every $n > N$, and it follows that $x_n \in V$ for every $n > N$. We have shown that $x_n \to x$.

□

Finally, we can state the sequential characterization of continuity.

Theorem 2.61. *Let X and Y be topological spaces. If $f : X \to Y$ is continuous at $x \in X$, then*

$$x_n \to x \Rightarrow f(x_n) \to f(x)$$

for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X. In words: for any sequence $\{x_n\}$ in X it holds that if $x_n \to x$, then the sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ in Y converges to $f(x)$, i.e. $f(x_n) \to f(x)$. Furthermore, the converse is true if X is first countable.

Proof. $\Rightarrow$: Assume $x_n \to x$ for a sequence $\{x_n\}$ in X. Let U be a neighbourhood of $f(x)$ in Y, then there exists a neighbourhood V of x such that $f(V) \subset U$. By the definition of convergence, there exists a $N > 0$ such that $n > N \Rightarrow x_n \in V$, thus $f(x_n) \in f(V) \subset U$, and it follows that $f(x_n) \to f(x)$.

$\Leftarrow$: Let now X be first countable, and let $\{B_n\}$ be a countable basis at x. Suppose $x_n \to x$ implies that $f(x_n) \to f(x)$ for all sequences $\{x_n\}$ in X. Assume that f is not continuous at x, then there exists a neighbourhood U of $f(x)$ such that for every neighbourhood V of x $f(V) \not\subset U$. In particular, this is true for the neighbourhood $\bigcap_{i=1}^n B_i$ of x for each $n \in \mathbb{N}$. We may therefore choose $x_n \in \bigcap_{i=1}^n B_i$ such that $f(x_n) \notin U$ for each $n \in \mathbb{N}$. We have shown that $f(x_n)$ does not converge to $f(x)$ even though $x_n \to x$, which is a contradiction. The function f must therefore be continuous at x. □

Remark 2.62. Note that the sequence lemma (Lemma 2.60) as well as the last theorem reduce to the familiar characterization of closed sets in terms of limit points, as well as the sequential definition of continuity in terms of convergent sequences, when applied to $\mathbb{R}^n$ with the metric (i.e. standard) topology.

3. Homeomorphisms and the quotient topology

We have introduced the objects of study in topology, topological spaces, as well as the basic tool to compare topological spaces, namely continuous functions. In more formal language one may say that we have defined a category of topological spaces and continuous maps. A central question is then which topological spaces should be considered equivalent, i.e. which topological spaces are isomorphic? One can argue that this question is the focus of the major part of activity in topology. In order to study the question, we first need to introduce the suitable notion of isomorphism namely homeomorphism.

3.1. Homeomorphisms

If the natural way to compare topological spaces are continuous maps, then an isomorphism must be a continuous map that is bijective and has a continuous inverse.

Definition 3.1. Let X and Y be topological spaces. A bijection $F : X \to Y$ is called a *homeomorphism* if both f and f^{-1} are continuous. If there exists a homeomorphism $f : X \to Y$ we say that X and Y are *homeomorphic*, and we write $X \simeq Y$, or sometimes $f : X \xrightarrow{\sim} Y$ to emphasize the homeomorphism f.

Quite often we are dealing with injective maps that are homeomorphisms onto their images.

Definition 3.2. Let X, Y be topological spaces. If $f : X \to Y$ is such that $f : X \to f(X)$ is a homeomorphism, then f is called an *embedding*.

■ There is the obvious equivalence: f is a homeomorphism $\Leftrightarrow$ f is a bijection and $f(U)$ is open iff U is open.
■ If $X \simeq Y$ then X and Y are indistinguishable as far as topological properties (i.e. properties defined purely in terms of open sets) go. Whenever X has (or does not have) a topological property then Y also has (or does not have) the same property, and vice versa.

Example 3.3.

(*i*) $(-1, 1) \simeq \mathbb{R}$ (the interval has the subspace topology) There are many possible homeomorphisms, let us take

$$\varphi : (-1, 1) \to \mathbb{R},\ x \mapsto \tan \frac{\pi x}{2}.$$

φ is clearly bijective and continuous (the relevant notion of continuity is the one familiar from calculus). Equally obvious is that $\varphi^{-1} : x \mapsto \frac{2}{\pi} \arctan x$ is continuous, and f is thus a homeomorphism.

(*ii*) In fact $(a, b) \simeq \mathbb{R}$ for any $a, b \in \mathbb{R}$, $a < b$.

(*iii*) Let $B^n := B(0,1)$ be the unit ball in $\mathbb{R}^n$, then $B^n \simeq \mathbb{R}^n$. We construct a homeomorphism φ as

$$\varphi : B^n \to \mathbb{R}^n,\ x \mapsto \frac{x}{1 - \|x\|}.$$

It is easily checked that φ is bijective with inverse

$$\varphi^{-1} : \mathbb{R}^n \to B^n,\ x \mapsto \frac{x}{1 + \|x\|}.$$

Again we trivially see that both φ and φ^{-1} are continuous in the classical sense, so φ is indeed a homeomorphism.

(*iv*) $B^n \simeq (-1,1) \times (-1,1) \times \cdots \times (-1,1) \subset \mathbb{R}^n$ It is possible, although a bit tricky, to find an explicit homeomorphism. Instead we can show that the box is homeomorphic to $\mathbb{R}^n$, after which the stated homeomorphism follows by transitivity of equivalence relations. The function

$$\varphi : (-1,1) \times \cdots \times (-1,1) \to \mathbb{R}^n,\ (x_1, ..., x_n) \mapsto (\tan \frac{\pi x_1}{2}, ..., \tan \frac{\pi x_n}{2})$$

furnishes such a homeomorphism for the same reasons as in the example (*i*).

■ From these examples we see clearly that topology does not care about length, as the real line is homeomorphic to a bounded interval, or about smoothness, as a ball is homeomorphic to a box.

3.2. The n-dimensional sphere

Definition 3.4. The space

$$S^n := \{x \in \mathbb{R}^{n+1} | x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}, \tag{3.1}$$

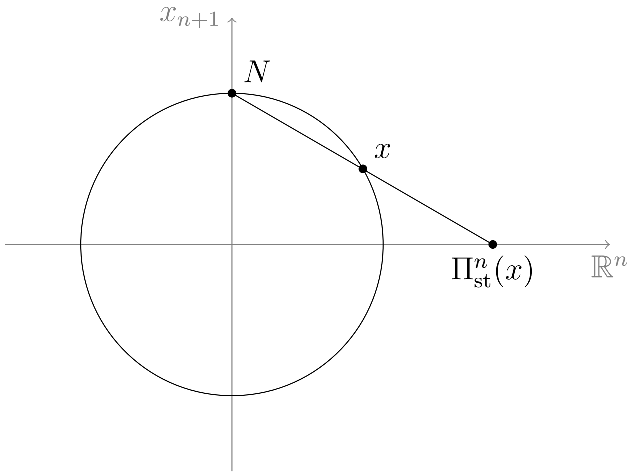
with the subspace topology, is called the *n-dimensional unit sphere*, or simply the n-sphere.

Proposition 3.5. *For any $p \in S^n$ we have $S^n \backslash \{p\} \simeq \mathbb{R}^n$.*

Define $N := (0, 0, ..., 1) \in S^n$ to be the "north pole" of the n-sphere.

Proof. We will construct an explicit homeomorphism $\Pi^n_{\text{st}} : S^n \backslash \{N\} \to \mathbb{R}^n$, known as the *stereographic projection* (this may be familiar from earlier courses in the case $n = 2$). The second part of the proof amounts to showing that $S^n \backslash \{p\} \simeq S^n \backslash \{N\}$ for any $p \in S^n$.

Define Π^n_{st} to take the point x to the intersection of the hyperplane $x_n = 0$ with the line passing through N and x, as illustrated in the following picture.



To obtain an explicit formula, let $x = (x', x_{n+1})$, i.e. $x' \in \mathbb{R}^n$. Then

$$\Pi^n_{\text{st}} : (x', x_{n+1}) \mapsto \frac{x'}{1 - x_{n+1}}.$$

It is easily checked that Π^n_{st} is invertible with inverse

$$(\Pi^n_{\text{st}})^{-1} : x' \mapsto (\frac{2x'}{\|x'\|^2 + 1}, 1 - \frac{2}{\|x'\|^2 + 1})$$

Clearly both Π^n_{st} and its inverse are continuous in the classical sense, and they are therefore homeomorphisms. We have shown that $S^n \backslash \{N\} \simeq \mathbb{R}^n$.

It seems intuitively clear that the same result holds if we replace N with an arbitrary point $p \in S^n$. In order to prove this we state a few results without proof; see e.g. [RÖ, Section 7.3]. First, any point $x \in S^n$ is mapped to any other point $y \in S^n$ by means of an invertible linear map of $\mathbb{R}^{n+1}$ that preserves the Euclidean distance, i.e. an element of the set (actually *group*) $O(n+1)$ of orthogonal transformations. Second, any linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous. It follows that there exists a homeomorphism from $\mathbb{R}^{n+1}$ to itself preserving S^n (hence restricting to a homeomorphism of S^n) and taking a point $p \in S^n$ to N. Composing with Π^n_{st} finally gives the proof of the general statement. □

■ Not every continuous bijection is a homeomorphism, as the following example shows.

Example 3.6. Define $f : [0, 1) \to S^1$, $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. Obviously f is bijective, with inverse $f^{-1} : (x, y) \mapsto \frac{\arcsin y}{2\pi}$. An open set in S^1 is a union of open intervals, in the obvious meaning of the word. It follows that the pre image $f^{-1}(U)$ of an open set U is a union of open intervals and possibly half-open intervals of the type $[0, a)$ for $a \in (0, 1)$, all of which are open in the subspace topology. It follows that f is continuous. However, $f([0, 1/2)$ is half-open, and not open, in S^1, thus f is not a homeomorphism.

■ There is, however, the following variation of the last example.

Example 3.7. Consider $\mathbb{R}/\mathbb{Z}$, i.e. the equivalence classes of the relation $x \sim y$ iff $x - y \in \mathbb{Z}$. An element in $\mathbb{R}/\mathbb{Z}$ is thus a set $\{a + \mathbb{Z}\}$. The set $[0, 1)$ forms a set of representatives of the equivalence classes, and we can informally (making sure we are careful) identify $\mathbb{R}/\mathbb{Z}$ with $[0, 1)$. We will not, however, endow the set of equivalence classes with the subspace topology following from this identification. Define $\gamma : \mathbb{R}/\mathbb{Z} \to S^1$, $\{a + \mathbb{Z}\} \mapsto (\cos 2\pi a, \sin 2\pi a)$; the map γ is clearly well-defined and bijective. Define a topology on $\mathbb{R}/\mathbb{Z}$ by demanding γ to be a homeomorphism, i.e. a subset $U \subset \mathbb{R}/\mathbb{Z}$ is open iff $\gamma(U) \subset S^1$ is open. This trivially defines a topology on $\mathbb{R}/\mathbb{Z}$. Identifying the latter with the set $[0, 1)$ the so defined topology is *not* the subspace topology, however. Rather, any open set containing 0 is contained in a set of the type $[0, a) \cup (b, 1)$ for $0 < a < b < 1$. In other words, the open sets behave as though we have glued the two end points of the interval together, which is exactly what has happened in $\mathbb{R}/\mathbb{Z}$, where 0 and 1 are identified.

■ Although it will not play a central role in this course, an important concept in topology is nevertheless that of embedded circles.

Definition 3.8. Let X be a topological space. A *simple closed curve* in X is an injective continuous map $\gamma : S^1 \to X$.

Remark 3.9. Note that the definition of simple closed curve does not require γ to be an embedding, i.e. a homeomorphism onto its image. Instead that is a result, one can indeed show that the inverse is continuous.

3.3. The quotient topology

We will go through a very useful way to explicitly construct new topological spaces from known spaces.

Definition 3.10. Let X and Y be topological spaces. A surjective map $f : X \to Y$ is called a *quotient map* if

$$U \subset Y \text{ open} \Leftrightarrow f^{-1}(U) \subset X \text{ open}.$$

■ A quotient map is not necessarily a homeomorphism since invertibility is not required.
■ For any surjective map $f : X \to Y$ there is an equivalence relation $\sim_f$ on X where $x \sim_f y$ iff $f(x) = f(y)$. Equivalence classes are sets $f^{-1}(p) \subset X$ for $p \in Y$, and there is a bijection between $X/\sim_f$ and Y taking $p \in Y$ to $f^{-1}(p)$

Definition 3.11. Let X be a topological space, $\sim$ an equivalence relation on X, and $p : X \to X/\sim$ the canonical surjection that maps any element x to its equivalence class $[x]$. The *quotient topology* on $X/\sim$ is defined as follows

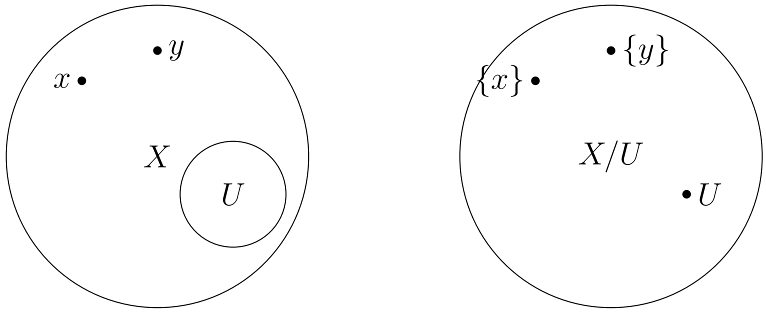
$$U \subset X/\sim \text{ open} \Leftrightarrow p^{-1}(U) \subset X \text{ open}.$$

It is an exercise to verify that this really defines a topology!

■ The quotient topology can be characterized by being the unique topology that turns $p : X \to X/\sim$ into a quotient map.

■ The quotient topology is the finest (largest) topology on $X/\sim$ such that p is continuous.

Example 3.12. Let X be a set and $U \subset X$ a subset. There is an equivalence relation on X defined as $x \sim y$ iff $x = y$ or $x, y \in U$.



Moving from X to X/U can be described as *collapsing* U *to a point.*

Example 3.13. Let $D^n := \overline{B(0,1)} = \{x \in \mathbb{R}^n | \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. The boundary is precisely S^{n-1}: $\partial D^n = \{x \in \mathbb{R}^n | \|x\| = 1\} = S^{n-1} \subset D^n$. What is the topology of $D^n/\partial D^n = D^n/S^{n-1}$? Intuitively we can think of collapsing the boundary to a point as "closing an open bag", where the result looks like a sphere, more precisely S^n.

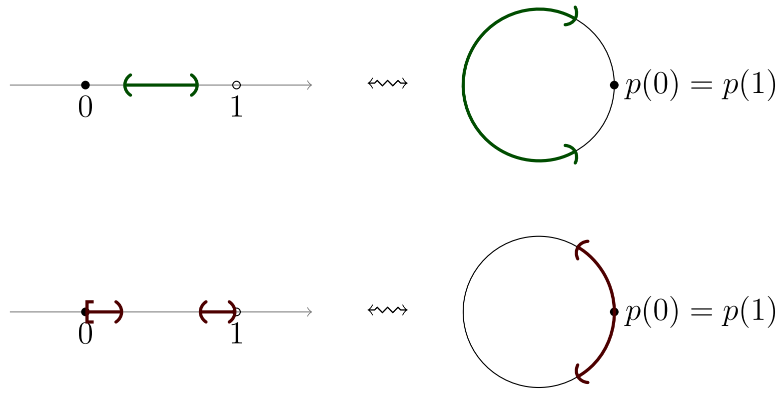
Proposition 3.14. *$D^n/\partial D^n \simeq S^n$.*

The proof is not complicated once we have introduced the notion of compactness, so we leave the proof until later.

Cutting and pasting topological spaces One nice consequence of the quotient topology is the possibility to construct topological spaces by gluing together other topological spaces in various ways.

Example 3.15. We start with a very simple example. For $x, y \in [0,1] \subset \mathbb{R}$, let $x \sim y$ if $x = y$ or $x = 0,\ \&\ y = 1$, and let $p : [0,1] \to [0,1]/\sim$ be the canonical surjection. We endow $[0,1]/\sim$ with the quotient topology. What is then an open set on this space? First, choosing representatives we can identify $[0,1]/\sim$ with $[0,1)$. For any $U \subset [0,1)$ such that $0 \notin U$, we have $p^{-1}(U) = U \subset [0,1]$. For $U \subset [0,1)$ such that $0 \in U$ we have $p^{-1}(U) = U \cup \{1\} \subset [0,1]$. It follows that $U \subset [0,1)$ is open in the quotient topology iff it is a union of intervals of the type $(a.b) \subset [0,1)$ and sets of the type $[0,a) \cup (b,1)$ where $0 < a < b < 1$.

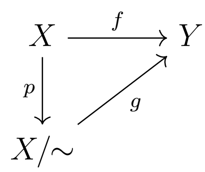
The identification we make when moving to $[0,1]/\sim$ are such that we glue the end of the interval to the beginning, which intuitively should result in a circle. Let us draw pictures of open sets after such an identification. An interval $(a,b) \subset [0,1)$ just looks like an open "circle interval", and the same is true for a set of form $[0,a) \cup (b,1)$.



We see that the open sets correspond exactly to the open sets of S^1. It is natural to conjecture that $S^1 \simeq [0,1]/\sim$. The bijection $f : [0,1]/\sim \to S^1$, $[x] \mapsto (\cos 2\pi x, \sin 2\pi x)$ is easily checked to be a homeomorphism since we understand the open sets in the quotient space. The details are left as an exercise.

A few technical results will aid us in identifying spaces homeomorphic to quotient spaces.

Lemma 3.16. *Let X be a topological space, and $\sim$ an equivalence relation on X. If $f : X \to Y$ is continuous and has the property that $f(x) = f(y)$ whenever $x \sim y$, then there exists a unique continuous map $g : X/\sim \to Y$ such that the following diagram commutes.*



Here $p : X \to X/\sim$ is the canonical surjection.

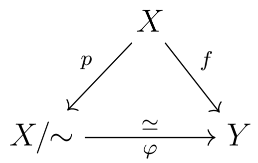
That a diagram commutes just means that any two ways that we can compose maps from one object to another results in the same map. The lemma can be reformulated as guaranteeing the existence of a continuous map g such that $f = g \circ p$.

Proof. The nontrivial claim of the theorem is actually that the map g is continuous. The existence of a map $g : X/\sim \to Y$ such that the diagram commutes is obvious; since $f(x) = f(y)$ if $x \sim y$ we can define the map g as $[x] \mapsto f(x)$, where x is an arbitrary representative of the equivalence class $[x]$. This map obviously has the property $f = g \circ p$. The uniqueness of a map g with this property is almost tautological, let us nevertheless spell it out in detail. Suppose that $g' : X/\sim \to Y$ is another map such that the diagram commutes, then we have $g \circ p = g' \circ p$. If $g' \neq g$, then there is some equivalence class $[x] \in X/\sim$ such that $g'([x]) \neq g([x])$. Since $f = g \circ p = g' \circ p$ we have $f(x) = g([x]) = g'([x])$, so $g' \neq g$ is a contradiction.

Assume that g is not continuous. Then there exists an open set $U \subset Y$ such that $g^{-1}(U) \subset X/\sim$ is not open. Since in the quotient topology $V \subset X/\sim$ is open iff

$p^{-1}(V)$ is open, it follows that $p^{-1}(g^{-1}(U)) \subset X$ is not open, but $f^{-1}(U)$ is open and $f^{-1}(U) = p^{-1}(g^{-1}(U))$, which is a contradiction. It follows that g is continuous. □

Proposition 3.17. *Let X, $\sim$ be as in Lemma 3.16, and let $f : X \to Y$ be a quotient map such that $\sim_f = \sim$. Then there exists a unique homeomorphism $\varphi : X/{\sim} \to Y$ such that the following diagram commutes.*



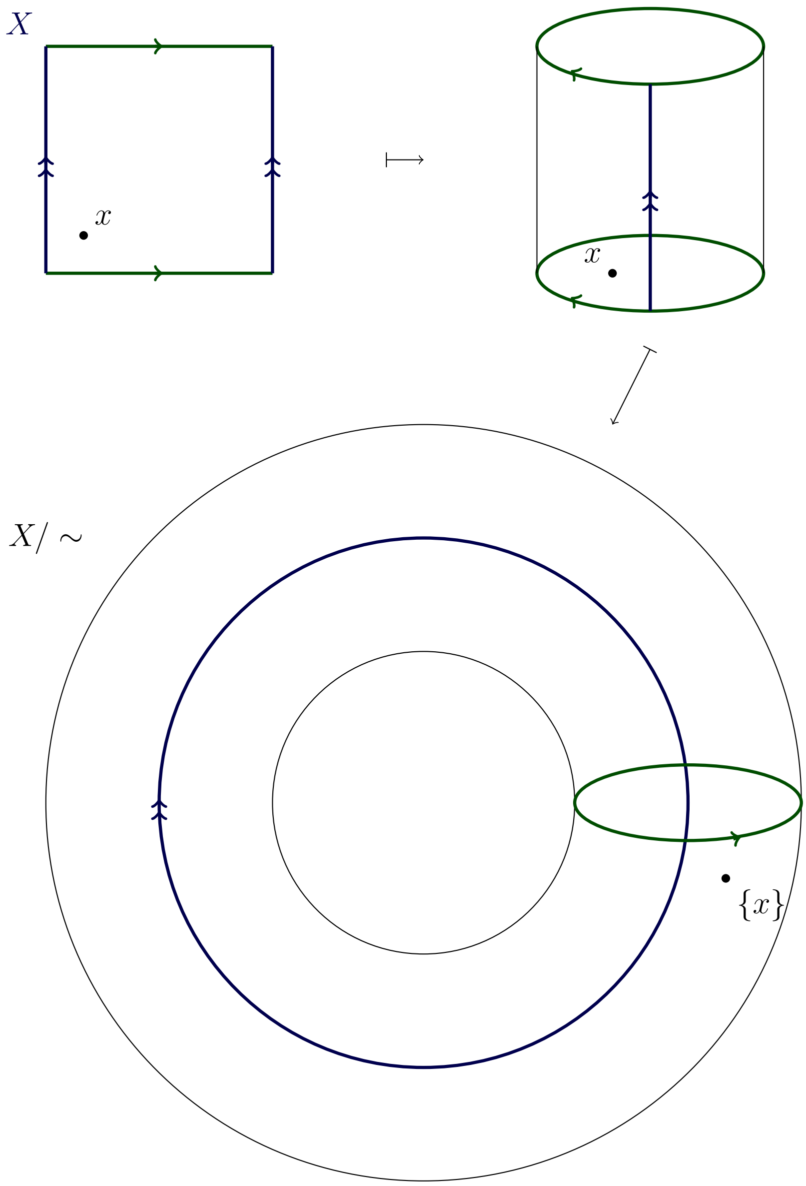
Proof. By Lemma 3.16 there exists a unique continuous φ such that the diagram commutes. We must show that φ is a homeomorphism. Since $\sim_f = \sim$ it follows that φ is a bijection between $X/{\sim}$ and $f(X)$, but since f is surjective the latter set is Y. It remains to show that $\varphi(U)$ is open whenever U is open. Choose an open set $U \subset X/{\sim}$, then $p^{-1}(U)$ is open in X. We have

$$\varphi(U) = (\varphi \circ p)(p^{-1}(U)) = f(U),$$

which is open since f is a quotient map. Thus φ is a homeomorphism. □

■ Proposition 3.17 can be used to show that $[0,1]/\sim$ from Example 3.15 really is homeomorphic to S^1. Consider the map $f : [0,1] \to S^1$, $x \mapsto (\cos 2\pi x, \sin 2\pi x)$. We see that $\sim_f = \sim$, where the latter is the equivalence relation in Example 3.15. It is very easy to verify that f is continuous. Let $U \subset S^1$ be any subset. If $f^{-1}(U)$ is open in $[0,1]$, then $f^{-1}(U)$ is a union of open intervals in $(0,1)$ and half open intervals ending on 0 or 1. Clearly U must then be a union of open "circle intervals", and $0 \inf^{-1}(U)$ iff $1 \in f^{-1}(U)$. In other words, if $f^{-1}(U)$ is open, then U is open and we have shown that f is a quotient map. By Proposition 3.17 it follows that $S^1 \simeq [0,1]/{\sim}$.

Example 3.18. The 2-torus T^2 (or just torus) can also be constructed using a quotient construction. Consider the closed square $X := [0,1] \times [0,1] \subset \mathbb{R}^2$. Introduce an equivalence relation on X as follows: $x \sim y$ if $x = y$, or $x = (p,0)$ & $y = (p,1)$, or $x = (0,p)$ & $y = (1,p)$. In the set $X/\sim$ the boundary edges are identified as to give a surface looking like a doughnut.



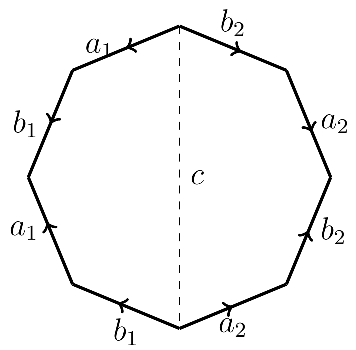
We endow $X/\sim$ with the quotient topology, and claim $T^2 \simeq X/\sim$. Consider the surjective map

$$f : X \to S^1 \times S^1,\ (x, y) \mapsto ((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y)).$$

The map f is continuous. Let $U \times V \subset S^1 \times S^1$ be an element of the basis of the product topology. It follows that $f^{-1}(U \times V)$ is a product $I \times J$ where I and J can be either open intervals $(a, b) \subset [0, 1]$, or sets of the form $[0, a) \cup (b, 1]$ where $0 < a < b < 1$. In either of the four different situations, the preimage is open. It is not difficult to generalise this result to show that the preimage of an arbitrary open set is open (check this!), and f is thus continuous. Next, consider $W \subset T^2$ and assume $f^{-1}(W)$ is open, i.e. a union of intersections of open discs with X. In order to be a preimage under f, however, certain conditions must hold. Most importantly, if it contains the point $(p, 0)$ then it must also contain the point $(p, 1)$, and analogously if it contains the point $(0, p)$ then it must contain the point $(1, p)$, and vice versa. It is then obvious (make sure that you understand why!) that W msut be open if its preimage under f is open, and therefore f is a quotient map. Finally, under the equivalence relation $\sim_f$, two points are equivalent $x \sim_f y$ if $x = y$ or $x = (p, 0)$ and $y = (p, 1)$, or $x = (0, p)$ and $y = (1, p)$ (or the analogous situation with x and y shifted). It follows that $\sim_f = \sim$, and Proposition 3.17 implies that $T^2 \simeq X/\sim$.

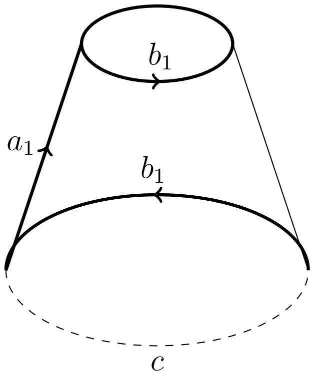
Example 3.19. The case of a torus can be generalised to a surface Σ_n with n holes (the torus corresponds to $n = 1$). Fix $n \in \mathbb{N}$ and consider a polygon $X_n \subset \mathbb{R}^2$ with $4n$ sides (a $4n$-gon). Choose an arbitrary corner as starting point, and label each edge moving counter-clockwise. The first edge is labeled a_1 and marked with an arrow in the counter-clockwise direction, the second edge is labeled b_1 and marked with an arrow in the same direction, the third edge is labeled a_1 again but marked with an arrow pointing clockwise, and the fourth edge is labeled b_1 and marked with a clockwise pointing arrow. Next, continue labeling the following four edges by a_2 and b_2 and arrows analogously. Continue until all sides of the polygon are labeled, the "last" side will be labeled b_n and marked with an arrow pointing in the clockwise direction. The labels of the edges defines an equivalence relation $\sim_n$ in the obvious way, i.e. the non-trivial relations are between points on edges with the same a or b labels, and the arrows indicate which points are equivalent. The rightmost point on the a_1 edge with counter clockwise arrow is equivalent to the leftmost point on the a_1 edge with clockwise arrow, and so on. The quotient space $\Sigma_n := X_n/\sim_n$ is a surface with n holes, or n "handles".

Let us look at the example $n = 2$. First draw an 8-gon X_2 and label the edges.

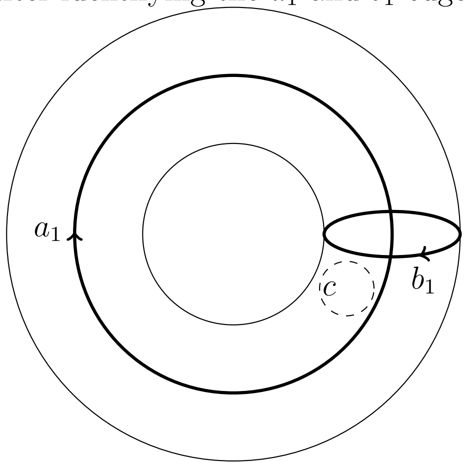


To see that $\Sigma_2 = X_2/\sim_2$ really is a surface with two handles, draw a dashed line separating the first four edges from the rest. In the picture this line is called c. Focus on the "left" part of the polygon and carry out the identifications to move to the quotient space, but forget about everything to the "right" in the polygon. The result looks like a torus but with one hole resulting from the line c. The remaining part of the surface is attached to that hole.

after identifying the a_1 edges

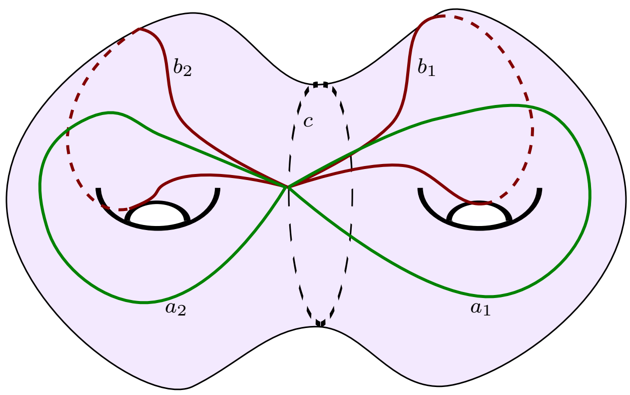


after identifying the a_1 and b_1 edges



the rest of X_2 is attached to the circle c

Of course we can do the same thing with the edges in the right side of c, and we get a similar torus with a hole whose boundary circle is labeled by c. It follows that $\Sigma_2 = X_2/\sim_2$ is obtained by joining the two tori along c, resulting in



4. Connectedness and path-connectedness

Perhaps the most basic topological property one can come up with has to do with separation; a given topological object may consist of one or two (or more) separated components (i.e. two apples next to each other "looks" topologically different from a single apple). It is possible to define several notions of connectedness of topological spaces, some closely tied to properties of familiar examples in $\mathbb{R}^n$, others more abstract. We will study two definitions, and also show precisely how they are related.

4.1. Connectedness

Definition 4.1. A *separation* of a topological space X is a decomposition $X = U \cup V$ such that U and V are non-empty, open, and satisfy $U \cap V = \emptyset$.

X is called *disconnected* if it has a separation
X is called *connected* if it does not have a separation

■ Note that being (dis)connected is a topological property, so $X \simeq Y$ implies that X is (dis)connected iff Y is.
■ If $X = U \cup V$ is a separation, then $X \backslash U = V$ and $X \backslash V = U$, so U and V are simultaneously open and closed.

Proposition 4.2. *X is connected iff $\emptyset$, X are the only subsets of X that are both open and closed.*

Proof. Assume $U \subset X$ is open and closed. Then $X \backslash U$ is also open and closed, and $U \cap X \backslash U = \emptyset$, so defining $V = X \backslash U$ we can write $X = U \cup V$ with both U and V disjoint and open. If both U and V are non-empty, this is a separation of X. □

Example 4.3.

(*i*) $\{(x, y)|y = 0\} \cup \{(x, y)|y = 1/x \ \& \ x > 0\} \subset \mathbb{R}^2$ is disconnected (by construction).

(*ii*) $\mathbb{Q} \subset \mathbb{R}$ is disconnected. A separation is given by

$$\mathbb{Q} = \mathbb{Q} \cap (-\infty, \pi) \bigcup \mathbb{Q} \cap (\pi, \infty) = \{q \in \mathbb{Q} | q < \pi\} \bigcup \{q \in \mathbb{Q} | q > \pi\}.$$

The two sets are disjoint, non-empty, and open by construction. The number π can be replaced with any other non-rational number.

(*iii*) $\mathbb{R}^n$ is connected. This fact is non-trivial, and we shall spend some time to prove it.

Let us show a number of useful technical results that are helpful when studying connectedness.

Lemma 4.4. *Let $X = U \cup V$ be a separation, and $Y \subset X$ be a subspace. If Y is connected, then $Y \subset U$ or $Y \subset V$.*

Proof. We have $Y = (Y \cap U) \cup (Y \cap V) =: U_0 \cup V_0$. If U_0 and V_0 are both non-empty, then this is a separation of Y since U_0 and V_0 are open (by definition of open in the subspace topology) and $U_0 \cap V_0 = (Y \cap U) \cap (Y \cap V) \subset U \cap V = \emptyset$. It follows then that either U_0 or V_0 must be empty, which implies that Y is fully contained in either U or V. □

Theorem 4.5.

(*i*) *Let $\{X_i\}_{i \in I}$ be a family of connected subspaces of X with a common point x, i.e. $\forall i \in I$: $x \in X_i$. Then $\bigcup_{i \in I} X_i$ is connected.*

(*ii*) *Let $Z \subset X$ be a connected subspace. If $Y \subset X$ is such that $Z \subset Y \subset \overline{Z}$, then also Y is connected (in the subspace topology).*

(*iii*) *If $f : X \to Y$ is continuous and X is connected, then $f(X) \subset Y$ is connected.*

(*iv*) *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. Then $\prod_{i\in I} X_i$ is connected iff X_i is connected $\forall i \in I$.*

Proof. (*i*): Assume that $\bigcup_{i\in I} X_i \subset X$ is disconnected with a separation $U \cup V$. Without loss of generality we can assume $x \in U$. Take $y \in V$, then there exists $i \in I$ such that $y \in X_i$, and $X_i = (U \cap X_i) \cup (V \cap X_i)$ is a separation of X_i (the two sets are open by construction, and non-empty since x and y are contained in the respective sets, and disjoint since U and V are disjoint).

(*ii*): Assume $Y = U \cup V$ is a separation of $Y \subset X$. By Lemma 4.4 Z is contained in U or V; assume without loss of generality that $Z \subset U$. This implies that $Y \subset \overline{Z} \subset \overline{U}$ (in this expression, $\overline{U}$ is the closure of U in X!). By the definition of the subspace topology there exists an open set $\widetilde{V}$ in X such that $V = Y \cap \widetilde{V}$, implying

$$U = Y \backslash V \subset X \backslash \widetilde{V},$$

where the last set is closed in X. It follows that $\overline{U} \subset X\backslash\widetilde{V} \subset X\backslash V$ and thus $Y \subset X\backslash V$, so $Y \cap V = \emptyset \Rightarrow V = \emptyset$. This is a contradiction, and it follows that Y is connected.

(*iii*): Assume $f(X) = U \cup V$ is a separation. Then

$$X = f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) =: U_0 \cup V_0.$$

Since f is continuous, both U_0 and V_0 are open. Furthermore, they are both non-empty since U and V are non-empty and contained in the image $\mathrm{Im}(f)$. Finally we see that

$$U_0 \cap V_0 = f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset,$$

so $U_0 \cup V_0$ is a separation of X, which is therefore disconnected. We have found a contradiction, and it follows that $f(X)$ must be connected.

(*iv*): The simple direction is $\Rightarrow$: By Definition 2.38, the canonical surjection $\pi_i : \prod_{j\in I} X_j \to X_i$ is continuous for each $i \in I$. Part (*iii*) of the theorem implies that X_i is connected for each $i \in I$ if $\prod_{i\in I} X_i$ is connected.

$\Leftarrow$: Let us first show that the theorem holds for finite products. It is easy to see (check this!) that $(X \times Y) \times Z \simeq X \times Y \times Z$, so it is enough to show for a product $X \times Y$. The idea is to use part (*i*). Choose an arbitrary point $(p, q) \in X \times Y$ and define for $x \in X$ the subspace $Z_x := (X \times \{q\}) \cup (\{x\} \times Y)$. Note that $(p, q) \in Z_x$ for every $x \in X$, and that $X \times Y = \bigcup_{x\in X} Z_x$. Furthermore, we have that $(x, q) \in X \times \{q\}$ and $(x, q) \in \{x\} \times Y$. Using the obvious identifications (homeomorphisms) $X \simeq X \times \{q\}$ and $Y \simeq \{x\} \times Y$, both of these spaces are connected by assumption, and by part (*i*) it follows that Z_x is connected for every $x \in X$. Again by part (*i*) it follows that $X = \bigcup_{x\in X} X_x$ is connected.

It remains to show the theorem for arbitrary, not only finite, families. The strategy is to first find (using part (*i*)) a connected subspace such that its closure is the full product, and then to use part (*ii*) to conclude that the product is connected. To this end, assume

I is infinite, let $x \in \prod_{i\in I} X_i$ be an arbitrary point, and denote its components by x_i, $i \in I$. If $J \subset I$ is a finite subset, i.e. $|J| < \infty$, then define the subspace X_J with factor $\{x_i\}$ for $i \notin J$, and factor X_j for $j \in J$. In other words, X_J is (canonically equivalent to) a finite product. If X_i is connected for each $i \in I$, then it follows from the first part of this proof that X_J is connected. Now, note that for every finite subset $J \subset I$, we have $x \in X_J$. Part (i) then implies that

$$Y := \bigcup_{\substack{J \in \mathscr{P}(I) \\ |J|<\infty}} X_J$$

is connected. Theorem 2.45 (ii) states $\overline{Y} = Y \cup \partial Y$. Pick any $y \in \prod_{i\in I} X_i$, and let U be an element in the basis of the product topology containing y. In particular, $U = \prod_{i\in I} U_i$ where $U_i = X_i$ unless $i \in K$ for some finite subset $K \subset I$ in which case U_i is an element of the basis of X_i containing y_i. Pick any finite set $J \subset I$ disjoint from K, i.e. $I \cap K = \emptyset$. Then $X_J \cap U \neq \emptyset$, so $Y \cap U \neq \emptyset$. Now assume that $y \in Y$, i.e. $y_i = x_i$ for all but finitely many $i \in I$. For every $i \in I$ there exists a $z_i \in U_i$ such that $z_i \neq y_i$; this defines a point $z \in U$ (with components z_i, $i \in I$) that is not contained in Y. In other words $((\prod_{i\in I} X_i)\backslash Y) \cap U \neq \emptyset$. Obviously it is then true that $Y \cap V \neq \emptyset$ and $((\prod_{i\in I} X_i)\backslash Y) \cap V \neq \emptyset$ for *any* neighbourhood V of y, not only basis elements. Since y was chosen arbitrarily, the statement holds for any point in $\prod_{i\in I} X_i$. In other words, $\partial Y = \prod_{i\in I} X_i$, and $\overline{Y} = \prod_{i\in I} X_i$. We have a sequence $Y \subset \prod_{i\in I} X_i \subset \overline{Y}$ where Y is connected, so by part (ii) it follows that $\prod_{i\in I} X_i$ is connected. □

■ From Theorem 4.5 (iv) we see that connectedness of $\mathbb{R}^n$ for $n > 1$ follows from connectedness of $\mathbb{R}$.
■ A *convex* subset of $\mathbb{R}$ is defined to be a subset $X \subset \mathbb{R}$ such that if $x, y \in X$ then $[x, y] \subset X$.

Theorem 4.6. $\mathbb{R}$ *(with the standard topology) is connected. Furthermore, a subset of* $\mathbb{R}$ *is connected iff it is convex.*

In order to prove this theorem, recall the following properties of $\mathbb{R}$.
■ $\mathbb{R}$ has the supremum property: any subset of $\mathbb{R}$ bounded from above has a least upper bound (a supremum).
■ if $x, y \in \mathbb{R}$ and $x < y$, then $\exists z \in \mathbb{R}$ such that $x < z < y$.
I refer to [M, §24] for more details.

Proof. Let $X \subset \mathbb{R}$ be convex (this is enough since $\mathbb{R}$ is convex), and assume $X = U \cup V$ is a separation. Choose $x \in U$ and $y \in V$. By convexity we then have $[x, y] \subset X$. Write

$$[x, y] = (U \cap [x, y]) \cup (V \cap [x, y]) = U_0 \cup V_0.$$

The subsets U_0 and V_0 of $[x, y]$ are non-empty, disjoint, and open (in the subspace topology). Thus they define a separation of $[x, y]$. Define $z := \sup(U_0)$.

Assume first that $z \in U_0$. It follows that $z \neq y$, so either $z = x$ or $x < z < y$. In either case there exists a number u, $z < u \leq y$, such that $[z, u) \subset U_0 \subset [x, y]$ (U_0 open implies that U_0 contains a neighbourhood of z, which in turn always contains such a half open interval). Now, let $v \in [z, u)$ such that $z < v$ (such a number always exists), and we have found a number $v \in U_0$ such that $v > c = \sup(U_0)$. This is a contradiction, so $z \notin U_0$.

Assume next that $z \in V_0$. Then $z = y$ or $x < z < y$. In either case, since V_0 is open we can find an interval $(u, z] \subset V_0$. If $z = y$ we have a contradiction since we can then find a smaller upper bound for U_0 than z. Thus $x < z < y$. Now, $(z, y]$ does not intersect U_0 since z is an upper bound for U_0, and it follows that

$$(u, y] = (u, z] \cup (z, y] \subset V_0,$$

which is again a contradiction since any number in $(u, z) \neq \emptyset$ is a smaller upper bound for U_0 than z. We have shown that $z \notin U_0 \cup V_0$, so our assumption that X is disconnected must be false. Any interval, ray, and $\mathbb{R}$ itself is convex, so we have indeed shown that any such subset is connected.

Assume $X \subset \mathbb{R}$ is not convex. Then there exist $x, y \in X$ such that $x < y$ and a z, $x < z < y$, such that $z \notin X$. I claim that $(X \cap (-\infty, z)) \cup (X \cap (z, \infty))$ is a separation of X. The two sets are non-empty since they contain x respectively y, open by the definition of subspace topology, and clearly disjoint. The proof is complete. □

Corollary 4.7. *$\mathbb{R}^n$ is connected for any $n \in \mathbb{N}$.*

The connectedness of $\mathbb{R}$ (and $\mathbb{R}^n$) implies many other results, we mention two. First, a generalised version of the intermediate value theorem.

Theorem 4.8 (The intermediate value theorem)**.** *Let $f : X \to Y$ be continuous and X connected. If $f(x) < r < f(y)$ for $x, y \in X$ and $r \in \mathbb{R}$, then there exists $z \in X$ such that $f(z) = r$.*

Proof. By Theorem 4.5 (*iii*) $f(X)$ is connected, and by Theorem 4.6 $f(X)$ is convex. Thus $[f(x), f(y)] \subset f(X)$ for every $x, y \in X$ such that $f(x) < f(y)$. It follows that any $f \in [f(x), f(y)]$ belongs to $\mathrm{Im}(f)$, so the theorem has been proved. □

The second consequence that we will mention is the connectedness of the n-sphere S^n.

Proposition 4.9. *S^n is connected for every $n \in \mathbb{N}$. Thus in particular, so is $T^n = S^1 \times S^1 \times \cdots \times S^1$.*

Proof. From Proposition 3.5 we know that $S^n \backslash \{p\} \simeq \mathbb{R}^n$, which is connected by Corollary 4.7, for any $p \in S^n$. Note that $\overline{S^n \backslash \{p\}} = S^n$ (this really is trivial, for instance since p is a boundary point). We can then write the sequence

$$S^n \backslash \{p\} \subset S^n \subset \overline{S^n \backslash \{p\}},$$

where $S^n \backslash \{p\}$ is connected. Theorem 4.5 (*ii*) now implies that S^n is connected. The connectedness of T^n follows from Theorem 4.5 (*iv*) and the connectedness of S^1. □

Let us also mention the following

Proposition 4.10. $\mathbb{R}^n \simeq \mathbb{R}^m$ *iff* $n = m$.

Proof. The statement when $m = 1$ is an exercise. The remaining cases are normally proved using homotopy groups in algebraic topology. □

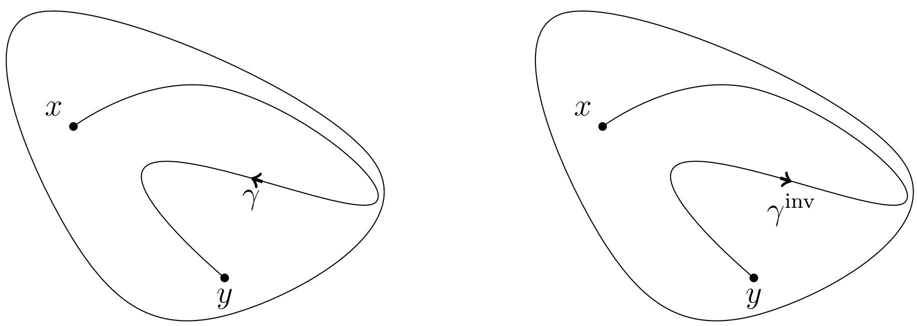
Connectedness properties can sometimes be used to distinguish topological spaces, as in the following proposition.

Proposition 4.11. *For any* $n \in \mathbb{N}$ *we have* $S^n \not\simeq \mathbb{R}^n$

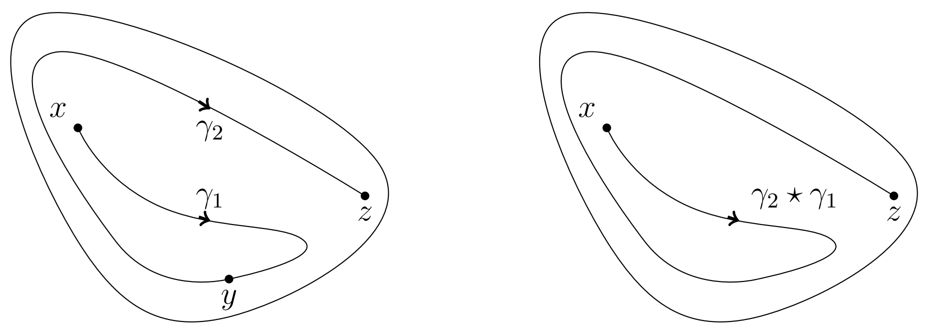
Proof. We again recall that for any $p \in S^n$ we have $S^n \backslash \{p\} \simeq \mathbb{R}^n$, which is connected, while for any $p \in \mathbb{R}$ we have $\mathbb{R}\backslash\{p\} = (-\infty, p) \cup (p, \infty)$, which is disconnected. If S^n and $\mathbb{R}$ were homeomorphic, then the spaces obtained by removing a point must also be homeomorphic, but they are not since they differ in connectivity. (In general one must take take more care since the result may depend on which point is removed, but in this example the result is, up to homeomorphism, independent of which point we remove.) □

4.2. Paths and path-connectedness

Another way to measure how connected a space is, is to determine which points can be connected by a path. To make this precise, we will have to give a precise definition of a path in a topological space, but let us first draw some pictures to see what kind of properties that we would expect from paths. A path γ has a start point and an end point (which may coincide). In particular, this means that a path has a direction; if necessary we can indicate the direction by an arrow. As a consequence, to every path γ we can associate the path going along the same steps as γ but in the other direction (i.e. backwards); call that the *inverse*, γ^{inv}, of γ.



Also, consider the situation where γ_1 is a path from x to y and γ_2 is a path from y to z. By *concatenating* the two paths, i.e. putting one path after the other, we get a path from x to z; denote the concatenated path by $\gamma_2 \star \gamma_1$.



Now, let us spell out the precise definitions of these notions.

Definition 4.12. Let X be a topological space and $x, y, z \in X$.

(i) A *path* from x to y is a continuous map

$$\gamma : [0,1] \to X : \ \gamma(0) = x, \ \gamma(1) = y.$$

(ii) If γ is a path from x to y, then the *inverse* of γ is the path from y to x defined through

$$\gamma^{\mathrm{inv}}(t) := \gamma(1-t), \ t \in [0,1].$$

Since γ^{inv} is the composition of two continuous maps, it is itself continuous.

(iii) If γ_1 is a path from x to y, and γ_2 is a path from y to z, then the *concatenation* $\gamma_2 \star \gamma_1$ is the path from x to z defined as

$$\gamma_2 \star \gamma_1(t) := \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

It is not obvious that the so obtained function is actually continuous, this needs proof.

Proof that the concatenation is continuous. We use the closed set characterisation of continuity. Let $F \subset X$ be closed, we then want to show that $(\gamma_2 \star \gamma_1)^{-1}(F) \subset [0,1]$ is closed. First, note that a closed subset in $[0,1]$ is just a closed subset of $\mathbb{R}$ contained in the closed unit interval. Next, we know that the restriction of $\gamma_2 \star \gamma_1$ to $[0,1/2]$ is continuous (being the composition of γ_1 with the continuous function $t \mapsto 2t$), the same holds for the restriction to $[1/2,1]$. Let $\tilde{\gamma}_1(t) := \gamma_1(2t)$ and $\tilde{\gamma}_2(t) = \gamma_2(2t-1)$, and define

$$V_1 := \tilde{\gamma}_1^{-1}(F) = (\gamma_2 \star \gamma_1)^{-1}(F) \cap [0,1/2] \subset [0,1/2]$$
$$V_2 := \tilde{\gamma}_2^{-1}(F) = (\gamma_2 \star \gamma_1)^{-1}(F) \cap [1/2,1] \subset [1/2,1].$$

Since $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are continuous, the sets V_1 and V_2 are closed, both as subsets of $[0,1/2]$ respectively $[1/2,1]$ and as subsets of $[0,1]$ (or of $\mathbb{R}$) . Furthermore, $(\gamma_2 \star \gamma_1)^{-1}(F) = V_1 \cup V_2 \subset [0,1]$, which is thus closed. It follows that the concatenation $\gamma_2 \star \gamma_1$ is continuous. $\square$

Definition 4.13. If every two points in a topological space X can be joined by a path, we say that X is *path-connected* (or *pathwise conencted*).

We will spend some time to investigate the precise relationship between the notions of connectedness and of path-connectedness. We start with

Theorem 4.14. *A path-connected space is connected.*

Proof. The proof is in fact simple. Let X be a path-connected space, and fix a point $x \in X$. For every point $y \in X$ choose a path γ_y from x to y. By Theorem 4.5 (iii) it follows that $\gamma_y([0,1])$ is connected; note that $x \in \gamma_y([0,1])$. Obviously $X = \bigcup_{y \in X} \gamma_y([0,1])$, and by Theorem 4.5 (i) it now follows that X is connected. □

Example 4.15.

(i) S^n is path-connected. It is an exercise to prove this.

(ii) $\mathbb{R}^n$ is path-connected. Let $x, y \in \mathbb{R}^n$, and define

$$\gamma : [0,1] \to \mathbb{R}^n,\ t \mapsto (1-t)x + ty.$$

Clearly γ is a path from x to y.

■ Note that in the last example, the path is indeed the straight line from x to y. A subset of $\mathbb{R}^n$ containing the straight line between any two of its points is called *convex*. It follows that a convex subset of $\mathbb{R}^n$ is path-connected and therefore connected.

Example 4.16.

(i) $\mathbb{H}^n := \{x \in \mathbb{R}^n | x_n > 0\} \subset \mathbb{R}^n$ is convex.

(ii) The ball $B(0,r) \subset \mathbb{R}^n$ is convex since for every $x, y \in B(0,r)$ we have

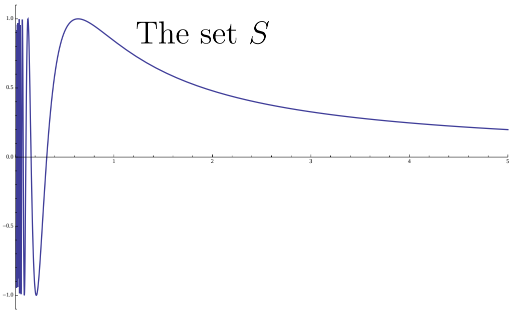
$$\|(1-t)x + ty\| \leq |1-t|\|x\| + |t|\|y\| < (1-t)r + tr = r.$$

It is worth noting that the same property holds for any metric norm in $\mathbb{R}^n$ since all we used was the triangle inequality. Thus, an open ball in $\mathbb{R}^n$ is convex in any metric.

■ Not every connected space is path-connected, however. A counterexample is the so-called "topologists sine curve".

Example 4.17 (The topologists sine curve)**.** Define the set

$$S := \{(x,y) \in \mathbb{R}^2 | y = \sin \frac{1}{x},\ x > 0\} \subset \mathbb{R}^2.$$



Defining the function $f : t \mapsto (t, \sin 1/t)$ we see that $S = f((0, \infty))$, and by Theorem 4.5 (i) it follows that S is connected. By Theorem 4.5 (ii) it follows that the closure $\overline{S}$ is also connected. It is easily checked that $\overline{S} = S \cup (\{0\} \times [-1, 1])$.

Assume that $\overline{S}$ is path-connected, and let γ be a path from $(0, 0)$ to a point $s \in S$; write $\gamma(t) = (x(t), y(t))$. Since γ is continuous, the canonical projections $\pi_{1/2}$ from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ are continuous, and the composition of two continuous functions is continuous, it follows that $x(t)$ and $y(t)$ are separately continuous. Since $\{0\} \times [-1, 1] = ([-1, 0] \times [-1, 1]) \cap \overline{S}$, and $[-1, 0] \times [-1, 1] \subset \mathbb{R}^2$ is closed, it follows that $\{0\} \times [-1, 1] \subset \overline{S}$ is closed. The set $C := \gamma^{-1}(\{0\} \times [-1, 1]) \subset [0, 1]$ is then also closed since γ is continuous, and it holds that $0 \in C$, $1 \notin C$. A closed subset of $\mathbb{R}$ has a maximal element; define $m := \max(C)$, i.e. $0 \leq m < 1$. Note that $x(m) = 0$, and that m is the largest element in $[0, 1]$ with that property.

We next construct a sequence $\{t_n\}_{n \in \mathbb{N}}$ in $[0, 1]$ such that $t_n \to m$. For every $n \in \mathbb{N}$ consider the interval $[x(m), x(m + 1/n)] = [0, x(m + 1/n)]$ (note that $x(m + 1/n) > 0$). Since x is continuous and $[m, m + 1/n] \subset \mathbb{R}$ is connected, it follows from Theorem 4.8 that for every $x_n \in (0, x(m + 1/n))$ there exists a $t_n \in (m, m + 1/n)$ such that $0 < x(t_n) < x(m + 1/n)$ and $x_n = x(t_n)$. Choose any such an x_n and t_n for every $n \in \mathbb{N}$. For any neighbourhood U of $m \in [0, 1]$ there exists some $N > 0$ such that $t_n \in (m - 1/n, m + 1/n) \cap [0, 1] \subset U$ for every $n > N$, and therefore $t_n \to m$.

Next, restrict the choice of x_n. For $n \in \mathbb{N}$ even (odd), define u_n to be the smallest even (odd) integer strictly larger than $\frac{1}{\pi x(m+1/n)}$. Choose $x_n = \frac{1}{\pi u_n}$ and t_n such that $x_n = (t_n)$. It follows that $y_n := y(t_n) = \sin 1/x_n = \sin \pi u_n = (-1)^n$. We have thus constructed a sequence $t_n \to m$ in $[0, 1]$ such that $\{y_n = y(t_n)\}_{n \in \mathbb{N}}$ is not convergent. It follows that $y(t)$ is not continuous, which is a contradiction. Thus $\overline{S}$ cannot be path-connected.

■ Having seen that not every connected space is path-connected, we would like to determine under what conditions we may draw such a conclusion.

Theorem 4.18. *A topological space X is path-connected iff it is connected and every point $x \in X$ has a path-connected neighbourhood*

Proof. $\Rightarrow$: This direction is trivial due to Theorem 4.14.

$\Leftarrow$: For a point $x \in X$, define the set $U_x := \{y \in X | \exists \text{ a path from } x \text{ to } y\}$. The condition that every point has a path-connected neighbourhood implies that $U_x \neq \emptyset$.

Let $y \in U_x$, and let V_y be a path-connected neighbourhood of y. Then $V_y \subset U_x$, since if $z \in V_y$, γ is a path from x to y, and γ' is a path from y to z, then $\gamma' \star \gamma$ is a path from x to z, and it follows that $z \in U_x$. We have shown that $U_x = \bigcup_{y \in U_x} V_y$, so U_x is open. Assume $X \backslash U_x \neq \emptyset$, then $X \backslash U_x$ is open since if $y \in X \backslash U_x$ then the (actually, any) path-connected neighbourhood $V_y \subset X \backslash U_x$ (otherwise there would exist a $z \in V_y \cap U_x$, implying that there exists a path from x to y contrary to the assumption), so $X \backslash U_x = \bigcup_{y \in X \backslash U_x} V_y$. We have shown that $X = U_x \cup X \backslash U_x$ is a separation, so X is disconnected. That is a contradiction, so it must follow that our assumption, namely that $X \backslash U_x$ is non-empty, was wrong. In other words, $X = U_x$ and X is thus path-connected. □

4.3. Connected components

■ When a space is disconnected, it would seem natural to talk separately of each "part" of the space.

Definition 4.19. Let X be a topological space, and introduce an equivalence relation on X as follows.

$$x \sim y \Leftrightarrow \exists \text{ a connected } U \subset X \text{ such that } x, y \in U.$$

The equivalence classes of $\sim$ are called the *connected components* of X; sometimes just the *components* of X.

■ The connected components are not necessarily the parts of a (perhaps iterated) separation of X. As we shall see, connected components may not even be open.

It is not completely trivial that $\sim$ is an equivalence relation.

Proof that $\sim$ is an equivalence relation. It is clear that every $x \in X$ lies in some connected subspace, for instance the subspace $\{x\} \subset X$, thus $x \sim x$ for every $x \in X$. If $x \sim y$ then of course $y \sim x$ since there is no order between the elements in the definition of $\sim$. Finally, assume that $x \sim y$ and $y \sim z$. In other words, there exists a connected subspace U containing x and y, and a connected subspace V containing y and z. Since U and V contain the common point y it follows from Theorem 4.5 (i) that $U \cup V$ is connected, and thus $x \sim z$ since $x, y \in U \cup V$. □

Proposition 4.20. *Let $\{C_i\}_{i \in I}$ be the set of connected components of a topological space X. Then*

(i) $X = \bigcup_{i \in I} C_i$ *and* $C_i \cap C_j$ *if* $i \neq j$.

(ii) *If* $Y \subset X$ *is connected, then* $Y \subset C_i$ *for some* $i \in I$.

(iii) *For each* $i \in I$, $C_i \subset X$ *is connected.*

Proof. (i) Trivial, this is true for the equivalence classes of any equivalence relation.

(ii) By the definition of $\sim$ all elements in Y are equivalent, so they belong to C_i for some $i \in I$.

(*iii*) Fix $x \in C_i$. For any $y \in C_i$ we have that $x, y \in U_y$ for some connected subspace U_y. Thus $C_i = \bigcup_{y \in C_i} U_y$, which is connected by Theorem 4.5 (*i*).

□

Example 4.21. Connected components of the subspace $\mathbb{Q} \subset \mathbb{R}$. Take any non-empty $X \subset \mathbb{Q}$. If $x, y \in X$ then there exists a non-rational number r, $x < r < y$. It follows that $X = (X \cap (-\infty, r)) \cup (X \cap (a, \infty))$ is a separation of X, so the only connected subspaces of $\mathbb{Q}$ are those consisting of one element, $\{x\}$. The connected components are thus $\{\{q\} | q \in \mathbb{Q}\}$. The subspace topology on $\mathbb{Q}$ is not discrete, so the singleton subsets are not open. This example therefore shows that connected components do not have to be open.

Remark 4.22. Since connectedness is a topological property, the properties of connected components are also topological. If two spaces are homeomorphic then in particular there must exist a bijection between the connected components of the two spaces. In particular, if the cardinalities of the sets of connected components of two spaces differ, then they cannot be homeomorphic.

5. Compactness and sequential compactness

■ Recall from calculus that there are some strong results that hold for functions defined on compact sets, which were defined as subsets of $\mathbb{R}^n$ that are both closed and bounded. Similar strong results hold for more general topological spaces that are compact, but it is necessary to give a much more abstract definition of compactness. We will also give a definition of so-called sequential compactness, formulated in terms of sequences, and show results for when the two concepts coincide (as in, for instance, $\mathbb{R}^n$).

5.1. Compactness

Definition 5.1. Let $X \equiv (X, \mathscr{T})$ be a topological space.

(*i*) A collection $\mathscr{U} \subset \mathscr{T}$ of open sets is called an *open cover* of X if $X = \bigcup_{U \in \mathscr{U}} U$.

(*ii*) X is called *compact* if every open cover $\mathscr{U}$ of X has a finite subcover, i.e. there exists $U_1, U_2, ..., U_n \in \mathscr{U}$ such that $X = U_1 \cup U_2 \cup \cdots \cup U_n$.

Example 5.2.

(*i*) Every finite space is compact, since there are only finitely many open sets to start with.

(*ii*) $\mathbb{R}$ is not compact. Take for instance the open cover $\mathscr{U} = \{(n - 1/2 - \epsilon, n + 1/2 + \epsilon) | n \in \mathbb{Z}\}$ for some small $\epsilon > 0$. It is clear that there cannot be any finite subcovers of $\mathscr{U}$.

(*iii*) $K = \{1/n | n \in \mathbb{N}\} \subset \mathbb{R}$ is not compact. Note that the subspace topology is discrete and the space is infinite, so the open cover $\{\{1/n\} | n \in \mathbb{N}\}$ does not have finite subcover.

(*iv*) $X := \{0\} \cup K \subset \mathbb{R}$ is compact. Note that any open set containing 0 must also contain an infinite number of points in K, more precisely it must contain all points in K smaller than $1/N$ for some $N \in \mathbb{N}$. To find a finite subcover it is therefore enough to take any subset containing 0, and then adding a finite number of subsets containing the remaining finite number of points.

(*v*) $(0, 1] \subset \mathbb{R}$ is not compact. The open cover $\{(1/n, 1] | n \in \mathbb{N}\}$ does not have a finite subcover.

(*vi*) $[0, 1] \subset \mathbb{R}$ is compact. This non-trivial fact will be proven later; for now let us assume that we know this.

As always we need a number of technical results to work with compactness.

Theorem 5.3. *A closed subspace of a compact space is compact.*

Proof. Let $Y \subset X$ be a closed subspace of a compact space X. Let $\mathscr{U}$ be an open cover of Y. For each $U \in \mathscr{U}$ we choose an open set $\widetilde{U} \subset X$ such that $U = Y \cap \widetilde{U}$. It follows that $Y \subset \bigcup_{U \in \mathscr{U}} \widetilde{U}$, and since $X \backslash Y$ is open $\{\widetilde{U} | U \in \mathscr{U}\} \cup X \backslash Y$ is an open cover of X. Since X is compact there exists a finite subcover, say $\{\widetilde{U}_1, \widetilde{U}_2, ..., \widetilde{U}_n\} \cup X \backslash Y$, and it still holds that $Y \subset \widetilde{U}_1 \cup \widetilde{U}_2 \cup \cdots \cup \widetilde{U}_n$. In other words, $\{U_1, U_2, ..., U_n\} \subset \mathscr{U}$ is a finite subcover, and it follows that Y is compact. □

We know that in $\mathbb{R}^n$ every compact set is also closed, the general situation is described by

Theorem 5.4. *A compact subspace of a Hausdorff space is closed.*

Proof. Let X be Hausdorff and $Y \subset X$ compact. We want to show that $X \backslash Y$ is open. Choose $x \in X \backslash Y$, then since X is Hausdorff, for each $y \in Y$ there exist neighbourhoods U_y of x and V_y of y such that $U_y \cap V_y = \emptyset$. The collection $\{Y \cap V_y\}_{y \in Y}$ is an open cover of Y, so there is a finite subcover $\{Y \cap V_{y_1}, Y \cap V_{y_2}, ..., Y \cap V_{y_n}\}$. Define $U_x := U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$, then U_x is a neighbourhood of x. We see that

$$\begin{aligned} Y \cap U_x &= Y \cap (U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}) \subset (V_{y_1} \cup V_{y_2} \cup \cdots V_{y_n}) \cap (U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}) \\ &= V_{y_1} \cap (U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}) \cup \cdots \cup V_{y_n} \cap (U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}) \\ &= \emptyset. \end{aligned}$$

In other words, for every $x \in X \backslash Y$ we can find a neighbourhood $x \in U_x \subset X \backslash Y$, so $X \backslash Y$ is open, and thus Y is closed. □

Example 5.5. $\mathbb{R}$ is Hausdorff, both $\{0\} \cup K$ and $[0, 1]$ are closed in $\mathbb{R}$.

Theorem 5.6. *Let X be a compact space and $f : X \to Y$ a continuous function. Then $f(X) \subset Y$ is compact.*

Proof. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover of $f(X)$. Since f is continuous, $\{f^{-1}(U_i)\}_{i \in I}$ is an open cover of X, and there exists a finite subcover $\{f^{-1}(U_{i_1}), ..., f^{-1}(U_{i_n})\}$ since X is compact. It follows immediately that $f(X) = U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$ so we have found a finite subcover of $\mathscr{U}$. Therefore $f(X)$ is compact. □

We are now in a position to prove the following very useful and powerful result.

Theorem 5.7. *Let X be a compact space and Y a Hausdorff space. If $f : X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. Since f is by assumption continuous and bijective, it remains to show that f^{-1} is continuous, i.e. that $f(U)$ is open if U is open, or equivalently that $f(F)$ is closed whenever F is closed. Let $F \subset X$ be a closed subspace. By Theorem 5.3 F is compact, and by Theorem 5.6 $f(F)$ is compact. Finally, by Theorem 5.4 it follows that $f(F)$ is closed, finishing the proof. □

Example 5.8.

(*i*) Assume again that we know $[0, 1]$ to be compact. Then S^1 is compact by Theorem 5.6, since $f : [0, 1] \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is continuous and surjective

(*ii*) Let X be a Hausdorff space, and let $f : S^1 \to X$ be continuous and injective. Then f is a homeomorphism onto its image $f(S^1)$. The image of S^1 in a Hausdorff space under a continuous injection is called a *simple closed curve.*

(*iii*) Compactness of X in Theorem 5.7 is crucial. Note that if we restrict f in (*i*) to $(0, 1]$ then it is a continuous bijection, and S^1 is of course still Hausdorff. The continuous bijection f, however, is no homeomorphism. Consider the open set $(1/2, 1] \subset (0, 1]$. The image $f((1/2, 1]) \subset S^1$ is not open, and f^{-1} is therefore not continuous.

■ In order to see how compactness behaves under products, we will first prove a technical result that is very useful in a wide variety of applications.

Lemma 5.9 (The tube lemma)**.** *Let X and Y be topological spaces where Y is compact, and let $N \subset X \times Y$ be open and containing $\{x_0\} \times Y$ for some $x_0 \in X$. Then there exists an open set $W \times Y$ ("the tube"), where $W \subset X$ is a neighbourhood of x_0, such that $\{x_0\} \times Y \subset W \times Y \subset N$.*

Proof. The idea is to construct the tube using a finite cover of Y. Since $N \subset X \times Y$ is open every $(x_0, y) \in N$ has a neighbourhood contained in N, so in particular we can choose a basis element $(x_0, y) \in U_y \times V_y \subset N$. Since Y is compact and $\{V_y\}_{y \in Y}$ form an open cover of Y, we can find a finite subcover $\{V_{y_1}, V_{y_2}, ..., v_{y_n}\}$. Take $W := U_{y_1} \cap U_{y_2} \cap \cdots \cap U_{y_n}$, this is a neighbourhood of x_0. We see

$$\{x_0\} \times Y \subset W \times Y = W \times V_{y_1} \cup \cdots \cup W \times V_{y_n} \subset N,$$

completing the proof. □

■ Note that the tube lemma is not valid without the compactness condition on Y. Take for instance $X = Y = \mathbb{R}$ and let $N \subset \mathbb{R} \times \mathbb{R}$ be the open set below $1/x$ for $x > 0$, below $-1/x$ for $x < 0$, and together with the y-axis, $x = 0$. It is obviously impossible to find a tube around the y-axis (a *tubular neighbourhood* of the y-axis) contained in N.

Theorem 5.10 (Tychonov's theorem)**.** *Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The product $\prod_{i\in I} X_i$ is compact iff X_i is compact for every $i \in I$.*

Proof. $\Rightarrow$: Recall that each canonical projection $\pi_k : \prod_{i\in I} X_i \to X_k$ is continuous. Thus, since $\mathrm{Im}(\pi_k) = X_k$, it follows from Theorem 5.6 that X_k is compact for each $k \in I$.

$\Leftarrow$: I will here provide the proof for finite products. The full proof can be found in [M, §37]. Note that since $X \times Y \times Z \simeq (X \times Y) \times Z \simeq X \times (Y \times Z)$ it is enough to prove that $X \times Y$ is compact if X and Y are compact. The general statement for finite products then follows by induction.

Assume X, Y are both compact, and let $\mathscr{U}$ be an open cover of $X \times Y$. For any $x \in X$, $Y \simeq \{x\} \times Y \subset X \times Y$, so $\{x\} \times Y$ is compact with an open cover $\{(\{x\} \times Y) \cap U | U \in \mathscr{U}\}$. We can find a finite subcover $\{(\{x\} \times Y) \cap U_1^x, (\{x\} \times Y) \cap U_2^x, ..., (\{x\} \times Y) \cap U_n^x\}$, and it follows that $\{x\} \times Y \subset N_x := U_1^x \cup U_2^x \cup \cdots \cup U_n^x$. By the tube lemme (Lemma 5.9) there exists a neighbourhood W_x of x such that $\{x\} \times Y \subset W_x \times Y \subset N_x$ (where $W_x \times Y$ is open). Since $X = \bigcup_{x\in X} W_x$, and X is compact, there exist a finite subcover $\{W_{x_1}, W_{x_2}, ..., W_{x_m}\}$ of X. We can finally conclude that

$$X \times Y = (W_{x_1} \cup W_{x_2} \cup \cdots \cup W_{x_m}) \times Y = \bigcup_{j=1}^{m} W_{x_j} \times Y \subset \bigcup_{j=1}^{m} N_{x_j} = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n_j} U_i^{x_j}.$$

We have thus found a finite subcover of $\mathscr{U}$, and it follows that $X \times Y$ is compact. □

■ As with most basic topological properties, it is possible to characterize compactness using closed sets instead of open. Let's say that $\mathscr{C} \subset \mathscr{P}(X)$ has the *finite intersection property* if all finite subsets $\{C_1, C_2, ..., C_n\} \subset \mathscr{C}$ satisfy $C_1 \cap C_2 \cap \cdots \cap C_n \neq \emptyset$.

Proposition 5.11. *A topological space X is compact iff for any collection $\mathscr{C} \subset \mathscr{P}(X)$ of closed sets it holds that*

$$\mathscr{C} \text{ has the finite intersection property } \Rightarrow \bigcap_{C\in\mathscr{C}} C \neq \emptyset.$$

Proof. X is compact iff every open cover $\mathscr{U} = \{U_i\}_{i\in I}$ has a finite subcover; $X = U_1 \cup U_2 \cup \cdots \cup U_n$, where $U_1, U_2, ..., U_n \in \mathscr{U}$. Equivalently, there exists a finite subset of $\mathscr{C} := \{X \backslash U_i\}_{i\in I}$ with empty intersection;

$$(X\backslash U_1) \cap (X\backslash U_2) \cap \cdots \cap (X\backslash U_n) = X\backslash(U_1 \cup U_2 \cup \cdots \cup U_n) = \emptyset.$$

In other words, $\mathscr{U}$ lacks finite subcovers iff $\mathscr{C} = \{X\backslash U | U \in \mathscr{U}\}$ has the finite intersection property. A collection $\mathscr{C}$ of closed subsets of X is formed by the complements of the open sets of an open cover of X iff $\bigcap_{C\in\mathscr{C}} C = \emptyset$. Combining these facts we immediately obtain the proof of the proposition. □

5.2. Sequential compactness

■ It may be familiar from calculus that compactness can be characterized in terms of sequences. A similar result will not hold in general, instead we define a new property in terms of sequences.

Definition 5.12. A topological space X is called *sequentially compact* if every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X has a convergent subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$.

We show the following result to generalise the situation familiar from calculus.

Theorem 5.13. *Let X be a topological space.*

(*i*) *If X is first countable, then*

$$X \text{ is compact } \Rightarrow X \text{ is sequentially compact.}$$

(*ii*) *If X has a metric topology, then*

$$X \text{ is compact } \Leftrightarrow X \text{ is sequentially compact.}$$

Proof. Assume first that X is compact and first countable (recall that by Proposition 2.59 a metric space is first countable). Let $\{x_n\}$ be a sequence in X; we investigate two mutually exclusive situations:

1. $\exists x \in X$ such that for every neighbourhood U of x, $x_n \in U$ for infinitely many n. Let $\{B_i\}$ be a countable basis at x, then for every $k \in \mathbb{N}$ the set $\bigcap_{i=1}^{k} B_i$ is a neighbourhood of x, so $x_n \in \bigcap_{i=1}^{k} B_i$ for infinitely many n; chose one, x_{n_k}. In this way we construct a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. Every neighbourhood U of x contains some B_{N_U}, and thus

$$k > N \Rightarrow x_{n_k} \in \bigcap_{i=1}^{k} \subset B_{N_U} \subset U.$$

It follows that $x_{n_k} \to x$, i.e. we have constructed a convergent subsequence.

2. $\forall x \in X$, $\exists U_x \subset X$ neighbourhood of x such that $x_n \in U$ for at most finitely many $n \in \mathbb{N}$. Choose one neighbourhood U_x with that property for every $x \in X$. Then $\{U_x\}_{x\in X}$ is an open cover of X, and by compactness there exists a finite subcover $\{U_{x_1}, U_{x_2}, ..., U_{x_n}\}$. This, however, is impossible since each of the elements U_{x_i} contains only finitely many x_n, $n \in \mathbb{N}$. It follows that $\{x_n\}$ has a convergent subsequence.

$\Leftarrow$: Assume now that X is a sequentially compact metric space. Fix a $r > 0$, then $\{B_d(x,r)\}_{x\in X}$ is an open cover of X. Suppose there is no finite subcover, we can then find a sequence $\{x_n\}$ with the property that $x_{n+1} \notin \bigcup_{i=1}^{n} B_d(x_i, r)$, otherwise there would exist a finite subcover. It follows that $d(x_m, x_{m-k}) > r$ for every $k = 1, 2, ..., m-1$, thus $d(x_m, x_n) > r$ for all $m \neq n$. We have shown that $\{x_n\}$ has no convergent subsequence which is a contradiction, thus there must exist a finite subcover $\{B_d(x_i, r)\}_{i=1}^{n}$.

Let $\mathscr{U} = \{U_i\}_{\in I}$ be an open cover of X. I claim that there exists a $r > 0$ such that for every $x \in X$, $B_d(x, r) \subset U_i$ for some $i \in I$. A finite subcover $B_d(x_\alpha, r)\}_\alpha$ then defines a finite subcover $\{U_{i_\alpha}\}_\alpha$ of $\mathscr{U}$ by $B_d(x_\alpha, r) \subset U_{i_\alpha}$.

Suppose that for every $r > 0$ there exists some point $x \in X$ such that $B_d(x, r) \not\subset U_i$ for any $i \in I$. For each $n \in \mathbb{N}$, choose x_n such that $B_d(x_n, 1/n) \not\subset U_i$ for any $i \in I$. By assumption there exists a convergent subsequence $\{x_{n_i}\}$ to the so constructed sequence $\{x_n\}$, say $x_{n_k} \to x$. For some $j \in I$, $x \in U_j$, and since U_j is open it follows that $\exists N > 0 :$ $B_d(x, 1/N) \subset U_j$. By convergence of the subsequence, $\exists K > 0 : d(x_{n_k}, x) < \frac{1}{2N}\ \forall k > K$. Take k large enough so that $n_k > 2N$. For any $y \in B_d(x_{n_k}, 1/n_k)$ we then have

$$d(y, x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \frac{1}{n_k} + \frac{1}{2N} < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}.$$

But then $B_d(x_{n_k}, 1/n_k) \subset U_j$, which is a contradiction. Thus we have shown that for every $r > 0$ there exists some $x \in X$ with the property that $B_d(x, r) \subset U_i$ for some $i \in I$. □

Remark 5.14. The "long line" (see e.g. one of the project presentations) is a counter example of a first countable sequentially compact space that is not compact, showing that a property stronger than first countability is required for sequential compactness to imply compactness.

5.3. Compact subsets of $\mathbb{R}^n$

■ We will now derive some results about compactness in $\mathbb{R}^n$. In particular we will rediscover the familiar characterisation of compactness, as closedness and boundedness. These results will, however, also be useful in showing that other topological spaces, like n-spheres or n-tori, are compact, as they are constructed from (pieces of) $\mathbb{R}^n$.

Definition 5.15. Let (X, d) be a metric space. A set $U \subset X$ is called *bounded* if $U \subset B(x, r)$ for some $x \in X$, $r > 0$.

Theorem 5.16. *Let $K \subset \mathbb{R}^n$. The following statements are equivalent.*

1. *K is compact*
2. *K is sequentially compact*
3. *K is closed and bounded*

Remark 5.17.

(i) Note that 1. and 2. are statements about topological spaces, whereas 3. is a statement about a mere subset. in the two first points, the subset K is thus given the subspace topology, whereas the formulation the third statement is independent of this.

(*ii*) The equivalence 1. $\Leftrightarrow$ 3. is known as the Heine-Borel theorem, while the equivalence 2. $\Leftrightarrow$ 3. is called the Bolzano-Weierstrass theorem.

In order to prove this theorem we will employ several earlier results as well as some new. Observe first that the equivalence between 1. and 2. follows immediately from the more general result of Theorem 5.13. The implication 2. $\Rightarrow$ 3. will be showed to folllow from the sequence lemma (Lemma 2.60), while the implication 3. $\Rightarrow$ 1. will follow from a combination of Tychonov's theorem (Theorem 5.10) and the following result.

Theorem 5.18. *The subspace* $[0,1] \subset \mathbb{R}$ *is sequentially compact.*

Corollary 5.19. *The subspace* $[a,b] \subset \mathbb{R}$ *is compact for every* $a, b \in \mathbb{R}$*:* $a \leq b$

The proof of the corollary follows from Theorem 5.13 together with the existence of a homeomorphism $\varphi : [0,1] \to [a,b]$ when $a < b$ (the case $a = b$ is trivial). Such a homeomorphism is not difficult to find, and the proof is left as a simple exercise.
■ The proof of Theorem 5.18 makes use of a property of $\mathbb{R}$ known as *completeness*. This will on the one hand follow immediately by the construction of the real numbers. Since this is not included in the course this year, however, I show in Appendix A that completeness of $\mathbb{R}$ follows from the least upper bound property.

Definition 5.20. Let (X, d) be a metric space, endowed with the metric topology.

(*i*) A sequence $\{x_n\}$ is called a *Cauchy sequence* if

$$\forall \epsilon > 0,\ \exists N > 0 :\ m, n > N \Rightarrow d(x_m, x_n) < \epsilon.$$

(*ii*) X is called *complete* if every Cauchy sequence in X converges.

■ It follows easily from the triangle inequality that any convergent sequence is Cauchy. Not every Cauchy sequence converges, however, as the following two examples show.

Example 5.21.

(*i*) $\mathbb{Q} \subset \mathbb{R}$ is not complete. This is true since we can find sequences of rational numbers with non-rational limits (take for example the sequence $x_n = \sum_{i=0}^{n} \frac{1}{n!}$, where $x_n \to e$).

(*ii*) $(0,1) \subset \mathbb{R}$ is not complete. Take the sequence $x_n = 1/n$. For any $\epsilon > 0$, let N be the smallest integer larger than $1/\epsilon$, and take $m = N + i$, $n = N + j$. It follows that

$$\left| \frac{1}{x_m} - \frac{1}{x_n} \right| = \left| \frac{1}{N+i} - \frac{1}{N+j} \right| = \frac{1}{N} \left| \frac{1}{1+i/N} - \frac{1}{1+j/N} \right| < \epsilon,$$

and $\{x_n\}$ is thus a Cauchy sequence, but is clearly not convergent (in $(0,1) \subset \mathbb{R}$).

Proof of Theorem 5.18. We want to show that every sequence has a convergent subsequence. To this end, let $\{x_n\}$ be a sequence in $[0,1]$. Define $A_i^j = [\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}] \subset [0,1]$, $i \geq 0$, $0 \leq j \leq 2^{i+1}-1$. Observe that $A_i^j = A_{i+1}^{2j} \cup A_{i+1}^{2j+1}$.

Note that $[0,1] = A_0^0 \cup A_0^1$, thus there must be an infinite number of elements of $\{x_n\}$ in A_0^0 or in A_0^1; choose that subinterval and define x_{n_1} to be the first element of the sequence found in that subinterval. If both subintervals contain infinitely many elements of $\{x_n\}$, then simply choose one of them. Next, write the chosen subinterval (A_0^0 or A_0^1) as a union of new subintervals ($A_1^0 \cup A_1^1$ or $A_1^2 \cup A_1^3$), and go through the same process again. In other words, choose a subinterval containing infinitely many elements of the sequence $\{x_n\}$, and pick the first one of these as x_{n_2}. Iterate the procedure to define a subsequence $\{x_{n_i}\}$. Note that if $k \geq i+1$ then $x_{n_k} \in A_i^{j_i}$ for some j_i not depending on k. It follows that

$$k, l \geq i+1 \Rightarrow |x_{n_k} - x_{n_l}| \leq \frac{1}{2^{i+1}},$$

and $\{x_{n_i}\}$ is thus a Cauchy sequence. Since $\mathbb{R}$ is complete, $x_n \to x \in \mathbb{R}$. By the sequence lemma (Lemma 2.60) it follows that $x \in [0,1]$. In other words, we have found a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that is convergent (in $[0,1] \subset \mathbb{R}$), and the subspace $[0,1]$ is therefore sequentially compact. □

Proof of Theorem 5.16. Let us first show that the statement 2. implies 3. (as already mentioned, the equivalence between the statements 1. and 2. follows directly from Theorem 5.13). Assume therefore that $K \subset \mathbb{R}^n$ is compact, and our task is to show that K is closed and bounded.

Closed: Let $x \in \overline{K}$. By the sequence lemma there exists a sequence $\{x_n\}$ in K such that $x_n \to x \in \mathbb{R}$ (note that the convergence is that in $\mathbb{R}^n$). Since K is sequentially compact, there is a subsequence $\{x_{n_i}\}$ converging in K, say $x_{n_i} \to a \in K$ (here, the convergence is that in K as a subspace of $\mathbb{R}^n$). The inclusion map $\imath : K \hookrightarrow \mathbb{R}^n$ is continuous (this is an easy exercise to prove), implying that the sequence $\imath(x_{n_i}) = x_{n_i} \to \imath(a) = a$, where the convergence is in $\mathbb{R}^n$. Since limits are unique for metric spaces (Propositions 2.56 and 2.57), and since any subsequence of a convergent sequence has the same limit (Proposition 2.52), it follows that $a = x$. We have shown that $x \in K$ for every $x \in \overline{K}$, and therefore K must be closed.

Bounded: Assume that K is not bounded. Then, for each $n \in \mathbb{N}$ there exists some $x_n \in K \backslash B(0,n)$. Define in this way a sequence $\{x_n\}$, and let $\{x_{n_i}\}$ be any subsequence. Suppose the latter is Cauchy, and let $N > 0$ be such that $i, j > N \Rightarrow d(x_{n_i}, x_{n_j}) < 1$. Fix $i > N$, and choose $j > N$ such that $n_j > \|x_{n_i}\| + 1$ (this is possible since $n_{i+1} > n_i$). We calculate

$$1 < n_j - \|x_{n_i}\| \leq \|x_{n_j}\| - \|x_{n_i}\| = d(x_{n_j}, 0) - d(x_{n_i}, 0) \leq d(x_{n_j}, x_{n_i})$$

The first step is tautological, the second step is clear since by the procedure to define the sequence x_n we have $\|x_n\| \geq n$. The last step follows from the triangle inequality. We have reached a contradiction, and it follows that $\{x_{n_i}\}$ cannot be a Cauchy series; in particular it cannot be convergent, so the assumption that K is not bounded is wrong.

Let us now show that the statement 3. implies 1. Assume that K is bounded, then there exists $x \in \mathbb{R}^n$ and $r > 0$ such that $K \subset B(x, r)$. Define $N := \|x\| + r$. If $y \in B(x, r)$, then

$$\|y\| \leq \|x\| + \|y - x\| = N - r + \|y - x\| < N,$$

and since $|y_i| \leq \|y\|$, for $i = 1, 2, ..., n$, it follows that

$$B(x, r) \subset [-N, N] \times [-N, N] \times \cdots \times [-N, N] =: Y.$$

By Theorems 5.18 and 5.10 the space Y is compact, and since $K \subset Y$ is closed, by Theorem 5.3 it follows that K is even compact. Thus the proof is complete. □

The results that have just been showed can be invoked to show compactness of various other spaces that are constructed from $\mathbb{R}^n$ in one way or another.

Proposition 5.22.

(i) *The n-sphere S^n is compact*

(ii) *The n-torus T^n is compact*

(iii) *The surface Σ_n with n holes from Example 3.19 is compact*

Proof. (i): For every $\epsilon > 0$ we note that $S^n \subset B(0, 1 + \epsilon) \subset \mathbb{R}^{n+1}$, so the n-sphere is bounded. It is easy to check that $\overline{S^n} = S^n$, so the n-sphere is also closed. By Theorem 5.16 it follows that the n-sphere is compact.

(ii): From (i) we know that the circle S^1 is compact. Tychonov's theorem (Theorem 5.10) then implies that the n-torus $T^n := S^1 \times S^1 \times \cdots \times S^1$ is compact.

(iii): Recall that $X_n \subset \mathbb{R}^2$ is a regular $4n$-gon, in particular a closed and bounded subset of $\mathbb{R}^2$. As a space with the subspace topology it is therefore compact by Theorem 5.16. The canonical surjection $p_n : X_n \to \Sigma_n = X_n/\sim_n$ is continuous (this follows immediately from the definition of the quotient topology), so p_n is a continuous map from a compact space X_n. Theorem 5.6 then implies that $\text{Im}(p_n) = p_n(X_n) = \Sigma_n$ is compact. □

We are finally in a position where we can easily prove Proposition 3.14.

Proof of Proposition 3.14. We want to show $D^n/\partial D^n \simeq S^n$, where $D^n = \overline{B(0, 1)} \subset \mathbb{R}^{n+1}$ is the closed unit ball. We note first that D^n is closed and bounded, hence it is a compact space. Since the canonical surjection $p : D^n \to D^n/\partial D^n$ is continuous, it follows by Theorem 5.6 that $D^n/\partial D^n$ is also compact. Next, construct an explicit bijection $\varphi : D^n/\partial D^n \to S^n$. Recall from Example 3.3 that there is a homeomorphism $\varphi_n : B^n \to \mathbb{R}^n$, and from (the proof of) Proposition 3.5 that there is a homeomorphism $\Pi^n_{\text{st}} : S^n \backslash \{N\} \to \mathbb{R}^n$ (the stereographic projection). Define the homeomorphism

$$\widetilde{\psi}_n := (\Pi^n_{\text{st}})^{-1} \circ \varphi : B(0, 1) \to S^n \backslash \{N\},$$

and extend this in the obvious way to a map $\psi_n : D^n \to S^n$. In other words, $\psi_n|_{B(0,1)} = \widetilde{\psi}_n$, and for each $x \in \partial D^n$, let $\psi_n : x \mapsto N$. Since the restriction to $B(0,1)$ is a homeomorphism, it is in particular continuous. Let us show that ψ_n is also continuous at all $x \in \partial D^n$. If $0 < \epsilon < 1$, then the set $A^\epsilon := \{y \in D^n | \|y\| > 1 - \epsilon\}$ is open. By the definitions of φ and Π^n_{st} it is quickly verified that

$$\psi_n(A^\epsilon) = \{y \in S^n | \frac{1-2\epsilon}{1-2\epsilon+2\epsilon^2} < y_{n+1} \leq 1\}.$$

It is easy to see that for any open ball $B \subset \mathbb{R}^{n+1}$ containing N, we can find a small enough ϵ such that $A^\epsilon \subset B$. If $U := S^n \cap \widetilde{U}$ is a neighbourhood of $N \in S^n$, it then follows that there is a $\epsilon > 0$ such that $A^\epsilon \subset U$. Since $x \in A^\epsilon$ for each $x \in \partial D^n$, we have shown that ψ_n is continuous at every $x \in \partial D^n$, and ψ_n is thus continuous. Lemma 3.16 now implies that the map $\widehat{\psi}_n : D^n/\partial D^n \to S^n$ defined by $\widehat{\psi}_n([x]) = \psi_n(x)$ is the unique continuous map with the property that $\psi_n = \widehat{\psi}_n \circ p$, where $p : D^n \to D^n/\partial D^n$ is the canonical surjection. In particular, $\widehat{\psi}_n$ is a continuous bijection. Since S^n is Haudorff (a subspace of a Hausdorff space is Hausdorff, see Problem Sheet 1), Theorem 5.7 implies that $\widehat{\psi}_n : D^n/\partial D^n \to S^n$ is a homeomorphism. □

5.4. Local compactness and one-point compactification

As we have seen, compactness is a very strong property which implies a number of powerful theorems. Some of these results may hold even under a weaker condition than compactness, however; it is sometimes enough that a space is in a certain sense locally compact.

Definition 5.23. A space X is called *locally compact* if every $x \in X$ has a neighbourhood U contained in a compact subspace $K \subset X$, i.e. $x \in U \subset K \subset X$.

Example 5.24. (*i*) If X is compact then X is locally compact.

(*ii*) Any open or closed subset of $\mathbb{R}^n$ (in particular, $\mathbb{R}^n$ itself) is locally compact.

Proposition 5.25. $\mathbb{Q} \subset \mathbb{R}$ *is not locally compact.*

Proof. We show that compact sets in $\mathbb{Q}$ have empty interior, and thus cannot contain neighbourhoods.

Let $K \subset \mathbb{Q}$ be compact, then by Theorem 5.4 (note that $\mathbb{Q} \subset \mathbb{R}$ is Hausdorff) K is closed (i.e. $\overline{K} = K$ in $\mathbb{Q}$). Suppose $x \in \mathring{K}$, which is an open set, then there exists an open interval $(a,b) \subset \mathbb{R}$ such that $x \in (a,b) \cap \mathbb{Q} \subset \mathring{K} \subset K$. It follows that $[a,b] \cap \mathbb{Q} \subset K$ (since $\overline{K} = K$, and $\overline{(a,b) \cap \mathbb{Q}} = [a,b] \cap \mathbb{Q}$). By Theorem 5.3, $[a,b] \cap \mathbb{Q} \subset \mathbb{Q}$ is a compact subspace. Let $r \in (a,b)$ be non-rational. Then

$$\mathscr{U} = \{([a, r-1/n) \cup (r+1/n, b]) \cap \mathbb{Q} \big| n \in \mathbb{N}\}$$

is an open cover of $[a,b] \cap \mathbb{Q}$. Clearly there exists no finite subcover of $\mathscr{U}$, which is a contradiction. It follows that $\mathring{K} = \emptyset$. □

■ Another important, and frequently occurring, example of a non-locally compact space is provided by an infinite dimensional normed vector space of the type ℓ^p.

Definition 5.26. A *normed vector space* (V, ρ) is a vector space V (over $\mathbb{R}$ or $\mathbb{C}$) together with a *norm* on V, i.e. a function $\rho : V \to \mathbb{R}_{\geq 0}$ satisfying

(N1) $\rho(\lambda v) = |\lambda|\rho(v)\ \forall v \in V,\ \lambda \in \mathbb{R}\ (\mathbb{C})$

(N2) $\rho(u + v) \leq \rho(u) + \rho(v)\ \forall u, v \in V$

(N3) $\rho(v) = 0 \Leftrightarrow v = 0$

If ρ is a norm on V, then $d_\rho : V \times V \to \mathbb{R}_{\geq 0}$ defined by

$$d_\rho : (u, v) \mapsto \rho(u - v),$$

is a metric on V satisfying

$$\begin{aligned} d_\rho(\lambda u, \lambda v) &= |\lambda| d_\rho(u, v) && \forall u, v \in V,\ \lambda \in \mathbb{R}\ (\mathbb{C}) \\ d_\rho(u, v) &= d_\rho(u + w, v + w) && \forall u, v, w \in V. \end{aligned}$$

Proof that d_ρ is a metric. Let (V, ρ) be a normed vector space, and let d_ρ be the function defined above. Obviously $d_\rho(u, v) = 0$ iff $u - v = 0$, i.e. iff $u = v$; thus d_ρ satisfies $(M1)$. Second, $d_\rho(v, u) = \rho(v - u) = \rho(-(u - v)) = \rho(u - v) = d_\rho(u, v)$, so d_ρ satisfies $(M2)$. Finally, $d_\rho(u, v) = \rho(u - v) = \rho((u - w) + (w - v)) \leq \rho(u - w) + \rho(w - v) = d_\rho(u, w) + d_\rho(w, v)$, so d_ρ satisfies the triangle inequality $(M3)$. □

■ A normed vector space (V, ρ) becomes a topological space, or more precisely a *topological vector space*, with the metric topology given by d_ρ.
■ We note that $\mathbb{R}^n$ with the Euclidean norm becomes a normed vector space, and the norm topology is nothing but the standard topology. Finite dimensional vector spaces are not the typical case of interest, however.

Definition 5.27. Consider the set

$$\mathbb{R}^\infty := \{f : \mathbb{N} \to \mathbb{R}\} = \{\{x_n\}_{n\in\mathbb{N}} | x_n \in \mathbb{R}\}$$

of sequences in $\mathbb{R}$. This becomes a vector space over $\mathbb{R}$ in the obvious way, i.e. under component-wise addition and scalar multiplication. For $p \in [1, \infty)$ and a sequence $x \in \mathbb{R}^\infty$ define the *p-norm*

$$\|x\|_p := \left(\sum_{i\in\mathbb{N}} |x_i|^p \right)^{1/p},$$

and the ∞-*norm*

$$\|x\|_\infty := \sup_{n\in\mathbb{N}} \{|x_n|\}.$$

For any $1 \leq p \leq \infty$ define the subspace $\ell^p := \{x \in \mathbb{R}^\infty | \|x\|_p < \infty\}$. It is easily checked that $(\ell^p, \|\cdot\|_p)$ is a (infinite dimensional) normed vector space for every $1 \leq p \leq \infty$. For brevity write only ℓ^p for the corresponding normed vector space.

Proposition 5.28. *The topological space ℓ^p is not locally compact for any $1 \leq p \leq \infty$.*

Proof. For simplicity, denote the metric given rom the p-norm by d_p. Let $U \subset \ell^p$ be a neighbourhood of $0 \in \ell^p$. Then there exists a $\epsilon > 0$ such that $B_{d_p}(0, \epsilon) \subset U$. Define for every $n \in \mathbb{N}$ an element $\delta_n \in \ell^p$ by

$$\delta_n := \{x_m\}_{m \in \mathbb{N}}, \text{ where } x_m = \delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}.$$

Writing δ_n as an "∞-tuple", it looks like

$$\delta_n = (\underbrace{0, 0, ..., 0}_{n-1}, 1, 0, ...).$$

We have $\|\delta_n\|_p = 1$, so $\delta_n \in \ell^p$ for every $n \in \mathbb{N}$ and every $1 \leq p \leq \infty$. Consider the sequence $\{y_n := \frac{\epsilon}{2}\delta_n\}_{n \in \mathbb{N}}$ in $B_{d_p}(0, \epsilon) \subset U$. Choose arbitrary $m, n \in \mathbb{N}$ and calculate

$$d_p(y_m, y_n) = \frac{\epsilon}{2} d_p(\delta_m, \delta_n) = \frac{\epsilon}{2}\|\delta_m - \delta_n\|_p = \frac{\epsilon}{2} \begin{cases} 2^{1/p} & 1 \leq p < \infty \\ 1 & p = \infty \end{cases} \geq \frac{\epsilon}{2}.$$

It follows that $\{y_n\}$ has no convergent subsequence, so U cannot be contained in a compact subspace $K \subset \ell^p$. The space ℓ^p is therefore not locally compact. □

Definition 5.29. Let X be Hausdorff. The *one-point compactification* of X is the space $\widehat{X} := X \sqcup \{\star\}$ where a set $U \subset \widehat{X}$ is open iff $U \subset X$ is open, or if $U = \{\star\} \cup X \backslash K$ where $K \subset X$ is compact. It is an exercise to show that these open sets indeed form a topology.

Proposition 5.30. *$\widehat{X}$ is compact.*

Proof. Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open cover of $\widehat{X}$. For some $j \in I$, $\star \in U_i$ so $U_j = \{\star\} \cup X \backslash K$ for a compact $K \subset X$. Note that $\{U_i \cap K\}$ is an open cover of K, and since K is compact there is a finite subcover $\{U_{i_1} \cap K, U_{i_2} \cap K, ..., U_{i_n} \cap K\}$ such that j is not among i_1, ..., i_n. The set $\{U_{i_1}, U_{i_2}, ..., U_{i_n}, U_j\}$ is then a finite subcover of $\mathscr{U}$, so $\widehat{X}$ is indeed compact. □

Proposition 5.31. *If X is locally compact Hausdorff, then $\widehat{X}$ is Hausdorff.*

Proof. Let $x, y \in \widehat{X}$ be two distinct points. If $x, y \in X$ then we are done since X is Hausdorff, so assume $x \in X$ and $y = \star$. Let $U \subset X$ be a neighbourhood of x, and $K \subset X$ a compact subspace containing U (this exists since X is locally compact). In other words, $x \in U \subset K \subset X \subset \widehat{X}$. Since $\star \in \{\star\} \cup X \backslash K$ and U is disjoint from the latter open set, we have found a separation of x and $\star$ by disjoint open sets. □

Example 5.32. If X is compact, then $\widehat{X}$ is simply X together with an open point $\{\star\}$.

Proposition 5.33. *If X is compact Hausdorff, then $\widehat{X \backslash \{pt\}} \simeq X$ for any $pt \in X$.*

Proof. Exercise. □

Example 5.34. It follows immediately from Proposition 5.33 that $\widehat{\mathbb{R}^n} \simeq S^n$.

Definition 5.35. A continuous map $f : X \to Y$ is called *proper* if $f^{-1}(K)$ is compact whenever $K \subset X$ is compact.

Proposition 5.36. *Let X and Y be Hausdorff. If $f : X \to Y$ is a proper map, then $\hat{f} : \widehat{X} \to \widehat{Y}$ defined by $\hat{f}|_X = f$, $\hat{f}(\star_X) = \star_Y$, is continuous.*

Proof. If $U \subset Y$ is open, then $\hat{f}^{-1}(U) \subset X$ is obviously open since f is continuous. Let $K_Y \subset Y$ be a compact subspace so $\{\star_Y\} \cup Y \backslash K_Y \subset \widehat{Y}$ is open. Then $\hat{f}^{-1}(\{\star_Y\} \cup Y \backslash K_Y) = \{\star_X\} \cup f^{-1}(Y \backslash K_Y)$. Since f is proper we have that $K_X := f^{-1}(K_Y)$ is a compact subspace of X. Furthermore, $f^{-1}(Y \backslash K_Y) = X \backslash K_X$ so that $\hat{f}^{-1}(\{\star_Y\} \cup Y \backslash K_Y) = \{\star_X\} \cup X \backslash K_X$, i.e. $\hat{f}$ is continuous. □

6. Separation and countability axioms

Let us now introduce the last pieces of separation axioms to characterise topological spaces.

6.1. Separation axioms – part 2

Definition 6.1. A space X is called *regular* is for any closed $F \subset X$ and any point $x \in X \backslash F$ there exist open $U, V \subset X$ such that $x \in U$, $F \subset V$, and $U \cap V = \emptyset$. In words, X is regular if any closed subset F of X and any point the complement of F can be separated by disjoint neighbourhoods. A regular T_1-space is called T_3. A common alternative name is *regular Hausdorff.*

- If X is T_1, then $\{x\}$ is closed for any $x \in X$, so T_3 implies T_2.

Proposition 6.2. *If X is T_3 and $F \subset X$ is closed, then X/F is Hausdorff.*

Proof. Exercise □

Proposition 6.3. *Let (X, d) be a metric space with the metric topology. Then X is regular.*

Proof. Let $F \subset X$ be closed, and $x \in X \backslash F$. Since $X \backslash F$ is open, there exists a $\epsilon > 0$ such that $B_d(x, \epsilon) \subset X \backslash F$. Furthermore

$$x \in B_d(x, \epsilon/3) \subset B_d(x, \epsilon) \subset X \backslash F.$$

I claim that $\partial B_d(x, \epsilon/3) \subset \{y \in X | d(x, y) = \epsilon/3\}$: take $y \in \partial B_d(x, \epsilon/3)$, and consider $B_d(y, r)$ for some $r > 0$. Then there exist $u, v \in B_d(y, r)$ such that $d(x, u) < \epsilon/3$ and $d(x, v) > \epsilon/3$. It follows that $d(x, y) \leq d(x, z) + d(y, z) < \epsilon/3 + r$. This has to be true for any $r > 0$ so we conclude that $d(x, y) \leq \epsilon/3$. However, if $d(x, y) < \epsilon/3$ then there exists an $r > 0$ such that $B_d(y, r) \subset B_d(x, \epsilon/3)$, contradicting the assumption that $y \in \partial B_d(x, \epsilon/3)$, so $d(x, y) = \epsilon/3$. There is the sequence

$$x \in B_d(x, \epsilon/3) \subset \overline{B_d(x, \epsilon/3)} \subset B_d(x, \epsilon) \subset X \backslash F.$$

Summarising we conclude that $x \in B_d(x, \epsilon/3)$ and $F \subset X \backslash \overline{B_d(x, \epsilon/3)}$, i.e. we have found disjoint open sets separating x and F. □

Proposition 6.4. *The space $\mathbb{R}_K$, i.e. the real line with the K-topology, is Hausdorff but not regular (thus not T_3).*

Proof. Recall that the K-topology is strictly finer than the standard topology, so every subset of $\mathbb{R}$ that is open in the standard topology is open also in $\mathbb{R}_K$. Since the standard topology is Hausdorff, also $\mathbb{R}_K$ is therefore Hausdorff. The set $K \subset \mathbb{R}_K$ is closed since $\mathbb{R} \backslash K$ is open by definition. The point 0 and K cannot be separated by disjoint open sets, however, so $\mathbb{R}_K$ is not regular (and also not T_3). □

■ It follows that the inclusion of the class of T_3-spaces in the class of Hausdorff spaces is strict; Hausdorff does not imply T_3.

Definition 6.5. A topological space X is called *normal* if any two disjoint closed subsets, $F_1, F_2 \subset X$, can be separated by disjoint open sets $U_1, U_2 \subset X$, i.e. $U_1 \cap U_2 = \emptyset$ and $F_i \subset U_i$, $i = 1, 2$. A normal T_1-space is called T_4.

■ Since T_1 implies that points are closed, we conclude the sequence of implications $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. Equivalently we have the sequence of inclusions of classes of topological spaces $T_4 \subset T_3 \subset T_2 \subset T_1 \subset T_0$.

Proposition 6.6. *Let (X, d) be a metric space. Then X is normal (and therefore T_4).*

Proof. Let $x \in X$ and $F \subset X$ and define the distance $d(x, F) := \{\inf\{d(x, y) | y \in F\}$. Suppose $F_1, F_2 \subset X$ are closed and disjoint. Let $x \in F_1$, $y \in F_2$, and choose $\epsilon_x > 0$, $\epsilon_y > 0$ such that $0 < \epsilon_x < d(x, F_2)$ respectively $0 < \epsilon_y < d(y, F_1)$. Assume $z \in B_d(x, \epsilon_x/2) \cap B_d(y, \epsilon_y/2)$ and calculate

$$d(x, y) \leq d(x, z) + d(z, y) < \epsilon_x/2 + \epsilon_y/2 \leq \max(\epsilon_x, \epsilon_y).$$

This is a contradiction since, by definition, $d(x, y) \geq \max(\epsilon_x, \epsilon_y)$, so it follows that $B_d(x, \epsilon_x/2) \cap B_d(y, \epsilon_y/2) = \emptyset$. This holds for any $x \in F_1$ and any $y \in F_2$, and thus $F_1 \subset \bigcup_{x \in F_1} B_d(x, \epsilon_x/2)$, $F_2 \subset \bigcup_{y \in F_2} B_d(y, \epsilon_y/2)$ is a separation of F_1 and F_2 by disjoint open subsets. □

Proposition 6.7. *Let X be a compact Hausdorff space. Then X is normal, and therefore T_4.*

Proof. Exercise. □

■ The property of being T_4 is strictly stronger than T_3, as the following example shows.

Proposition 6.8. *Let $S = \mathbb{R}_l \times \mathbb{R}_l$ be the product of $\mathbb{R}_l$ (see Example 2.12) with itself, S is usually called the Sorgenfrey plane. Then S is T_3 but not T_4.*

Proof. See [M, §31 Example 3] □

Let us finally collect some properties of T_2- and T_3-spaces in a single theorem.

Theorem 6.9.

(*i*) *A subspace of a Hausdorff space is Hausdorff.*

(*ii*) *A product of Hausdorff spaces is Hausdorff.*

(*iii*) *A subspace of a T_3-space is T_3.*

(*iv*) *A product of T_3-spaces is T_3.*

Proof. Both (*i*) and (*ii*) are exercises.

(*iii*): Let $Y \subset X$ be a subspace, where X is regular Hausdorff. By (*i*) Y is Hausdorff, so in particular points are closed in Y. Let $y \in Y$ be a point and $F \subset Y$ a closed subset not containing y. Let $\overline{F} \subset X$ be the closure of F in X, then $\overline{F} \cap Y = F$, and it follows that $y \notin \overline{F}$. By the T_3 property of X there exist disjoint open subsets $U', V' \subset X$ such that $x \in U'$ and $\overline{F} \subset V'$; the intersections $U := U' \cap Y$ and $V := V' \cap Y$ are then disjoint open subsets of Y separating y and F.

(*iv*): Let $\{X_i\}_{i\in I}$ be a family of T_3-spaces, and let $X := \prod_{i\in I} X_i$. By (*ii*), X is Hausdorff, so in particular points are closed in X. Let $x = \{x_i\}_{i\in I}$ be a point in X, and let $U \subset X$ be a neighbourhood of x. Choose a basis element $\prod_{i\in I} U_i$ inside U, and containing x. Choose for each $i \in I$ a neighbourhood V_i of x_i such that $x_i \in \overline{V_i} \subset U_i$, this exists since X_i is T_3; if $U_i = X_i$ choose $V_i = X_i$. Then $V := \prod_{i\in I} V_i$ is a neighbourhood of x, and by Proposition 2.46 $\overline{V} = \prod_{i\in I} \overline{V_i}$. We have provided a neighbourhood V of x so that $x \in \overline{V} \subset U$, and by Lemma 6.14 X is T_3.

□

6.2. Second countability

Recall Definition 2.58 of first countable, which is a local property. There is a natural stronger condition.

Definition 6.10. A topological space X is called *second countable* if it has a countable basis.

■ Obviously every second countable space is also first countable.

Example 6.11. R^n is second countable. Showing this is an exercise.

Theorem 6.12. *Let X be a second countable space. Then* (*i*) *Every open cover of* X *has a countable subcover.* X *is then called* Lindelöf.

(*ii*) *There is a countable dense subset of* X. X *is then called* separable.

Proof. (*i*): Let $\mathscr{U}$ be an open cover of X, and let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis. Construct a countable subcover $\{U_n\} \subset \mathscr{U}$ as follows. For each $n \in \mathbb{N}$ define U_n to be $\emptyset$ if B_n is not contained in any $U \in \mathscr{U}$, and otherwise U_n is chosen to be one $U \in \mathscr{U}$ that contains B_n. If $x \in X$ then there exists some $x \in U_n$ for some $n \in \mathbb{N}$. There exists at least one n such that $x \in B_n$, and if $U \subset X$ is open then for every $x \in U$ there exists some $m \in \mathbb{N}$ such that $x \in B_m \subset U$. Therefore $\{U_n\}$ is an open cover of X.

(*ii*): For each $n \in \mathbb{N}$ choose some $x_n \in B_n$. For each $x \in X \backslash \{x_1, x_2, ...\}$, note that any neighbourhood $U \subset X$ of x contains some B_n, so $x_n \in U$. Now, note that $x \in \overline{U}$ iff every neighbourhood of x intersects U. This follows immediately from $\overline{U} = U \cup \partial U$. It follows that $X = \overline{\{x_1, x_2, ...\}}$. □

■ In fact, not every metric space is second countable. Consider for instance the discrete metric (inducing the discrete topology) on an uncountable set X; the collection $\{\{x\}\}_{x\in X}$ is an open cover with no countable subcover.

Theorem 6.13. *If* X *is a second countable* T_3*-space, then* X *is normal (and therefore also* T_4*).*

To prove this theorem we need the following lemma.

Lemma 6.14. *Let* X *be a* T_1*-space.*

(*i*) X *is* T_3 *iff for each* $x \in X$ *and every neighbourhood* U *of* x*, there exists a neighbourhood* V *of* x *such that* $x \in \overline{V} \subset U$.

(*ii*) X *is* T_4 *iff for each closed* $F \subset X$ *and every open* $U \subset X$ *containing* F*, there exists an open* $V \subset X$ *such that* $F \subset V \subset \overline{V} \subset U$.

Proof. (*i*) $\Rightarrow$: Assume X is T_3. Let $x \in X$ and let U be a neighbourhood of x; the set $B := X \backslash U$ is then closed and does not contain x. By assumption there are disjoint open subsets $V, W \subset X$ such that $x \in V$ and $B \subset W$. Note that $\overline{V} \cap B = \emptyset$ since for $y \in B$, W is a neighbourhood of y disjoint from V, so $y \notin \partial V$. In other words, $\overline{V} \subset U$.

$\Leftarrow$: Let $x \in X$, and let $B \subset X$ be a closed subset not containing x. The set $X \backslash B$ is open, and a neighbourhood of x,, so by assumption there exists a neighbourhood V of x such that $x \in \overline{V} \subset U$. We have that $B \subset X \backslash \overline{V}$ and $V \cap X \backslash \overline{V} = \emptyset$, so we have found disjoint open sets separating x and B.

(*ii*): The proof is copied almost verbatim from (*i*) by simply replacing $x \in X$ by a closed subset $F \subset X$. □

Proof of Theorem 6.13. Let X be T_3 with a countable basis $\{B_n\}$, and consider disjoint closed subsets $F_1, F_2 \subset X$. Every $x \in F_1$ has a neighbourhood U_x disjoint from F_2 by the T_3 property. By Lemma 6.14 (*i*) there exists a neighbourhood V of x such that $\overline{V} \subset U$. Choose B_n containing x such that $B_n \subset V$. Choosing one such basis element for each

$x \in F_1$ yields a countable subset $\{B^1_k\}_{k\in\mathbb{N}} \subset \{B_n\}_{n\in\mathbb{N}}$ covering F_1, whose closures do not intersect F_2. Define $\widetilde{U}_1 := \bigcup_{k\in\mathbb{N}} B^1_k$. In the same way, choose a countable subset $\{B^2_j\}$ covering F_2 and whose closures do not intersect F_1; define $\widetilde{U}_2 := \bigcup_{j\in\mathbb{N}} B^2_j$. The so constructed sets $\widetilde{U}_1$ and $\widetilde{U}_2$ are open, and such that $F_i \subset \widetilde{U}_i$, $i = 1, 2$, but they are not necessarily disjoint.

Define for each $k \in \mathbb{N}$ $U'_k := B^1_k \setminus \left(\bigcup_{j=1}^k \overline{B^2_j}\right)$, and for each $j \in \mathbb{N}$ $V'_j := B^2_j \setminus \left(\bigcup_{k=1}^j \overline{B^1_k}\right)$. We claim that $U_1 := \bigcup_k U'_k$, and $U_2 := \bigcup_j V'_j$ are open, disjoint, and contain F_1 respectively F_2. Obviously each U'_k resp. V'_j is open, being the difference between an open set and a closed set (i.e. the intersection between two open sets), thus U_1 and U_2 are both open. The open set U_i still covers F_i, for $i = 1, 2$, since the points removed from $\widetilde{U}_i$ are not in F_i. Finally, let $x \in U_1 \cap U_2$, then there exist natural numbers k, j such that $x \in U'_k \cap V'_j$. Suppose $k \le j$, then by the definition of U'_k $x \in B^1_k$, and since $k \le j$ $x \notin V'_j$. This is a contradiction, and a similar contradiction occurs if $j \le k$. Thus we must have that $U_1 \cap U_2 = \emptyset$. □

6.3. The Urysohn lemma

Finally we will state a technical lemma that is central in the proof of many important theorems in point set topology.

Lemma 6.15 (Urysohn lemma)**.** *Let X be a T_4-space, $F_1, F_2 \subset X$ disjoint closed subsets, and $[a, b] \subset \mathbb{R}$ a closed interval. Then there exists a continuous map $f : X \to \mathbb{R}$ such that $f(x) = a$ for every $x \in F_1$ and $f(x) = b$ for every $x \in F_2$.*

Proof. See Munkres. □

One important theorem that follows from the Urysohn lemma is the so called Urysohn metrisation theorem. A topological space $(X, \mathscr{T})$ is called metrisable if there exists a metric d on X such that $\mathscr{T}$ is the metric topology induced by d.

Theorem 6.16. *Let X be a second countable T_3-space, then X is metrisable.*

Proof. See Munkres. □

7. Manifolds and embeddings

In much of topology (for instance in all of differential topology, and all topological applications in Riemannian geometry) one is only interested in a particular kind of topological spaces called manifolds (or slight generalisations such as manifolds with boundary).

7.1. Manifolds

Definition 7.1. An *n-manifold* (or an n-dimensional manifold) is a second countable Hausdorff space X such that each $x \in X$ has a neighbourhood U_x homeomorphic to $\mathbb{R}^n$.

■ It follows that manifolds can be, literally, viewed as being constructed by patching together overlapping open balls in $\mathbb{R}^n$.

Example 7.2.

(*i*) $\mathbb{R}^n$ is an n-manifold

(*ii*) S^n is an n-manifold. A point $p \in S^n$ has a neighbourhood homeomorphic to $\mathbb{R}^n$ given by $S^n \backslash \{\tilde{p}\}$, where $\tilde{p}$ is the point antipodal to p. A concrete homeomorphism is given by the stereographic projection with $\tilde{p}$ in the north pole.

(*iii*) The n-torus T^n is an n-manifold.

(*iv*) Each surface $\Sigma_g = X_g / \sim_g$ with g handles from Example 3.19 is a 2-manifold.

7.2. Embeddings of manifolds

We will now state and prove what will be our main result concerning the point set topology of manifolds.

Theorem 7.3. *If X is a compact n-manifold, then X can be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$.*

■ The theorem actually holds without the compactness condition, but the proof is then considerably more complicated.

We need some preparation in order to prove Theorem 7.3.

Definition 7.4. Let X be a topological space and $f : X \to \mathbb{R}$ a function (not necessarily continuous). The *support* of f, $\text{supp}(f)$, is defined as

$$\text{supp}(f) := \overline{f^{-1}(\mathbb{R} \backslash \{0\})}.$$

In words, the support of f is the closure of the preimage of the real numbers minus $\{0\}$, i.e. it is the closure of the set of $x \in X$ such that $f(x) \neq 0$.

■ Note that by the definition of support, every $x \in X \backslash \text{supp}(f)$ has a neighbourhood not intersecting $\text{supp}(f)$. In other words, every $x \in X \backslash \text{supp}(f)$ has a neighbourhood on which f vanishes.

Definition 7.5. Let $\{U_1, U_2, ..., U_n\}$ be an open cover of X. A family $\{\phi_i\}$ of continuous functions $\phi_i : X \to [0, 1]$, $i = 1, ..., n$ is called a *partition of unity* dominated by $\{U_i\}$ if

1. $\text{supp}(\phi_i) \subset U_i$ for each $i = 1, ..., n$.

2. $\sum_{i=1}^{n} \phi_i(x) = 1$ for each $x \in X$.

Theorem 7.6 (Existence of finite partitions of unity)**.** *Let X be a T_4-space, and let $\{U_1, ..., U_n\}$ be a finite open covering of X. Then there exists a partition of unity dominated by $\{U_i\}$.*

Proof. Let us first show the existence of a smaller open covering $\{V_i\}$ such that $\overline{V}_i \subset U_i$ for each $i = 1, ..., n$. The set $A_i = X \backslash (U_2 \cup \cdots \cup U_n)$ is closed and contained in U_1 (since $\{U_i\}$ covers X). By Lemma 6.14 (*ii*) we can choose an open set V_1 containing A_1 and such that $\overline{V_1} \subset U_1$. The set $\{V_1, U_2, ..., U_n\}$ is still an open covering of X. If $\{V_1, ..., V_{k-1}, U_k, U_{k+1}, ..., U_n\}$ covers X, then $A_k := X \backslash (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n)$ is closed and contained in U_k, so again by Lemma 6.14 (*ii*) we can choose an open set V_k containing A_k and such that $\overline{V_k} \subset U_k$; the set $\{V_1, ..., V_k, U_{k+1}, ..., U_n\}$ still covers X. Going through this procedure n times gives an open covering $\{V_1, ..., V_n\}$ with the stated properties.

Now, start with an open covering $\{U_1, ..., U_n\}$, and produce a new smaller covering $\{V_1, ..., V_n\}$, and iterating the procedure once more another open covering $\{W_1, ..., W_n\}$ with the property that $\overline{W_i} \subset V_i$ for each $i = 1, ..., n$. By the Urysohn Lemma (Lemma 6.15) we can choose a continuous function $f_i : X \to [0, 1]$ for each $i = 1, ..., n$ such that $f_i(\overline{W_i}) = \{1\}$, and $f_i(X \backslash V_i) = \{0\}$. It follows that $\text{supp}(f_i) \subset \overline{V_i} \subset U_i$. Since $\{W_i\}$ covers X it follows that $f(x) := \sum_{i=1}^{n} f_i(x) > 0$ for each $x \in X$; define therefore

$$\phi_i(x) := \frac{f_i(x)}{f(x)},\ i = 1, ..., n.$$

We have $\text{supp}(\phi_i) = \text{supp}(f_i) \subset U_i$ for each $i = 1, ..., n$, and $\sum_{i=1}^{n} \phi_i(x) = \sum_{i=1}^{n} \frac{f_i(x)}{f(x)} = 1$ for each $x \in X$. We have thus shown that $\{\phi_i\}$ is a partition of unity dominated by $\{U_i\}$. □

Proof of Theorem 7.3. Let $\{U_1, U_2, ..., U_k\}$ be a finite open cover of X such that each U_i is homeomorphic to (an open ball in) $\mathbb{R}^n$. Such a cover exists; for each $x \in X$ choose a neighbourhood V_x of x homeomorphic to $\mathbb{R}^n$, then $\{V_x\}_{x \in X}$ is an open cover of X, and by compactness there exists a finite subcover. For each $i = 1, ..., k$ choose an embedding $\varphi_i : U_i \to \mathbb{R}^n$.

Let $\{\phi_1, \phi_2, ..., \phi_k\}$ be a partition of unity dominated by $\{U_i\}$; this exists since X is compact Hausdorff and hence T_4. Define $A_i := \text{supp}(\phi_i)$, $i = 1, 2, ..., k$. Finally define functions $\psi_i : X \to \mathbb{R}^n$, $i = 1, 2, ..., k$ as follows

$$\psi_i : x \mapsto \begin{cases} \phi_i(x) \cdot \varphi_i(x) & x \in U_i \\ 0 & x \in X \backslash A_i \end{cases}.$$

Each ψ_i is continuous since it is continuous on U_i (this is easily verified using convergence of sequences) and $X \backslash A_i$ (trivial).

Define a function $F : X \to \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k \text{ times}} \times \underbrace{\mathbb{R}^n \times \cdots \mathbb{R}^n}_{k \text{ times}}$ as follows:

$$F : x \mapsto (\phi_1(x), \phi_2(x), ..., \phi_k(x), \psi_1(x), \psi_2(x), ..., \psi_k(x)).$$

By Theorem 2.40 F is continuous since each of the component functions is continuous. Since X is compact and $F(X) \subset \mathbb{R}^{(n+1)k}$ is Hausdorff, by Theorem 5.7 F is an embedding iff F is injective. Suppose $F(x) = F(y)$. Then $\phi_i(x) = \phi_i(y)$ and $\psi_i(x) = \psi_i(y)$ for every $i = 1, ..., k$. Since $\sum_{i=1}^k \phi_i(x) = 1$ there exists some i such that $\phi_i(x) > 0$, and hence $\phi_i(y) > 0$. It follows that $x, y \in U_i$. Now, $\psi_i(x) = \phi_i(x) \cdot \varphi_i(x) = \psi_i(y) = \phi_i(y) \cdot \varphi_i(y)$. This implies that $\varphi_i(x) = \varphi_i(y)$, but φ_i is injective so it follows that $x = y$. We have shown that F is injective, and hence an embedding. □

8. Homotopy and the fundamental group

In this section we will finally develop the tools necessary to distinguish, topologically, the two-sphere S^2 from the two-torus, T^2. One way to understand this easily makes use of the property of simply connectedness familiar from, perhaps, complex analysis. A space will be called simply connected if every closed curve in the space can be contracted to a point. To make the relevant notion of "contract" precise we must introduce the concept of homotopy. Homotopies of paths then lead to the fundamental group (and the more general fundamental groupoid), which in particular can be used to determine if a space is simply connected or not. The fundamental group(oid) behaves well under homeomorphisms, making it possible to use it as a topological invariant.

8.1. Homotopies of functions and paths

Definition 8.1.

(i) Let $f, g : X \to Y$ be two continuous maps. We say that f and g are *homotopic* if there exists a continuous map

$$F : X \times [0, 1] \to Y, \text{ such that } F(\cdot, 0) = f, \;\&\; F(\cdot, 1) = g.$$

The map F is called a *homotopy*, and we write $f \sim g$ to indicate that f is homotopic to g. If g is a constant map, then f is called *nulhomotopic*.

(ii) Two paths γ, γ' from x to y in X are called *path homotopic* if there exists a homotopy $F : [0, 1] \times [0, 1] \to X$ from γ to γ' with the property that $F(0, t) = x$, $F(1, t) = y$ for every $t \in [0, 1]$. Such a map F is called a *path homotopy*, and we will write $\gamma \sim_p \gamma'$ to indicate that γ and γ' are path homotopic.

■ Note: a path γ can be nulhomotopic only if it its startpoint coincides with its endpoint, $\gamma(0) = \gamma(1)$.

Lemma 8.2. *Homotopy $\sim$ and path homotopy $\sim_p$ are equivalence relations.*

Proof. Fix topological spaces X and Y, and consider continuous maps $f, g, h : X \to Y$. It should be proven that $\sim$ and $\sim_p$ are reflexive, symmetric, and transitive.

(reflexive) Define the map $F : X \times [0,1] \to Y$ by $F : (x,t) \mapsto f(x)$ for every $t \in [0,1]$. It is clear that F is continuous, and $F(x,0) = f(x) = F(x,1)$, so F is indeed a homotopy from f to itself, i.e. $f \sim f$. In case $f = \gamma$ is a path, i.e. $X = [0,1]$, then $F(0,t) = \gamma(0) = x$, $F(1,t) = \gamma(1) = y$ for every $t \in [0,1]$, so $\gamma \sim_p \gamma$ for every path γ.

(symmetric) Assume F is a homotopy from f to g. Define the map $G : X \times [0,1] \to Y$ through $G : (x,t) \mapsto F(x, 1-t)$. We have $G(x,0) = F(x,1) = g(x)$, $G(x,1) = F(x,0) = f(x)$. Moreover, $G = F \circ (id_X \times r)$, where $r : [0,1] \to [0,1]$ is the continuous function $r : (s,t) \mapsto (s, 1-t)$. Clearly, $id_X \times r$ is continuous, so it follows that G is continuous and hence a homotopy from g to f. In case f and g are paths and F a path homotopy, then G is automatically a path homotopy as well.

(transitive) Let F be a homotopy from f to g, and let G be a homotopy from g to h. Define a function $H : X \times [0,1] \to Y$ as

$$H : (x,t) \mapsto \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ G(x, 2t-1) & t \in [1/2, 1]. \end{cases}$$

H is well-defined since $F(x,1) = g(x) = G(x,0)$. Note that $X \times [0,1] = X \times [0,1/2] \cup X \times [1/2, 1] =: U_1 \cup U_2$, where the two subsets are closed. $H|_{U_1}$ and $H|_{U_2}$ are clearly continuous, so by the pasting lemma (Lemma 2.25) it follows that H is continuous. Finally, it is clear that in case F and G are path homotopies, also H is a path homotopy. □

■ If γ is a path, denote its path homotopy class by $[\gamma]$.

Example 8.3. Let $f, g : X \to \mathbb{R}^n$ be two continuous maps. Then $F : X \times [0,1] \to \mathbb{R}^n$ defined by $F : (x,t) \mapsto (1-t)f(x) + tg(x)$ is a homotopy from f to g. In other words, there is only one homotopy class of maps from X to $\mathbb{R}^n$. In case f and g are paths from x to y, then F defined as above is automatically a path homotopy, and it follows that there is only one equivalence class of paths from x to y in $\mathbb{R}^n$. Everything generalises without change if $\mathbb{R}^n$ is replaced with a convex subset of $\mathbb{R}^n$.

Example 8.4. Let γ and γ' be two paths from 1 to -1 in $\mathbb{R}^2 \backslash \{0\}$. More precisely, $\gamma : t \mapsto (\cos \pi t, \sin \pi t)$, and $\gamma' : t \mapsto (\cos \pi t, -\sin \pi t)$. Then $\gamma \not\sim_p \gamma'$, so $[\gamma] \neq [\gamma']$. The proof, however, is non-trivial.

The concatenation of paths induces a product on homotopy classes.

Proposition 8.5. *Let γ be a path from x to y, and let γ' be a path from y to z in a topological space X. Then the operation $[\gamma] \star [\gamma'] := [\gamma \star \gamma']$ on homotopy classes is well defined.*

Proof. Well defined means that the operation on homotopy classes is independent of representatives. Suppose F is a path homotopy from γ to $\widetilde{\gamma}$, and let G be a path homotopy from γ' to $\widetilde{\gamma}'$. We want to prove that $\gamma \star \gamma' \sim_p \widetilde{\gamma} \star \widetilde{\gamma}'$. Define a function $H : [0,1] \times [0,1] \to X$ as

$$H : (s,t) \mapsto \begin{cases} F(2s,t) & s \in [0,1/2] \\ G(2s-1,t) & s \in [1/2,1] \end{cases}.$$

Since $F(1,t) = y = G(0,t)$ for all $t \in [0,1]$, H is well defined on $[0,1] \times [0,1]$. Moreover, continuity of F and G implies that $H|_{[0,1/2]\times[0,1]}$ and $H|_{[1/2,1]\times[0,1]}$ are continuous, so by the pasting lemma (Lemma 2.25) H is continuous. Noting that $H(\cdot,0) = \gamma \star \gamma'$ and $H(\cdot,1) = \widetilde{\gamma} \star \widetilde{\gamma}'$ finishes the proof. □

The proposition above shows that there is partially defined product on homotopy classes of paths (as, of course, on paths themselves). This product satisfies a number of important properties.

Theorem 8.6. *Let X be a topological space. The operation $\star$ on homotopy classes of paths on X has the following properties.*

(*i*) *(Associativity) If $[\gamma] \star ([\gamma'] \star [\gamma''])$ is defined, then so is $([\gamma] \star [\gamma']) \star [\gamma'']$ and they coincide.*

(*ii*) *(Left and right units) For $x \in X$, let e_x denote the constant path at x, i.e. $e_x : s \mapsto x$ for each $s \in [0,1]$. If γ is a path in X from x to y, then*

$$[\gamma] \star [e_y] = [\gamma]$$
$$[e_x] \star [\gamma] = [\gamma].$$

In other words, the homotopy classes of constant paths act as units with respect to the product of homotopy classes.

(*iii*) *(Inverse) Let γ be a path in X from x to y; recall that the inverse path γ^{inv} traces out the same path as γ with the same "speed", but in the opposite direction (i.e. from y to x). We have*

$$[\gamma] \star [\gamma^{\text{inv}}] = [e_x]$$
$$[\gamma^{\text{inv}}] \star [\gamma] = [e_y].$$

In other words, $[\gamma^{\text{inv}}]$ acts as a left and right inverse for $[\gamma]$ with respect to the product $\star$.

■ Due to the last property we will often use the notation $[\gamma]^{-1} = [\gamma^{\text{inv}}]$.

Proof. (*ii*): First, show that $\gamma \star e_y \sim_p \gamma$. Recall that

$$\gamma \star e_y : s \mapsto \begin{cases} \gamma(2s) & s \in [0,1/2] \\ e_y(2s-1) = y & s \in [1/2,1]. \end{cases}$$

Define the function $F : [0,1] \times [0,1] \to X$ as follows

$$F : (s,t) \mapsto \begin{cases} \gamma((2-t)s) & s \in [0, \frac{1}{2-t}] \\ y & s \in [\frac{1}{2-t}, 1]. \end{cases}$$

The function F is well defined since $\gamma(1) = y$. Define $U_1 = \{(s,t) \in [0,1] \times [0,1] | s \in [0, \frac{1}{2-t}]\}$ and $U_2 = \{(s,t) \in [0,1] \times [0,1] | s \in [\frac{1}{2-t}, 1]\}$; both subsets are closed and $X = U_1 \cup U_2$. Note that $F|_{U_1} = \gamma \circ f$, where $f : [0,1] \times [0,1] \to [0,1]$, $(s,t) \mapsto (2-t)s$, is continuous, and $F|_{U_2} = y$, the constant function, is also continuous. By the pasting lemma (Lemma 2.25), F is continuous. Since $F(s,0) = \gamma \star e_y$ and $F(s,1) = \gamma$, it follows that F is a path homotopy from $\gamma \star e_y$ to γ. The proof of $e_x \star \gamma \sim_p \gamma$ is completely analogous.

(iii): Next, show that $\gamma \star \gamma^{\text{inv}} \sim_p e_x$. For each $t \in [0,1]$ define the path $\gamma_t : s \mapsto \gamma(ts)$ from x to $\gamma(t)$. It follows that $\gamma_0 = e_x$ and $\gamma_1 = \gamma$. Define a function $G : [0,1] \times [0,1] \to X$, $(s,t) \mapsto \gamma_t \star \gamma_t^{\text{inv}}(s)$; I claim that G is a homotopy from e_x to $\gamma \star \gamma^{\text{inv}}$. Clearly $G(\cdot, 0) = e_x$, and $G(\cdot, 1) = \gamma \star \gamma^{\text{inv}}$. Writing out the definition of G in detail we get

$$G : (s,t) \mapsto \begin{cases} \gamma_t(2s) = \gamma(2ts) & s \in [0, 1/2] \\ \gamma_t^{\text{inv}}(2s-1) = \gamma(2t(1-s)) & s \in [1/2, 1]. \end{cases}$$

In particular we can write $G = \gamma \circ g$, where $g : [0,1] \times [0,1] \to [0,1]$ is defined as

$$g : (s,t) \mapsto \begin{cases} 2st & s \in [0, 1/2] \\ 2t(1-s) & s \in [1/2, 1] \end{cases}.$$

The function g is continuous by the pasting lemma, so it follows that G is continuous, and hence a homotopy from e_x to $\gamma \star \gamma^{\text{inv}}$. The proof of $\gamma^{\text{inv}} \star \gamma \sim_p e_y$ is completely analogous.

(i): Finally, to show associativity let us first write out explicitly the definition of $\gamma \star (\gamma' \star \gamma'')$ and $(\gamma \star \gamma') \star \gamma''$ respectively.

$$\gamma \star (\gamma' \star \gamma'') : t \mapsto \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \gamma'(4t-2) & t \in [1/2, 3/4] \\ \gamma''(4t-3) & t \in [3/4, 1] \end{cases}$$

$$(\gamma \star \gamma') \star \gamma'' : t \mapsto \begin{cases} \gamma(4t) & t \in [0, 1/4] \\ \gamma'(4t-1) & t \in [1/4, 1/2] \\ \gamma''(2t-1) & t \in [1/2, 1] \end{cases}$$

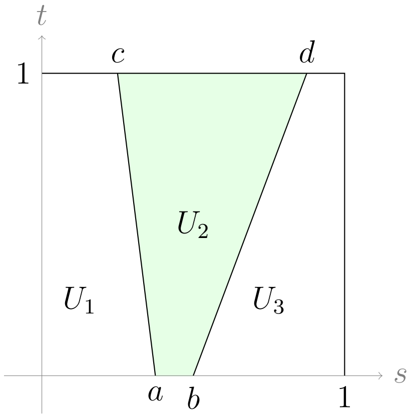
For any numbers a, b such that $0 < a < b < a$ define the function $\kappa : [0,1] \to X$ as

$$\kappa_{a,b} : t \mapsto \begin{cases} \gamma(\frac{1}{a}t) & t \in [0, a] \\ \gamma'(\frac{1}{b-a}(t-a)) & t \in [a, b] \\ \gamma''(\frac{1}{1-b}(t-b)) & t \in [b, 1]. \end{cases}$$

Now, let (a, b) and (c, d) be two pairs of numbers such that $0 < a < b < 1$ and $0 < c < d < 1$, and define a function $H_{(a,b)}^{(c,d)} : [0, 1] \times [0, 1] \to X$ as follows.

$$H_{(a,b)}^{(c,d)} : (s, t) \mapsto \kappa_{t(c-a)+a,t(d-b)+b}(s).$$

This is well defined since for any $t \in [0, 1]$ we have $t(d - b) + b - t(c - a) + a = t(d-c)+(1-t)(b-a) > 0$. Divide the square $[0, 1] \times [0, 1]$ into three overlapping closed subsets, U_1, U_2, and U_3, as indicated in the following figure.



We see that $H_{(a,b)}^{(c,d)}|_{U_i}$ is continuous for $i = 1, 2, 3$. By the pasting lemma (Lemma 2.25) the function $H_{(a,b)}^{(c,d)}$ is continuous. Since $H_{(a,b)}^{(c,d)}(s, 0) = \kappa_{a,b}$ and $H_{(a,b)}^{(c,d)}(s, 1) = \kappa_{c,d}(s)$ we have constructed a homotopy from $\kappa_{a,b}$ to $\kappa_{c,d}$. It is quickly verified that $\kappa_{1/2,3/4} = \gamma \star (\gamma' \star \gamma'')$, and $\kappa_{1/4,1/2} = (\gamma \star \gamma') \star \gamma''$. Finally, $H_{(a,b)}^{(c,d)}(0, t) = \gamma(0)$, $H_{(a,b)}^{(c,d)}(1, t) = \gamma''(1)$, so $H_{(a,b)}^{(c,d)}$ is a path homotopy, and we have shown associativity. □

8.2. The fundamental groupoid and fundamental group

The properties of Theorem 8.6 are precisely such that they define a structure known as a groupoid. Let us first define the notion of a groupoid abstractly, and then describe the groupoid we get by homotopy classes of paths on topological spaces, the so called fundamental groupoid.

Definition 8.7. A (small) *groupoid* $\mathscr{G}$ consists of a collection of data satisfying certain axioms. The data of $\mathscr{G}$ is

GD1 A set that we will, abusing notation, also denote by $\mathscr{G}$. (Objects)

GD2 For each $(x, y) \in \mathscr{G} \times \mathscr{G}$, a set $\mathscr{G}(x, y)$. (Arrows from x to y)

GD3 For each $x \in \mathscr{G}$, a distinguished element $e_x \in \mathscr{G}(x, x)$ (Identity)

GD4 For each $(x, y) \in \mathscr{G} \times \mathscr{G}$, a bijection $\mathscr{G}(x, y) \xrightarrow{\sim} \mathscr{G}(y, x)$, $g \mapsto g^{-1}$. (Inverse)

GD5 For each $(x, y, z) \in \mathscr{G} \times \mathscr{G} \times \mathscr{G}$, a function $\cdot : \mathscr{G}(x, y) \times \mathscr{G}(y, z) \to \mathscr{G}(x, z)$. (Product, or Composition)

These data are subject to the following axioms

GA1 For each $g \in \mathscr{G}(x,y)$, $h \in \mathscr{G}(y,z)$, $k \in \mathscr{G}(z,w)$, we have $g \cdot (h \cdot k) = (g \cdot h) \cdot k$. (Associativity)

GA2 For each $g \in \mathscr{G}(x,y)$, $e_x \cdot g = g = g \cdot e_y$. (Left and right unit property)

GA3 For each $g \in \mathscr{G}(x,y)$, $g \cdot g^{-1} = e_x$ and $g^{-1} \cdot g = e_y$.

Remark 8.8.

(*i*) As the prefix "small" indicates, this is not the most general type of groupoid that we have defined. A reader who is versed in category theory will notice that the structure we have defined is nothing but a small category whose arrows are all invertible. A general groupoid is then nothing but a (not necessarily small) category whose arrows are all invertible.

(*ii*) Note that $\mathscr{G}(x,y) = \emptyset$ is allowed by the definition when $x \neq y$. A groupoid $\mathscr{G}$ such that $\mathscr{G}(x,y) \neq \emptyset$ for all $x, y \in \mathscr{G}$ is called *connected*. If $\mathscr{G}$ is not connected, it is called disconnected.

The notion of a groupoid encompasses that of a group.

Definition 8.9. A *group* is is a groupoid $\mathscr{G}$ with one object, i.e. $\mathscr{G} = \{*\}$. This means that the structure of a group lies in its only set of arrows $\mathscr{G}(*,*)$, and their product. The traditional definition of a group, which is completely equivalent, states that a group consists of the following data

GrD1 A set G.

GrD2 A unit element $e \in G$.

GrD3 A bijection of G, $g \mapsto g^{-1}$.

GrD4 A product $m : G \times G \to G$.

It is customary to abbreviate the product as follows $gh := m(g,h)$ for $g, h \in G$. These data are subject to the following axioms

GrA1 $g(hk) = (gh)k$, $\forall g, h, k \in G$

GrA2 $eg = g = ge$, $\forall g \in G$

GrA3 $gg^{-1} = e = g^{-1}g$, $\forall g \in G$

It should be completely obvious that the two definitions of a group are equivalent.

Remark 8.10. If $\mathscr{G}$ is a groupoid and $x \in \mathscr{G}$, then the set $\mathscr{G}(x,x)$ carries a natural structure of a group. A groupoid can be understood as a collection of groups together with collections of "arrows" between pairs of groups.

Example 8.11.

(*i*) The integers $\mathbb{Z}$ forms a group with the product being addition, the unit $0 \in \mathbb{Z}$, and the inverse being $n \mapsto -n \in \mathbb{Z}$. The same product, unit, and inverse gives $\mathbb{Q}$, $\mathbb{R}$, ad $\mathbb{C}$ structures of groups. All of these groups have the property that $xy = yx$ for all elements x and y; they are called *Abelian*

(*ii*) The set $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ with multiplication, 1, and $z \mapsto 1/z$ is a group. So is $\mathbb{R}^\times := \mathbb{R}\backslash\{0\}$.

(*iii*) The set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n is a group with addition.

(*iv*) The set $GL(n, \mathbb{R})$ of invertible $(n \times n)$-matrices with matrix multiplication forms a group. The unit is the $(n \times n)$-unit matrix.

Definition 8.12. Let X be a topological space.

(*i*) The *fundamental groupoid* $\Pi(X)$ of X has X itself as the set of objects, and for $x, y \in X$, the set of arrows $\Pi(X)(x, y)$ is the set of homotopy classes of paths from x to y, i.e. $\Pi(X)(x, y) = \{[\gamma] | \gamma \text{ is a path from } x \text{ to } y\}$. The product is the product $\star$, whereas the unit and inverse bijections are of course $[e_x]$ for $x \in X$ respectively $[\gamma] \mapsto [\gamma]^{-1} = [\gamma^{\mathrm{inv}}]$. It follows immediately from Theorem 8.6 that $\Pi(X)$ is a groupoid.

(*ii*) Let $x \in X$. The *fundamental group* $\pi_1(X, x)$ of X with base point x is the group $\Pi(X)(x, x)$ of homotopy classes of closed paths starting and ending in x.

Let us introduce the natural notion of maps between groups, in particular the natural notion of isomorphism.

Definition 8.13.

(*i*) Let G and H be groups. A map $\varphi : G \to H$ is called a *group homomorphism* if $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$, i.e. if φ is compatible with the products of G and H. One can show that the definition implies that $\varphi(e_G) = e_H$, and that $\varphi(g^{-1}) = \varphi(g)^{-1}$. A bijective group homomorphism is called a group isomorphism, and we write $G \simeq H$ if there exists an isomorphism from G to H.

(*ii*) Let $\mathscr{G}$ and $\mathscr{H}$ be groupoids. A *groupoid homomorphism* consists of a map $\Phi : \mathscr{G} \to \mathscr{H}$, and for each $x, y \in \mathscr{G}$ a map $\Phi : \mathscr{G}(x, y) \to \mathscr{H}(\Phi(x), \Phi(y))$, satisfying $\Phi(g \cdot h) = \Phi(g) \cdot \Phi(h)$ for every $g \in \mathscr{G}(x, y)$ and $h \in \mathscr{G}(y, z)$. One can show that $\Phi(e_x) = e_{\Phi(x)}$, and $\Phi(g^{-1}) = \Phi(g)^{-1}$, for $g \in \mathscr{G}(x, y)$. A groupoid homomorphism that is bijective on objects (i.e. $\Phi : \mathscr{G} \to \mathscr{H}$ is a bijection) and arrows (i.e. $\Phi : \mathscr{G}(x, y) \to \mathscr{H}(\Phi(x), \Phi(y))$ is a bijection for all $x, y \in \mathscr{G}$) is called a groupoid isomorphism.

Remark 8.14. If $\Phi : \mathscr{G} \to \mathscr{H}$ is a groupoid homomorphism (isomorphism), then for every $x \in \mathscr{G}$ the map $\varphi_x := \Phi : \mathscr{G}(x, x) \to \mathscr{H}(\Phi(x), \Phi(x))$ is a group homomorphism (isomorphism).

Proposition 8.15. *Let $\mathscr{G}$ be a groupoid and $x, y \in \mathscr{G}$ two objects such that $\mathscr{G}(x, y) \neq \emptyset$. Then the groups $\mathscr{G}(x, x)$ and $\mathscr{G}(y, y)$ are isomorphic.*

Proof. Let $f \in \mathscr{G}(x, y)$, and define a map $\varphi_f : \mathscr{G}(x, x) \to \mathscr{G}(y, y)$ by $\varphi_f : g \mapsto f^{-1} \cdot g \cdot f$. We will show that φ_f is a group isomorphism. Let $g, h \in \mathscr{G}(x, x)$, then

$$\varphi_f(g \cdot h) = f^{-1} \cdot g \cdot h \cdot f = f^{-1} \cdot g \cdot f \cdot f^{-1} \cdot h \cdot f = \varphi_f(g) \cdot \varphi_f(h),$$

so φ_f is a group homomorphism. Furthermore, the map $\varphi_{f^{-1}}$ maps $\mathscr{G}(y, y)$ to $\mathscr{G}(x, x)$, and it is easily verified that $\varphi_f \circ \varphi_{f^{-1}} = id_{\mathscr{G}(y,y)}$, $\varphi_{f^{-1}} \circ \varphi_f = id_{\mathscr{G}(x,x)}$, so it follows that φ_f is a bijection, hence an isomorphism. □

Corollary 8.16. *If X is a path connected topological space, then $\pi_1(X, x) \simeq \pi_1(X, y)$ for any two base points $x, y \in X$. This observation justifies the name "the" fundamental group.*

■ If the groupoid $\mathscr{G}$ is connected, then all groups $\mathscr{G}(x, x)$ are isomorphic. If $\mathscr{G}$ is disconnected, then every two $\mathscr{G}(x, x)$ and $\mathscr{G}(y, y)$ such that $\mathscr{G}(x, y) \neq \emptyset$ (we say that x and y are connected) are isomorphic. Moreover, if the objects x, y, z are connected, then $\mathscr{G}(x, y)$ and $\mathscr{G}(y, z)$ are in bijection. It could therefore make sense to consider a weaker form of equivalence between groupoids than isomorphism, since the "essential" information in a groupoid is largely contained in the groups $\mathscr{G}(x, x)$. A groupoid homomorphism $\Phi : \mathscr{G} \to \mathscr{H}$ such that for every object $y \in \mathscr{H}$ there exists an object $x \in \mathscr{G}$ such that $\mathscr{H}(\Phi(x), y) \neq \emptyset$, and such that the map $\Phi : \mathscr{G}(x, y) \to \mathscr{H}(\Phi(x), \Phi(y))$ is a bijection for every $x, y \in \mathscr{G}$, is called an equivalence of groupoids. Equivalence is more central than isomorphism for groupoids, but it will not play an important role in in this course.

We can now state precisely what we mean by a topological space X being simply connected.

Definition 8.17. Let X be a path connected topological space, and let $x \in X$. If the fundamental group $\pi_1(X, x)$ is trivial, i.e. consists only of the unit element $[e_x]$, then X is called *simply connected*.

Example 8.18. It follows immediately from Example 8.3 that the fundamental group $\pi_1(\mathbb{R}^n, x) = \{[e_x]\}$ of $\mathbb{R}^n$ (with base point x) is trivial, thus $\mathbb{R}^n$ is simply connected.

Let us now show some of the main properties of the fundamental group(oid).

Theorem 8.19. *Let $f : X \to Y$ be a continuous map.*

(i) The map $f_* : \Pi(X) \to \Pi(Y)$ defined as f on objects, and as $[\gamma] \mapsto [f \circ \gamma]$ on arrows (i.e. homotopy classes of paths) is a groupoid homomorphism.

(ii) If f is a homeomorphism, then f_* is a groupoid isomorphism.

Proof.

(i): Let us first check that f_* is well defined. Suppose F is a path homotopy from γ to γ', both of which are paths in X. In other words, $F : [0,1] \times [0,1] \to X$ is continuous with the properties $F(\cdot, 0) = \gamma$, $F(\cdot, 1) = \gamma'$, $F(0,t) = \gamma(0)$, $F(1,t) = \gamma(1)$ for all $t \in [0,1]$. The function $f \circ F : [0,1] \times [0,1] \to Y$ is continuous with the properties $(f \circ F)(\cdot, 0) = f \circ \gamma$, $(f \circ F)(\cdot, 1) = f \circ \gamma'$, $(f \circ F)(0,t) f \circ \gamma(0)$, $(f \circ F)(1,t) = f \circ \gamma(1)$, i.e. $f \circ F$ is a homotopy from $f \circ \gamma$ to $f \circ \gamma'$. Thus f_* is a well defined function on homotopy classes.

Next, consider a concatenated path $\gamma \star \gamma'$, and its homotopy class $[\gamma \star \gamma']$, in X. By definition of the concatenation we have

$$\gamma \star \gamma' : t \mapsto \begin{cases} \gamma(2t) & t \in [0, 1/2] \\ \gamma'(2t-1) & t \in [1/2, 1] \end{cases},$$

so we have for $f \circ (\gamma \star \gamma')$:

$$(f \circ (\gamma \star \gamma')) : t \mapsto \begin{cases} f \circ \gamma(2t) & t \in [0, 1/2] \\ f \circ \gamma'(2t-1) & t \in [1/2, 1] \end{cases}.$$

The right hand side just shows that $f \circ (\gamma \star \gamma') = (f \circ \gamma) \star (f \circ \gamma')$. It follows that

$$f_*([\gamma] \star [\gamma']) = f_*([\gamma \star \gamma']) = [f \circ (\gamma \star \gamma')] = [f \circ \gamma] \star [f \circ \gamma'] = f_*([\gamma]) \star f_*([\gamma']).$$

The map f_* is thus a groupoid homomorphism.

(ii): If f is a homeomorphism, then $f : X \to Y$ is a bijection, and the map $f^{-1} : Y \to X$ is continuous so we can construct the groupoid homomorphism $(f^{-1})_* : \Pi(Y) \to \Pi(X)$. It is easily checked that $(f^{-1})_* = f_*^{-1}$ when restricted to any $\Pi(X)(x,y)$ or $\Pi(Y)(f(x), f(y))$. It follows that f_* is bijective on objects and arrows, hence a groupoid isomorphism. □

Corollary 8.20. *By Remark 8.14 it follows that f_* restricts to group homomorphisms of fundamental groups, from $\pi_1(X,x)$ to $\pi_1(Y, f(x))$, which are isomorphisms if f is a homeomorphism.*

The following property, expressing so called functoriality of the fundamental groupoid construction, will not be of immediate use to us, but we include it due to its importance.

Lemma 8.21. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $(g \circ f)_* = g_* \circ f_*$. Moreover, $(id_X)_* = id_{\Pi(X)}$ (the identity groupoid homomorphism).*

Proof. Trivial (but check that you understand why it is trivial!). □

8.3. Covering spaces and examples

We will finally determine some examples of fundamental groups and groupoids. To do so it will be necessary to introduce the language of covering spaces and covering maps. Let us first, however, give a result that is intuitively obvious.

Proposition 8.22. *If $n \geq 2$ then the n-sphere S^n is simply connected, i.e. for any $p \in S^n$ we have $\pi_1(S^n, p) = \{[e_p]\}$.*

Sketch of proof. Let γ be a loop (based in $\gamma(0)$) in S^n, and suppose that $p \in S^n \backslash \text{Im}(\gamma)$. We know that $S^n \backslash \{p\} \simeq \mathbb{R}^n$, which is simply connected. Hence γ is nulhomotopic in $S^n \backslash \{p\} \subset S^N$, and hence also in S^n. This argument is not a complete proof, however, since there may be space-filling curves such that $\text{Im}(\gamma) = S^n$. Let $p \in S^n$ be an arbitrary point distinct from $\gamma(0)$, and let U be a neighbourhood of p that does not contain $\gamma(0)$ (this exists since S^n is T_1). Then $\gamma^{-1}(U) \subset (0,1) \subset [0,1]$ is open, and therefore a union of open intervals. Let $\{(a_i, b_i)\}_{i \in I}$ denote this collection of open intervals. Since $\{p\} \subset S^n$ is closed and since $[0,1]$ is compact, the subspace $\gamma^{-1}(p) \subset (0,1) \subset [0,1]$ is compact. Moreover, $\gamma^{-1}(p) \subset \bigcup_{i \in I}(a_i, b_i)$, so there is a finite subset (a_1, b_1), ..., (a_k, b_k) whose union contains $\gamma^-1(p)$. Let (a_i, b_i) be an interval such that $p \in \gamma((a_i, b_i))$, and denote by γ_i the restriction of γ to $[a_i, b_i]$. Then $f([a_i, b_i]) \subset \overline{U}$, and $\gamma(a_i), \gamma(b_i) \in \partial U$. Without loss of generality we may assume that U is homeomorphic to an open ball in $\mathbb{R}^n$. This is a convex subspace, and we may therefore use a straight line homotopy (as in Example 8.3) to homotopy γ_i away from p. This homotopy can be extended to a homotopy from γ. Iterate this procedure for each $i = 1, ..., k$ to get a homotopy from γ to γ' s.t. $p \notin \text{Im}(\gamma')$. □

■ A result known as the van Kampen theorem simplifies the proof of this proposition considerably. This theorem, however, would lead us further into group theory, and we will therefore not include it here.

Let us now turn to covering spaces with the goal of determining the fundamental group of S^1.

Definition 8.23. A continuous surjection $p : E \to B$ is called a *covering map* if every $b \in B$ has a neighbourhood U_b with the property $p^{-1}(U_b) = \sqcup_{\alpha \in A_b} V_\alpha$ where $p|_{V_\alpha} : V_\alpha \to U_b$ is a homeomorphism, i.e. the pre image of U_b is a disjoint union of open sets V_α, each of which is homeomorphic (via the restriction of p) to U_b. E is then called a *covering space* of B.

Example 8.24. The map $p : \mathbb{R} \to S^1$, $t \mapsto (\cos 2\pi t, \sin 2\pi t)$, is a covering map, and $\mathbb{R}$ is a covering space of S^1. Take for instance the point $(1,0) \in S^1$, and a neighbourhood U in the form of an open circle-interval containing $(1,0)$. The pre image $p^{-1}(U)$ is then a union of an infinite family of open intervals in $\mathbb{R}$, each being a neighbourhood of a single integer.

Theorem 8.25. $\pi_1(S^1, (1,0)) \simeq \mathbb{Z}$

To prove this theorem we need some preparation, more precisely we need results concerning liftings of maps to S^1.

Definition 8.26. Let $p : E \to B$ be a covering map, and let $f : X \to B$ be a continuous map. A map $\tilde{f} : X \to E$ such that the following diagram commutes

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

is called a *lifting* of f. In other words, a lifting $\tilde{f}$ of f satisfies $f = p \circ \tilde{f}$.

There are the following strong results on liftings of paths and homotopies.

Lemma 8.27 (Path lifting lemma)**.** *Let $p : E \to B$ be a covering map, and let $\gamma : [0,1] \to B$ be a path in B starting at $\gamma(0) = b \in B$. For any $e \in p^{-1}(\{b\})$ there is a unique lifting $\widetilde{\gamma}$ of γ starting at $\widetilde{\gamma}(0) = e$.*

Lemma 8.28 (Homotopy lifting lemma)**.** *Let $p : E \to B$ be a covering map, and let $F : [0,1] \times [0,1] \to B$ be a continuous map such that $F(0,0) = b \in B$. For any $e \in p^{-1}(\{b\})$ there is a unique lifting $\widetilde{F}$ of F with the property $\widetilde{F}(0,0) = e$. If F is a path homotopy, then $\widetilde{F}$ is a path homotopy.*

Definition 8.29. Let $p : E \to B$ be a covering map, and let $b \in B$. For each $e \in p^{-1}(\{b\})$ we define a map $\Phi_e : \pi_1(B, b) \to p^{-1}(\{b\})$, called a *lifting correspondence*, as follows. For a loop γ based in b representing $[\gamma] \in \pi_1(B, b)$, let $\widetilde{\gamma}$ be the unique lifting of γ with starting point e. The map Φ_e is defined as $\Phi_e : [\gamma] \mapsto \widetilde{\gamma}(1) \in p^{-1}(b)$. To see that Φ_e is well defined, let F be a path homotopy from γ to γ'. By the Homotopy lifting lemma (Lemma 8.28) there exists a unique lifting $\widetilde{F}$ of F such that $\widetilde{F}(0,0) = e$, and $\widetilde{F}$ is a path homotopy. It is quickly verified that $\widetilde{F}(\cdot, 0) = \widetilde{\gamma}$ and $\widetilde{F}(\cdot, 1) = \widetilde{\gamma}'$, so $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are path homotopic, and thus have the same end point $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$. The map Φ_e is therefore well defined.

Proposition 8.30. *Let $p : E \to B$ be a covering map, let $b \in B$, and let $e \in p^{-1}(\{b\})$. If E is path connected, then Φ_e is surjective. If E is simply connected, then Φ_e is bijective.*

Proof. Suppose E is path connected. For any $q \in p^{-1}(\{b\})$ we can then choose a path $\widetilde{\gamma}_q$ from e to q. Then $\widetilde{\gamma}_q$ is a lifting of $p \circ \widetilde{\gamma}_q$, which is a loop based in b. Thus $\Phi_e : [p \circ \widetilde{\gamma}_q] \mapsto q$, and it follows that Φ_e is surjective. Suppose that E is simply connected, i.e. path connected with trivial fundamental group. Then there is only one path homotopy class of paths from e to any $q \in p^{-1}(\{b\})$. To see this, let γ and γ' be two paths from e to q. Then $\gamma \star \gamma'^{\text{inv}}$ is a loop based in e which is therefore homotopic to the constant map, and thus $[\gamma] = [\gamma] \star ([\gamma']^{-1} \star [\gamma']) = [\gamma \star \gamma'^{\text{inv}}] \star \gamma' = [\gamma']$. Suppose $\Phi_e([\gamma]) = \Phi_e([\gamma'])$, then $\widetilde{\gamma}(1) = \widetilde{\gamma}'(1)$, where the latter liftings both start at e. It follows that they are homotopic; let $\widetilde{F}$ be a path homotopy from $\widetilde{\gamma}$ to $\widetilde{\gamma}'$. Then $F := p \circ \widetilde{F}$ is a path homotopy from $p \circ \widetilde{\gamma} = \gamma$ to $p \circ \widetilde{\gamma}' = \gamma'$, hence Φ_e is injective. □

Proof of Theorem 8.25. Consider the covering map $p : \mathbb{R} \to S^1$ from Example 8.24; we have $p^{-1}(\{(1,0)\}) = \mathbb{Z} \subset \mathbb{R}$. Since $\mathbb{R}$ is simply connected, Proposition 8.30 implies that $\Phi_e : \pi_1(S^1, (1,0)) \to p^{-1}(\{(1,0)\})$ is bijective. For any $m \in \mathbb{N}$ define the path $\gamma_m : t \mapsto (\cos 2\pi mt, \sin 2\pi mt)$, it has a lifting $\widetilde{\gamma}_m : t \mapsto mt$ starting in 0. By the uniqueness of the lifting (Path lifting lemma) it follows that $\Phi_0 : [\gamma_m] \mapsto m$. We conclude that every element in the fundamental group based in $(1,0)$ lies in the homotopy class of some γ_m.

Let us finally show that Φ_0 is a group homomorphism (hence isomorphism). For $m, n \in \mathbb{Z}$ consider $[\gamma_m] \star [\gamma_n] = [\gamma_m \star \gamma_n]$. We can explicitly construct the unique lift of $\gamma_m \star \gamma_n$ starting in 0. Let $\widetilde{\gamma}_n^m$ be the unique lift of γ_n starting in m, i.e. mapping $t \mapsto nt + m$; the path $\widetilde{\gamma}_m \star \widetilde{\gamma}_n^m$ is a path from 0 to $m+n$. Applying the projection p we get $p \circ (\widetilde{\gamma}_m \star \widetilde{\gamma}_n^m) = (p \circ \widetilde{\gamma}_m) \star (p \circ \widetilde{\gamma}_n^m) = \gamma_m \star \gamma_n$. We have shown that $\widetilde{\gamma}_m \star \widetilde{\gamma}_n^m$ is the unique lifting of $\gamma_m \star \gamma_n$ starting in 0, and therefore $\Phi_0([\gamma_m] \star [\gamma_n]) = m+n = \Phi_0([\gamma_m]) + \Phi_0([\gamma_n])$; Φ_0 is a group isomorphism. □

In order to prove the two lemmas 8.27 and 8.28 let us first prove a result that we so far neglected, the "Lebesgue number lemma":

Lemma 8.31 (Lebesgue number lemma)**.** *Let (X, d) be a compact metric space, and let $\mathscr{U}$ be an open covering of X. Then there exists a $\delta > 0$ such that every subset $A \subset X$ of diameter (the diameter of A is the supremum of the set of distances between points in A) less than δ is contained in some $U \in \mathscr{U}$. Such a number δ is called a* Lebesgue number *for the covering $\mathscr{U}$.*

Proof. Let $\mathscr{U}$ be an open covering of X. If $X \in \mathscr{U}$, then any positive number is a Lebesgue number for $\mathscr{U}$; assume therefore that $X \notin \mathscr{U}$. Let $\{U_1, ..., U_n\}$ be af finite subcover of $\mathscr{U}$, and define $C_i := X \backslash U_i$, $i = 1, ..., n$, as well as $f : X \to \mathbb{R}$ by the following expression.

$$f : x \mapsto \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

Recall that $d(x, C_i) := \inf\{d(x,y) | y \in C_i\}$. Let $x \in X$ and choose i such that $x \in U_i$. Let $\epsilon > 0$ be such that $d(x, \epsilon) \subset U_i$, then $d(x, C_i) \geq \epsilon$ so $f(x) \geq \epsilon/n$. It is easily shown that $d(x, A)$ is continuous for any $A \subset X$, and it follows that f is continuous. But since X is compact, $\mathrm{Im}(f) = f(X)$ is then compact and bounded from below (as just shown), so $\mathrm{Im}(f)$ must have a minimum $\delta > 0$. Let $A \subset X$ have diameter less than δ and choose $a \in A$, then $A \subset Y_a := \{y \in X | d(a, y) < \delta\}$. Let $m \in \{1, ..., n\}$ be the index such that $\max\{d(a, C_1), d(a, C_2), ..., d(a, C_n)\}$, then $\delta \leq f(a) \leq d(a, C_m)$. It follows that $Y_a \subset U_m$, so $A \subset U_m$. In other words, we have found a Lebesgue number δ for $\mathscr{U}$. □

Proof of Lemma 8.27. Let $\mathscr{U}$ be an open covering of B by open sets U such that $p^{-1} = \sqcup_{\alpha \in A} V_\alpha$, and where $p|_{V_\alpha}$ is a homeomorphism for every $\alpha \in A$. The collection $\{\gamma^{-1}(U) | U \in \mathscr{U}\}$ is an open covering of $[0,1]$; choose a finite subcover $\{\gamma^{-1}(U_1), ..., \gamma^{-1}(U_n)\}$. By the Lebesgue number lemma (Lemma 8.31) it is possible to choose a subdivision

$0 = s_0 < s_1 < s_2 < ... < s_k = 1$ of $[0, 1]$ such that for each $i = 1, ..., k$, $[s_{i-1}, s_i] \subset \gamma^{-1}(U_j)$ for some $j = 1, ..., n$. In other words, $\gamma([s_{i-1}, s_i]) \subset U_j$.

Choose $e \in p^{-1}(\{b\})$, and define the lift $\widetilde{\gamma}$ step by step in the subdivision. First, say $\widetilde{\gamma}(0) = e$, and suppose that $\widetilde{\gamma}(s)$ has been defined for $s \in [0, s_i]$. Define $\widetilde{\gamma}$ on $[s_i, s_{i+1}]$ as follows. We know that $\gamma([s_i, s_{i+1}]) \subset U_j$ for some $j = 1, ..., n$, and that $p^{-1}(U_j) = \sqcup_{\alpha \in A} V_\alpha$ with $p|_{V_\alpha}$ a homeomorphism. There exists one $\alpha \in A$ such that $\widetilde{\gamma}(s_i) \in V_\alpha$; define $\widetilde{\gamma}$ for $s \in [s_i, s_{i+1}]$ by

$$\widetilde{\gamma}(s) := (p|_{V_\alpha})^{-1} (\gamma(s)).$$

With this definition, $\widetilde{\gamma}$ is continuous on each $[s_i, s_{i+1}]$, and by the pasting lemma (Lemma 2.25) it follows that $\widetilde{\gamma}$ is continuous, and therefore a path. By construction, $\widetilde{\gamma}$ is a lift of γ. One shows that the so constructed lift is unique by assuming that there is another lift of γ starting from the same point e, and then showing that the two lifts must coincide for any $s \in [s_i, s_{i+1}]$, and for every $i = 0, ..., n-1$. This is left as an exercise. □

Proof of Lemma 8.28. Define first $\widetilde{F}(0, 0) := e$, and use the path lifting lemma (Lemma 8.27) to extend $\widetilde{F}$ to $\{0\} \times [0, 1]$ as well as to $[0, 1] \times \{0\}$. We continue as follows. It follows again by the Lebesgue number lemma (Lemma 8.31) that it is possible to choose subdivisions $0 = s_0 < s_1 < ... < s_m = 1$, $0 = t_0 < t_1 < ... < t_n = 1$ such that each rectangle $R_{i,j} := [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into an open set U of B such that $p^{-1}(U) = \sqcup_{\alpha \in A} V_\alpha$ where $p|_{V_\alpha}$ is a homeomorphism. Define again $\widetilde{F}$ step by step, starting with the rectangle $R_{1,1}$ and continuing with the rectangles $R_{i,1}$, $i = 2, ..., m$, after which we continue with the rectangles $R_{i,2}$ etc. Assume that $\widetilde{F}$ has been extended to the set A consisting of all rectangles "before" the rectangle R_{i_0,j_0}, as well as on $\{0\} \times [0, 1]$ and $[0, 1] \times \{0\}$. Let $U \subset B$ be an open set of the type described earlier and containing $F(R_{i_0,j_0})$; $\widetilde{F}$ has already been defined on $C = A \cap R_{i_0,j_0}$, which is connected and must therefore be contained in one of the slices V_α. Let $p_\alpha := p_{V_\alpha}$, and define $\widetilde{F}|_{R_{i_0,j_0}} := p_\alpha^{-1} \circ F|_{R_{i_0,j_0}}$. This defines $\widetilde{F}$ on $[0, 1] \times [0, 1]$. We leave it as a simple exercise to check the uniqueness, and that it is a path homotopy if F is a path homotopy. □

A. Regarding completeness of $\mathbb{R}$

There are in fact many equivalent ways to express the completeness of $\mathbb{R}$. In these notes we have tacitly assumed that the real numbers have the least upper bound property, and as a consequence we can prove that they are complete.

Proposition A.1. *The real numbers are complete, i.e. every Cauchy sequence in $\mathbb{R}$ is convergent.*

Proof. Note first that if $\{x_n\}$ is a bounded monotone non-decreasing sequence, then $\lim x_n = \sup\{x_n\}$; analogously, if $\{x_n\}$ is a bounded monotone non-increasing sequence, then $\lim x_n = \inf\{x_n\}$. Therefore every bounded monotone sequence in $\mathbb{R}$ is convergent. Let $\{x_n\}$ be a bounded sequence, and define $y_m := \sup_{n>m}\{x_n\}$, $z_m := \inf_{n>m}\{x_n\}$. Clearly the resulting sequences are monotone and bounded ($\{y_n\}$ is non-increasing, while $\{z_n\}$ is non-decreasing), so the corresponding limits $\limsup x_n$ and $\liminf x_n$ exist.

If $\{x_n\}$ is a Cauchy sequence, then it is bounded: Choose $\epsilon > 0$, then there exists an integer $N > 0$ such that $|x_n - x_m| < \epsilon$ for every $m, n > N$. For $n > N$ it follows that

$$|x_n| - |x_{N+1}| \leq |x_n - x_{N+1}| < \epsilon \Rightarrow |x_n| < |x_{N+1}| + \epsilon.$$

Defining $m_0 := \max\{|x_1|, |x_2|, ..., |x_N|\}$ we conclude that $|x_n| \leq \max\{m_0, |x_{N+1}| + \epsilon\}$ for every $n \in \mathbb{N}$, so $\{x_n\}$ is bounded.

Let $\{x_n\}$ be a Cauchy sequence and define $S := \limsup x_n$; we then also have $S = \inf\{y_n\}$. For every $\epsilon > 0$ there exists some positive integer M such that $S \leq y_n \leq S + \epsilon$ for every $n > M$. Since $\{x_n\}$ is Cauchy, there exists a positive integer M' such that $|x_k - x_l| < \epsilon$ for every $k, l > M'$, and in particular it follows that $|y_k - x_l| < \epsilon$ for $k, l > M'$. Let $N = \max\{M + 1, M' + 1\}$, then

$$|x_k - S| \leq |x_k - x_N| + |x_N - y_N| + |y_N - S| < 3\epsilon \text{ for } k > N.$$

It follows that $x_n \to S$, so every Cauchy sequence is convergent.

□

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