Quantum representations of mapping class groups with a view towards surface dynamics

Uppsala universitet

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Notation

- Let $\Sigma = \Sigma_{g,n}$ denote a closed oriented surface of genus g with n marked points.

Definition

The mapping class group of Σ is

$$\mathrm{MCG}(\Sigma) = \mathrm{Homeo}^{+}(\Sigma)/\text{isotopy}.$$

Homeomorphisms are assumed to preserve the set of marked points.

Example

- $\mathrm{MCG}(\Sigma_{0,0}) = \{\mathrm{id}\}$.
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- Let $G = \mathrm{SU}(N)$, and let M be an (oriented connected framed) closed 3-manifold.
- Let $\mathcal{A} \cong \Omega^1(M, \mathfrak{g})$ be the space of connections in $G \times M \to M$, and let $\mathcal{G} \cong C^\infty(M, G)$ be the group of gauge transformations acting on $\mathcal{A}$.
- Define the Chern–Simons functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

- For $g \in \mathcal{G}$, we have $\mathrm{CS}(g^*A) - \mathrm{CS}(A) \in \mathbb{Z}$, and we can consider

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The Chern–Simons partition function

- Let $k \in \mathbb{N}$ (called the *level*) and define the *Chern–Simons partition function*

$$Z_k^{\mathrm{phys}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \,\mathrm{CS}(A)} \mathcal{D}A \in \mathbb{C}.$$

- Assume that M contains a framed oriented link L, and choose for every component L_i of L a finite dimensional representation R_i of $G = \mathrm{SU}(N)$. Set

$$Z_k^{\mathrm{phys}}(M, L, R) = \int_{\mathcal{A}/\mathcal{G}} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) e^{2\pi i k \,\mathrm{CS}(A)} \mathcal{D}A.$$

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A possible construction

Theorem (Reshetikhin–Turaev, 1991)

One can construct a topological invariant Z_k of 3-manifolds, called the quantum invariant, which behaves under gluing (or surgery) the way Z_k^{phys} is supposed to do.

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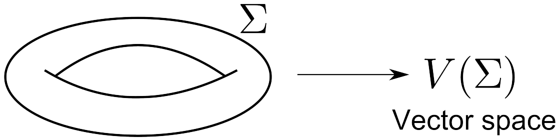

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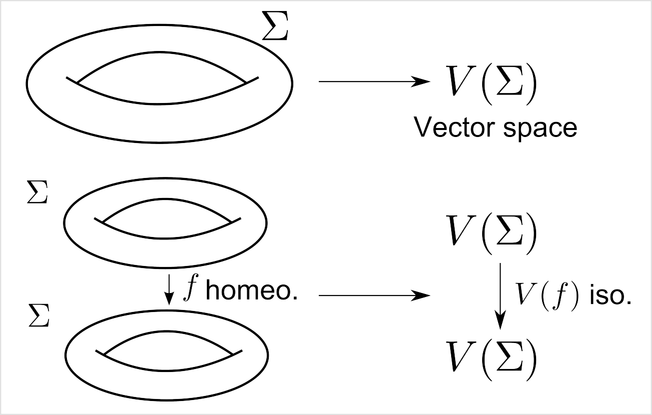

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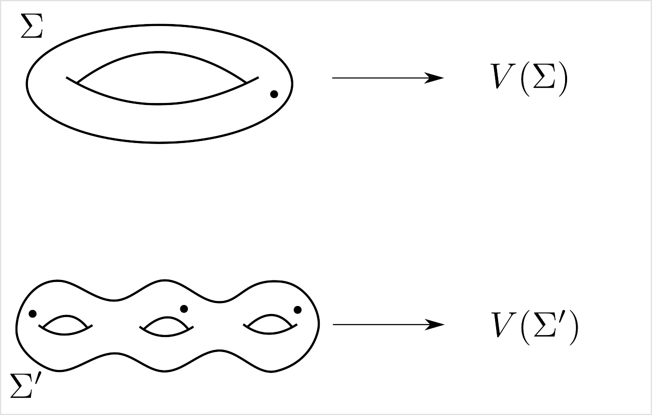

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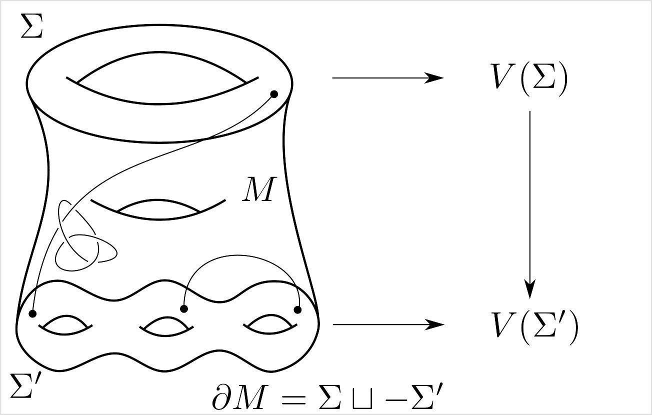

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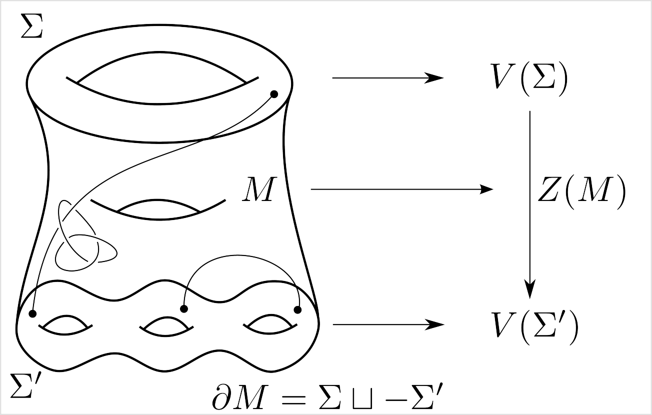

Quantum representations

The data (Z_k, V_k) satisfies a number of axioms.

Example

Let $\varphi : \Sigma \to \Sigma$ be a homeomorphism and consider the *mapping cylinder* and the *mapping torus*

$$C\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$
$$T_\varphi = \Sigma \times [0, 1]/((x, 0) \sim (\varphi(x), 1)).$$

Then $Z_k(C_\varphi) : V_k(\Sigma) \to V_k(\Sigma)$ depends on φ only up to isotopy. Define the (projective) *quantum representations* $\rho_k : \mathrm{MCG}(\Sigma) \to \mathrm{PAut}(V_k(\Sigma))$ by $\rho_k([\varphi]) = Z_k(C_\varphi)$. Furthermore, $Z_k(C_\varphi) = V_k(\varphi)$ and $Z_k(T_\varphi) = \operatorname{tr} Z_k(C_\varphi) = \operatorname{tr} \rho_k([\varphi])$.

Goal

Describe $\rho_k(f)$ for $f \in \mathrm{MCG}(\Sigma)$.

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Constructing quantum representations

Several equivalent approaches to the construction of quantum representations exist:

- Categorical/combinatorial through modular functors: (V_k, ρ_k) obtained from representation theory of $U_q(\mathfrak{sl}_N)$, the skein theory of the Kauffman bracket/HOMFLYPT polynomial, ...
- Conformal field theory: the monodromy of the WZW connection in the sheaf of conformal blocks.
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Let $f = \mathrm{id} \in \mathrm{MCG}(\Sigma_{g,0})$, $G = \mathrm{SU}(2)$. Then

$$\begin{aligned} Z_k(T_{\mathrm{id}}) &= Z_k(\Sigma_{g,0} \times S^1) = \mathrm{tr}\,\rho_k(\mathrm{id}) = \dim V_k(\Sigma_g) \\ &= \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g} \in \mathbb{N}. \end{aligned}$$

This is the *Verlinde formula*. For example,

$$\begin{aligned} \dim V_k(\Sigma_{0,0}) &= 1, \\ \dim V_k(\Sigma_{1,0}) &= k+1, \\ \dim V_k(\Sigma_{2,0}) &= \frac{1}{6}(k+1)(k+2)(k+3). \end{aligned}$$

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Isotopy invariant dynamics

What dynamical information do mapping classes contain?

Theorem (Nielsen–Thurston)

A mapping class $\varphi \in \mathrm{MCG}(\Sigma)$ is either

- *finite order,*
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$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \quad f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u).$$

Here, λ *is called the stretch factor of* φ.

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The NT classification vs. quantum reps

Are quantum reps sensitive to the trichotomy? Logically, yes, as $\bigcap_{k=1}^{\infty} \ker \rho_k = \{\mathrm{id}\}$ (for $G = \mathrm{SU}(2), n = 0, g > 2$, Andersen, Freedman–Walker–Wang).

Conjecture (Andersen–Masbaum–Ueno '06)

Let $2g + n > 2$, and let $\varphi \in \mathrm{MCG}(\Sigma_{g,n})$ be a pseudo-Anosov. Then there exists k_0 s.t. $\rho_k(\varphi)$ has infinite order for $k > k_0$.

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Theorem (Egsgaard, SFJ '13)

The AMU conjecture holds true for homological pseudo-Anosovs $\varphi \in \mathrm{MCG}(\Sigma_{0,n})$: those with only even-pronged singularities. Furthermore, stretch factors may be determined as k-limits of eigenvalues of ρ_k for these.

Steps in proof.

- Cook up representations $\tilde{\rho}_q$, $q \in \mathbb{C}$, such that $\rho_k = \tilde{\rho}_{\exp(2\pi i/(k+N))}$ for large k.
- Lemma: It suffices to show that the spectral radius of $\tilde{\rho}_q(\varphi)$ is greater than 1 for a $q \in \mathrm{U}(1)$.
- Main result: $\tilde{\rho}_{-1}$ is essentially an exterior power of the lifted action on homology of a double cover of Σ.

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- Main result: $\tilde{\rho}_{-1}$ is essentially an exterior power of the lifted action on homology of a double cover of Σ.

□

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Theorem (Egsgaard, SFJ '13)

The AMU conjecture holds true for homological pseudo-Anosovs $\varphi \in \mathrm{MCG}(\Sigma_{0,n})$: those with only even-pronged singularities. Furthermore, stretch factors may be determined as k-limits of eigenvalues of ρ_k for these.

Steps in proof.

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Thanks ...

... for listening!