Witten–Reshetikhin–Turaev invariants of mapping tori via skein theory

QGM Nielsen Retreat 2011

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Centre for Quantum Geometry of Moduli Spaces

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Notation

- Let $G = \mathrm{SU}(2)$, and let M be an (oriented connected framed) closed 3-manifold.
- Let $\mathcal{A}$ be the space of connections in $G \times M \to M$, and let $\mathcal{G}$ be the group of gauge transformations.
- Define the Chern–Simons functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

- For $g \in \mathcal{G}$, we have $\mathrm{CS}(g^* A) - \mathrm{CS}(A) \in \mathbb{Z}$, and we can consider

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The Chern–Simons partition function

- Let $k \in \mathbb{N}$ (called the *level*) and define the *Chern–Simons partition function*

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Theorem (Reshetikhin–Turaev, 1991)

One can construct a topological invariant Z_k of 3-manifolds, called the quantum SU(2)*-invariant, which behaves under gluing (or surgery) the way Z_k^{phys} is supposed to do.*

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For a closed oriented 3-manifold M,

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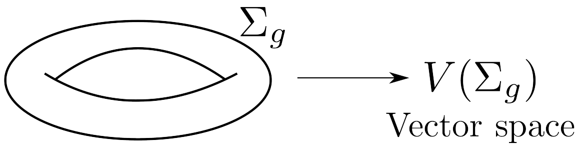

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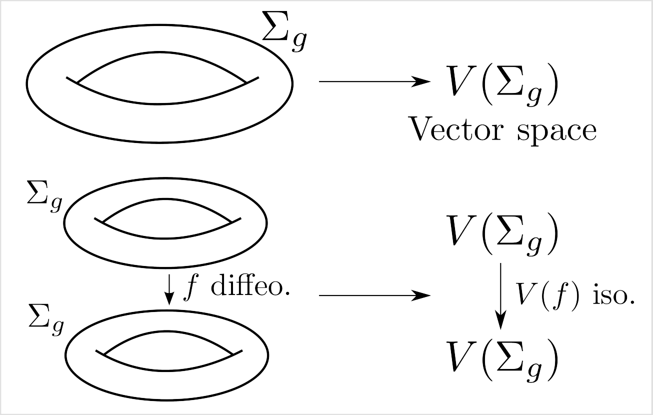

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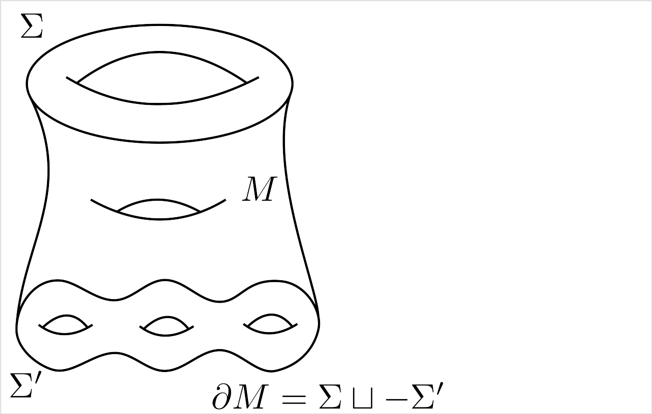

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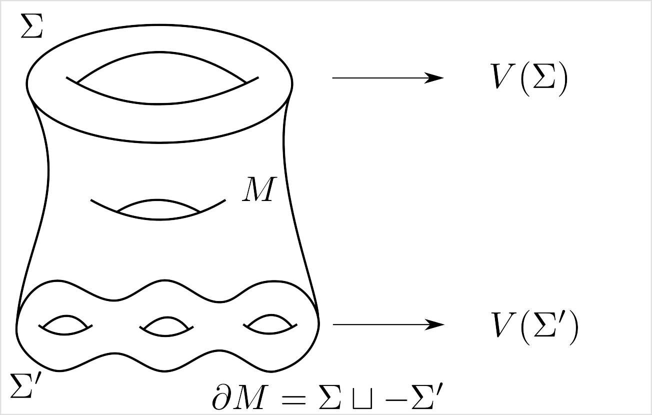

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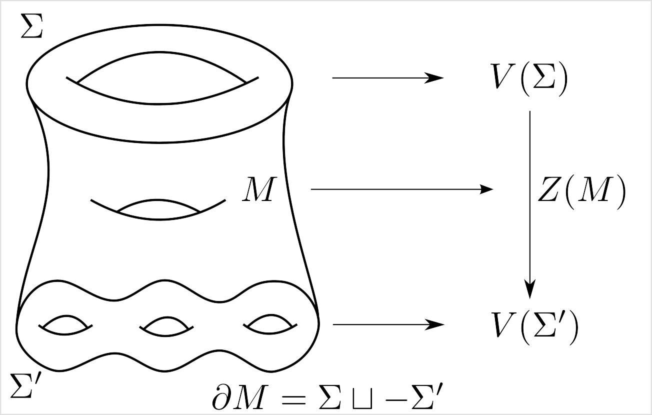

Quantum representations

The data (Z_k, V_k) satisfies a number of axioms.

Example

Let $\varphi : \Sigma \to \Sigma$ be a diffeomorphism and consider the *mapping cylinder* and the *mapping torus*

$$M_\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$
$$T_\varphi = \Sigma \times [0, 1]/((x, 0) \sim (\varphi(x), 1)).$$

Then $Z_k(M_\varphi) : V_k(\Sigma) \to V_k(\Sigma)$ depend on φ only up to isotopy. Define the *quantum representations* $\rho_k : \mathrm{MCG}(\Sigma) \to \mathrm{Aut}(V_k(\Sigma))$ by $\rho_k([\varphi]) = Z_k(M_\varphi)$. Furthermore, $Z_k(M_\varphi) = V_k(\varphi)$ and $Z_k(T_\varphi) = \operatorname{tr} Z_k(M_\varphi) = \operatorname{tr} \rho_k([\varphi])$.

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A Dehn twist

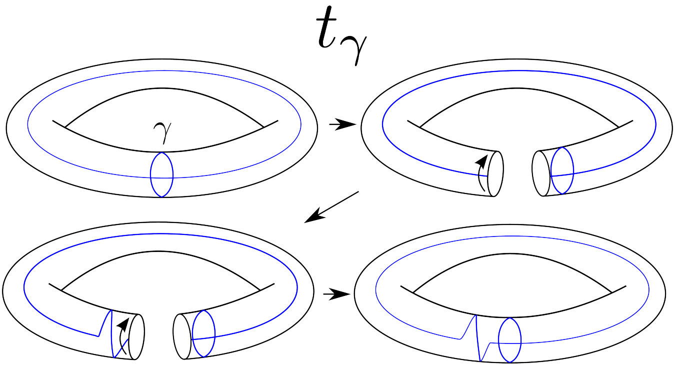

Figure: The Dehn twist t_γ about a curve γ.

The Dehn–Lickorish theorem

Theorem (Dehn–Lickorish)

The mapping class group $\mathrm{MCG}(\Sigma)$ *is generated by a certain finite set of Dehn twists about curves in* Σ*.*

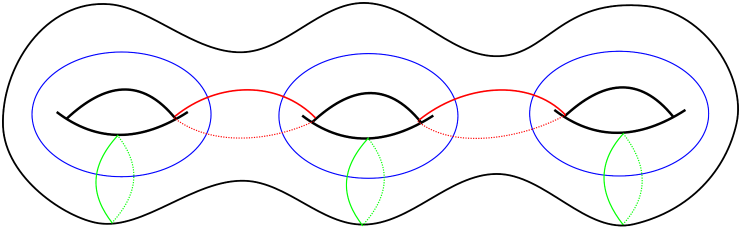

Figure: The Dehn–Lickorish generators in a genus 3 surface.

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Let $f = \mathrm{id} \in \mathrm{MCG}(\Sigma_g)$. Then

$$Z_k(T_{\mathrm{id}}) = Z_k(\Sigma_g \times S^1) = \operatorname{tr} \rho_k(\mathrm{id}) = \dim V_k(\Sigma_g)$$
$$= \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g} \in \mathbb{N}.$$

This is the *Verlinde formula*. For example,

$$\dim V_k(S^2) = 1,$$
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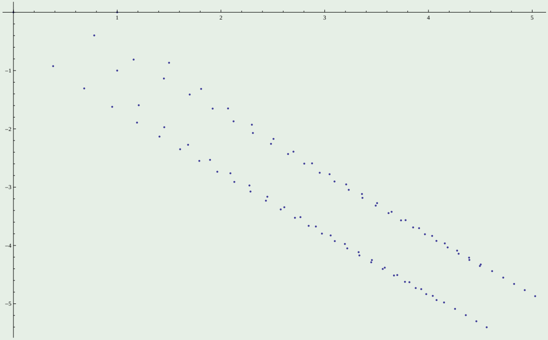

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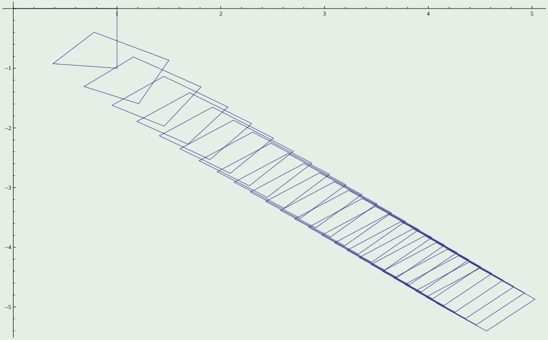

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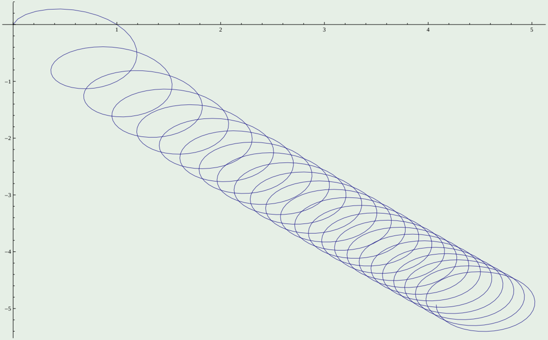

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Asymptotic expansion conjecture

Recall that the partition function looked like

$$Z_k^{\mathrm{phys}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \,\mathrm{CS}(A)} \mathcal{D}A.$$

Let $\mathcal{M}$ be the moduli space of flat connections on a 3-manifold M, and let $0 = c_0, c_1, \ldots, c_n$ be the values of CS on $\mathcal{M}$.

Conjecture (The asymptotic expansion conjecture)

There exist $d_j \in \frac{1}{2}\mathbb{Z}$, $b_j \in \mathbb{C}$, $a_j^l \in \mathbb{C}$ for $j = 0, \ldots, n$, $l \in \mathbb{N}_0$ such that $Z_k(M)$ has the asymptotic expansion

$$Z_k(M) \sim_{k\to\infty} \sum_{j=0}^{n} e^{2\pi i k c_j} k^{d_j} b_j \left(1 + \sum_{l=1}^{\infty} a_j^l k^{-l/2}\right)$$

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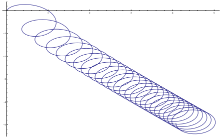
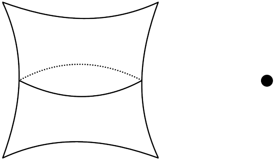
The example revisited



$$Z_{k-2}(T_{t_\gamma}) = e^{\frac{\pi i}{2k}} \left(\sqrt{k/2}e^{-\pi i/4}e^{2\pi i k 0} - \frac{e^{2\pi i k 3/4}}{2} - \frac{1}{2} \right).$$

$\mathcal{M}(T_{t_\gamma})$:



$c_0 = 0$ $\quad$ $c_1 = \frac{3}{4}$

$d_0 = \frac{1}{2}$ $\quad$ $d_1 = 0$

Theorems

Theorem (Jeffrey, '92)

The AEC holds for every mapping torus T_f of a torus diffeomorphism $f \in \mathrm{MCG}(S^1 \times S^1) \cong \mathrm{SL}(2,\mathbb{Z})$ with $|\mathrm{tr}(f)| > 2$ as well as for lens spaces.

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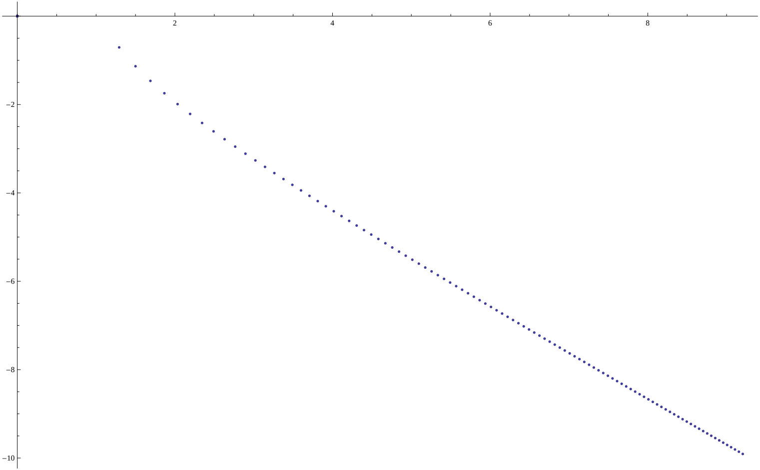

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 2$.

Pretty pictures

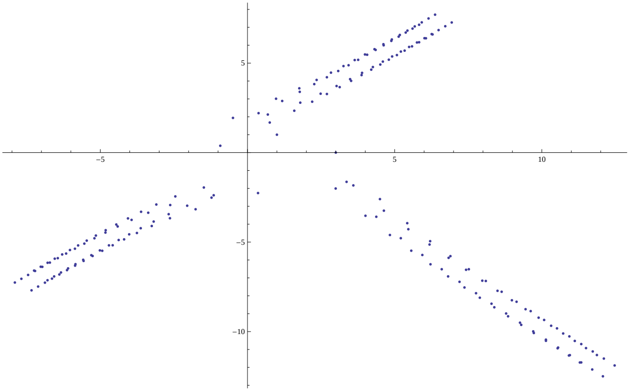

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 3$.

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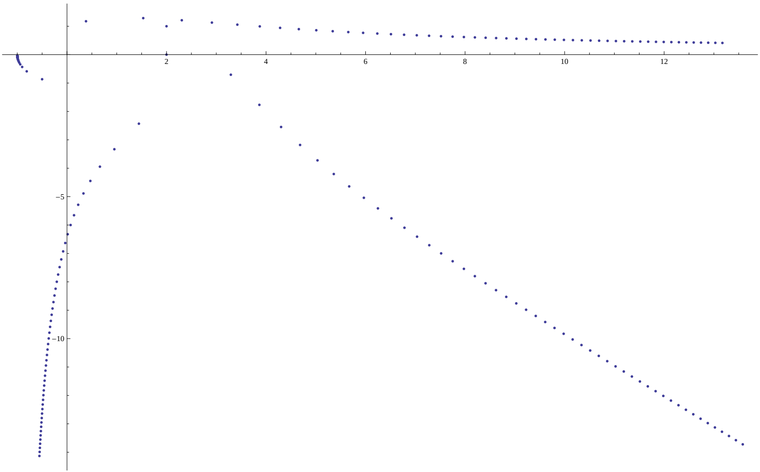

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 4$.

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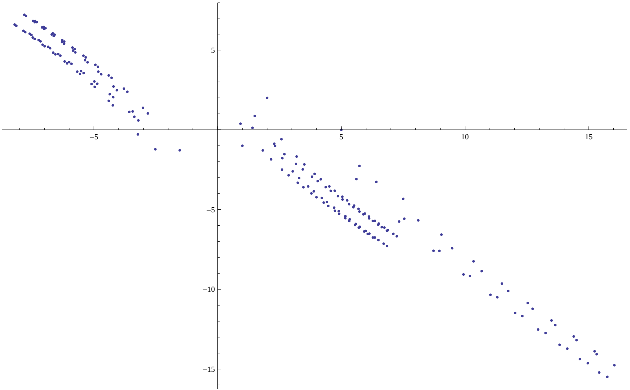

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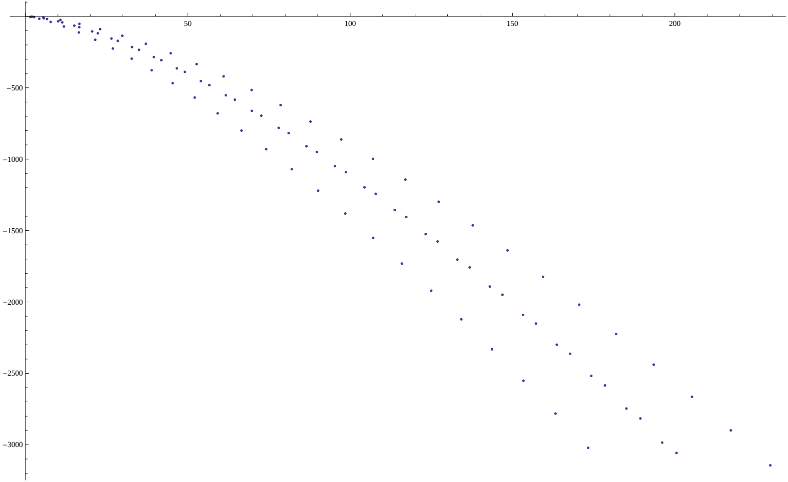

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 2, m = 1$.

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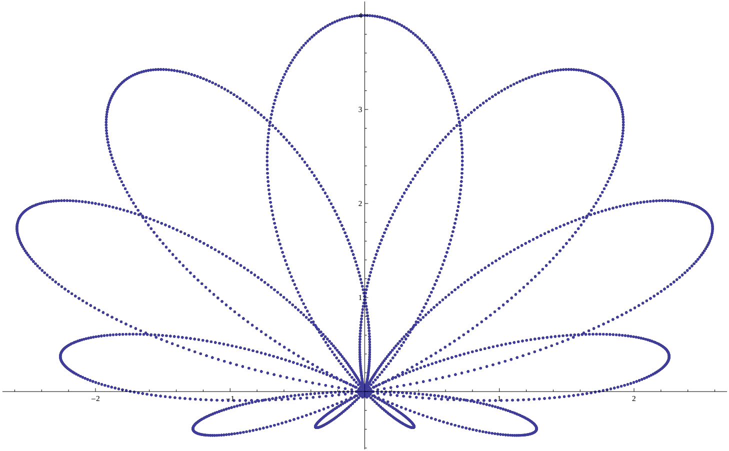

Figure: Pretending that the power m is a continuous parameter at level $k = 3$.

Thanks ...

... for listening!