Semi-classical properties of the quantum representations of mapping class groups

PhD defence

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Notation

- Let $G = \mathrm{SU}(N)$, and let M be an (oriented connected framed) closed 3-manifold.
- Let $\mathcal{A} \cong \Omega^1(M, \mathfrak{g})$ be the space of connections in $G \times M \to M$, and let $\mathcal{G} \cong C^\infty(M, G)$ be the group of gauge transformations acting on $\mathcal{A}$.
- Define the Chern–Simons functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

- For $g \in \mathcal{G}$, we have $\mathrm{CS}(g^* A) - \mathrm{CS}(A) \in \mathbb{Z}$, and we can consider

$$\mathrm{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$$

The Chern–Simons partition function

- Let $k \in \mathbb{N}$ (called the *level*) and define the *Chern–Simons partition function*

$$Z_k^{\mathrm{phys}}(M) = \int_{\mathcal{A}_P/\mathcal{G}_P} \exp(2\pi i k\, \mathrm{CS}([A]))\, \mathcal{D}A \in \mathbb{C}.$$

Witten '89: This defines a topological invariant.

Topological quantum field theory

Theorem (Reshetikhin–Turaev, 1991)

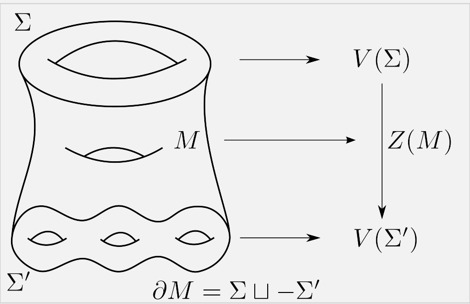
One can construct a topological invariant Z_k of 3-manifolds, called the quantum invariant, which behaves under gluing (or surgery) the way Z_k^{phys} is supposed to do.

Goal

Understand large k asymptotics of $Z_k(M)$ in the case where M is a mapping torus.

Topological quantum field theory

Reshetikhin and Turaev proved that the invariant Z_k is part of a $2+1$-dimensional topological quantum field theory (Z_k, V_k):



Quantum representations

The data (Z_k, V_k) satisfies a number of axioms.

Example

Let $\varphi : \Sigma \to \Sigma$ be a diffeomorphism and consider the *mapping cylinder* and the *mapping torus*

$$M_\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$
$$T_\varphi = \Sigma \times [0,1]/((x,0) \sim (\varphi(x), 1)).$$

Then $Z_k(M_\varphi) : V_k(\Sigma) \to V_k(\Sigma)$ depend on φ only up to isotopy. Define the (projective) *quantum representations* $\rho_k : \mathrm{MCG}(\Sigma) \to \mathrm{PAut}(V_k(\Sigma))$ by $\rho_k([\varphi]) = Z_k(M_\varphi)$. Furthermore, $Z_k(M_\varphi) = V_k(\varphi)$ and $Z_k(T_\varphi) = \mathrm{tr}\, Z_k(M_\varphi) = \mathrm{tr}\, \rho_k([\varphi])$.

Revised goal

Describe $\rho_k(f)$ for $f \in \mathrm{MCG}(\Sigma)$.

Constructing quantum representations

Several equivalent approaches to quantum representations exist:

- Categorical/combinatorial through modular functors: (V_k, ρ_k) obtained from representation theory of $U_q(\mathfrak{sl}_N)$, the skein theory of the Kauffman bracket/HOMFLYPT polynomial, ...
- Geometric quantization ($g \geq 2$): $\mathcal{L} \to \mathcal{M}^d$ is the pre-quantum line bundle over the moduli space of flat $\mathrm{SU}(N)$-connections

$$\mathcal{M}^d = \mathrm{Hom}_d(\tilde{\pi}_1(\Sigma), \mathrm{SU}(N))/\mathrm{SU}(N),$$

$V_k = H^0(\mathcal{M}^d_\sigma, \mathcal{L}^k_\sigma)$, $\sigma \in \mathrm{Teich}(\Sigma)$. Then ρ_k is defined by the parallel transport of Hitchin connection in $H^0(\mathcal{M}^d, \mathcal{L}^k) \to \mathrm{Teich}(\Sigma)$.

A Dehn twist

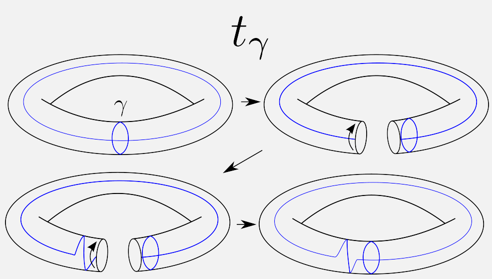


Figure: The Dehn twist t_γ about a curve γ.

The Dehn–Lickorish theorem

Theorem (Dehn–Lickorish)

The mapping class group $\mathrm{MCG}(\Sigma)$ *is generated by a certain finite set of Dehn twists about curves in* Σ.

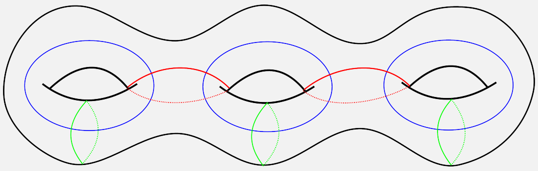


Figure: The Dehn–Lickorish generators in a genus 3 surface.

An algorithm

Quantum reps of Dehn–Lickorish generators implemented in PARI/GP by A'Campo–Masbaum.

Theorem (Masbaum)

For $g \geq 2$, $k \neq 1, 2, 4, 8$, *there exists* $\varphi \in \mathrm{MCG}(\Sigma_g)$ *s.t.* $\rho_k(\varphi)$ *has infinite order.*

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"Proof" for $k = 8$.

```
? init_su(8); init_boom([0],2); abs(subst(trace(lift(twA(1)^2*twB(1)^(-2))),A,exp(2*Pi*I*3/(4*8+8))))
su,  KLEVEL= 8, POL= A^16 - A^12 + A^8 - A^4 + 1
%15 = 9.2158645472653502432123429 10
```

□

Asymptotic expansion conjecture

Recall that the partition function looked like

$$Z_k^{\text{phys}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \, \mathrm{CS}(A)} \mathcal{D}A.$$

Let $\mathcal{M}$ be the moduli space of flat connections on a 3-manifold M, and let $0 = c_0, c_1, \ldots, c_n$ be the values of CS on $\mathcal{M}$.

Conjecture (The asymptotic expansion conjecture)

There exist $d_j \in \frac{1}{2}\mathbb{Z}$, $b_j \in \mathbb{C}$, $a_j^l \in \mathbb{C}$ *for* $j = 0, \ldots, n$, $l \in \mathbb{N}_0$ *such that* $Z_k(M)$ *has the asymptotic expansion*

$$Z_k(M) \sim_{k \to \infty} \sum_{j=0}^{n} e^{2\pi i r c_j} r^{d_j} b_j \left(1 + \sum_{l=1}^{\infty} a_j^l r^{-l/2}\right),$$

where $r = k + N$.

The AEC for Dehn twist bundles

Let γ in $S^1 \times S^1$ be non-trivial, and let t_γ be the Dehn twist about γ.

Theorem (Cor. 5.20 and Cor. 5.30)

The AEC holds for the mapping tori $T_{t_\gamma^b}$, $b \neq 0$, *when* $G = \mathrm{SU}(2)$ *or* $G = \mathrm{SU}(3)$.

Sketch of proof

Explicit calculation (Cor. 5.18 and Thm. 5.24) of $Z_k(T_{t_\gamma^b})$:

- $V_k(S^1 \times S^1)$ has natural basis vectors v_λ labelled by

$$\tilde{P}_k = \{\lambda \in \mathrm{int}(P_+) \cap \Lambda^w \mid \langle \lambda, \alpha_m \rangle < r\}.$$

  Here: Λ^w is the weight lattice, P_+ is the positive Weyl alcove, α_m maximal root.

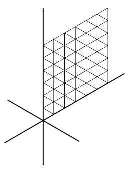
The AEC for Dehn twist bundles

Sketch of proof (cont.)

- Basis is set up (Lem. 4.4) such that $\rho_k(t_\gamma^b)$ is diagonal w.r.t. $\{v_\lambda\}$ with eigenvalues of the form

$$g(\lambda) = \exp\left(b \frac{\pi i}{r} \langle \lambda, \lambda \rangle\right).$$

- Jeffrey's quadratic reciprocity gives formula for $\sum_{\lambda \in \Lambda^w / 2rN\Lambda^w} g(\lambda)$.
- Evaluating $\sum_{\lambda \in \tilde{P}_k} g(\lambda)$ is combinatorics ($N = 2, 3$).



The AEC for Dehn twist bundles

Sketch of proof (cont.)

- On the other hand, $\rho \in \mathcal{M}$ may be considered as $\rho = [(A_1, A_2, A_3)] \in T \times T \times \mathrm{SU}(N)/\sim$, T maximal torus.
- Take a_1, a_2 s.t. $\exp(2\pi i a_j) = A_j$. There is a Weyl group element $w \in W$ s.t.

$$w t_\gamma^b (a_1, a_2) - (a_1, a_2) =: (\lambda, \mu) \in \Lambda^R \oplus \Lambda^R$$

- Then (Jeffrey)

$$e^{2\pi i \, \mathrm{CS}(\rho)} = \varepsilon(\lambda, \mu) e^{i\omega((a_1, a_2), (\lambda, \mu))/2},$$

  where $\varepsilon(\lambda_1, \lambda_2) \in \{\pm 1\}$ is a *theta-characteristic.*
- Now, match everything up (Prop. 5.28, Cor. 5.30).

□

Generalizing the result

- Assumption $N = 2, 3$ only used to simplify combinatorics.
- Evaluating $Z_k(T_{t_\gamma^b})$, $G = \mathrm{SU}(2)$, for $g = 1$ boils down to

$$\sum_{n=1}^{r-1} \exp\left(\frac{\pi i}{2r} b n^2\right)$$

Evaluating $Z_k(T_{t_\gamma^b})$, $G = \mathrm{SU}(2)$, for $g \geq 2$ boils down to

$$\sum_{n=1}^{r-1} p(n) \exp\left(\frac{\pi i}{2r} b n^2\right),$$

where p is a polynomial (Section 5.5).

The Nielsen–Thurston classification

Recall the notion of a pseudo-Anosov mapping class:

Theorem (Nielsen–Thurston)

A mapping class $\varphi \in \mathrm{MCG}(\Sigma_g)$ *is either*

- *finite order*
- *infinite order but has a power preserving an essential simple closed curve.*
- *pseudo-Anosov: there are transverse measured singular foliations* $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$, $\lambda > 1$ *and a homeo.* f, $[f] = \varphi$, *s.t.*

$$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s), \quad f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u).$$

  Here, λ *is called the stretch factor of* φ.

Asymptotics via geometric quantization

Consider $\mathcal{M} = \mathcal{M}^d$. Let $K_\sigma^{(k)}$ denote the Bergman kernel of the orthogonal projection $\pi_\sigma^k : C^\infty(\mathcal{M}, \mathcal{L}^k) \to H^0(\mathcal{M}_\sigma, \mathcal{L}_\sigma^k)$, i.e.

$$\pi_\sigma^k s(x) = \int_{\mathcal{M}} K_\sigma^{(k)}(x,y)s(y)\frac{\omega^n(y)}{n!(2\pi)^n},$$

for $s \in C^\infty(\mathcal{M}, \mathcal{L}^k)$. Suppose $\varphi \in \mathrm{MCG}(\Sigma_g)$ with

$$\mathrm{graph}(\varphi^*) \pitchfork \mathrm{diag} \subseteq \mathcal{M} \times \mathcal{M}.$$

Theorem (Karabegov–Schlichenmaier)

For y close to $x \in |\mathcal{M}|^\varphi$,

$$k^{-n}K_\sigma^{(k)}(x,y) \sim e^{k\chi(x,y)}b(x,y)$$

for functions χ, b defined close to (x,x), $b(x,x) = 1$, $n = \frac{1}{2}\dim\mathcal{M}$.

Asymptotics via geometric quantization

Theorem (Andersen, Thm. 6.7)

Let γ be a curve in $\mathrm{Teich}(\Sigma_g)$ from σ_0 to σ_1. Then there exists $g_\gamma \in C^\infty(\mathcal{M})$ s.t.

$$\left\| \mathrm{PT}_{\nabla^{\mathrm{Hitchin}}}(\gamma) - \pi_{\sigma_1}^k g_\gamma \pi_{\sigma_0}^k \right\| = O(1/k).$$

Let $L_x = \mathrm{Hess}_x(y \mapsto \chi(y, \varphi(y)))$.

Theorem (Thm. 6.8)

We have

$$\mathrm{tr}(\rho_k(\varphi)) \sim \sum_{x \in \mathrm{Fix}(\varphi:\mathcal{M}\to\mathcal{M})} \frac{\mathrm{tr}(\varphi : \mathcal{L}_x^k \to \mathcal{L}_x^k) g_\gamma(x) \exp(i\,\mathrm{sign}(L_x/4))}{\sqrt{|\det L_x|}}.$$

The AMU conjecture

Conjecture (AMU conjecture)

Let $\Sigma_{g,n}$ be a genus g surface with n coloured points, $2g + n > 2$, and let φ be a pseudo-Anosov. Then there exists k_0 s.t. $\rho_k(\varphi)$ has infinite order for $k > k_0$. Moreover, the ρ_k determine the stretch factor of φ.

Theorem (Thm. 6.32)

Let $G = \mathrm{SU}(2)$. The conjecture is true for orientable pseudo-Anosovs φ of the six punctured sphere.

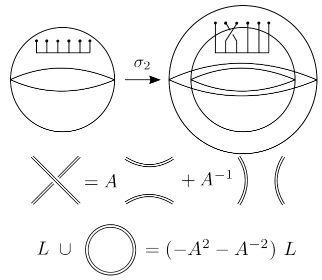
Proof

As in AMU: Tweak quantum reps to define reps of $\mathrm{MCG}(\Sigma_0^6)$ for which the statement holds.

The AMU conjecture

Proof (cont.)

- Let $\rho_A^{(S)}$ denote the generic action – induced by the action of B_6 on TL_6 – on the 5-dimensional space V of Kauffman skeins in B^3 meeting S^2 in six points.



The AMU conjecture

Proof (cont.)

- The representation $\rho_A = A^{3/5}\rho_A^{(S)}$ descends to a representation of $\mathrm{MCG}(\Sigma_0^6) = B_6/\sim$.
- On the other hand, with $A = q^{-1/4}$, $\sigma_i \mapsto -A^{-1}\rho_A^{(S)}(\sigma_i)$ is equivalent to Jones' Hecke algebra representation $\tilde{\rho}_q$ associated to ⊞ (see Wang).
- Kasahara: $\tilde{\rho}_{-1}$ is equivalent to the induced action on $\wedge^2 H_1(\Sigma_2, \mathbb{Z})/\omega\mathbb{Z} \otimes \mathrm{sgn}$ via Birman–Hilden.
- Specialize to $A = A_k$ with $A_k^2 \to -i$ as in AMU.
- For the pseudo-Anosovs of the claim, the stretch factor is the spectral radius of this homology action.

□

Fin