The AMU conjecture for punctured spheres

Discussion Meeting on Analytic and Algebraic Geometry Related to Bundles
Kerala School of Mathematics

Søren Fuglede Jørgensen
joint work with Jens Kristian Egsgaard
arXiv:1402.6059

Uppsala University

March 24th, 2014

Notation

- Let $G = \mathrm{SU}(N)$, and let M be an (oriented connected framed) closed 3-manifold.
- Let $\mathcal{A} \cong \Omega^1(M, \mathfrak{g})$ be the space of connections in $G \times M \to M$, and let $\mathcal{G} \cong C^\infty(M, G)$ be the group of gauge transformations acting on $\mathcal{A}$.
- Define the Chern–Simons functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

- For $g \in \mathcal{G}$, we have $\mathrm{CS}(g^*A) - \mathrm{CS}(A) \in \mathbb{Z}$, and we can consider

$$\mathrm{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$$

The Chern–Simons partition function

- Let $k \in \mathbb{N}$ (called the *level*) and define the *Chern–Simons partition function*

$$Z_k^{\mathrm{phys}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k\, \mathrm{CS}(A)} \mathcal{D}A \in \mathbb{C}.$$

- Assume that M contains a framed oriented link L, and choose for every component L_i of L a finite dimensional representation R_i of $G = \mathrm{SU}(N)$. Set

$$Z_k^{\mathrm{phys}}(M, L, R) = \int_{\mathcal{A}/\mathcal{G}} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) e^{2\pi i k\, \mathrm{CS}(A)} \mathcal{D}A.$$

Witten '89: This extends to a TQFT.

A possible construction

Theorem (Reshetikhin–Turaev, 1991)

One can construct a topological invariant Z_k of 3-manifolds, called the quantum invariant, which behaves under gluing (or surgery) the way Z_k^{phys} is supposed to do.

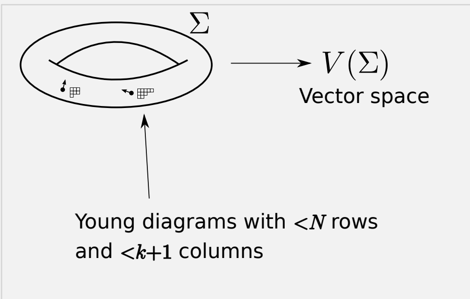
Source of inspiration

For a closed oriented 3-manifold M,

$$Z_k^{\mathrm{phys}}(M) = Z_k(M).$$

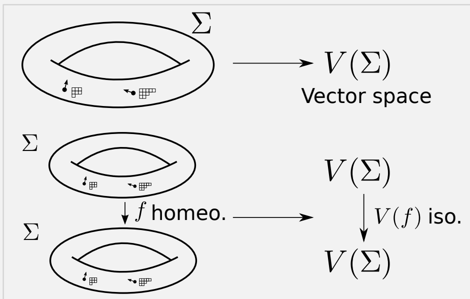
Topological quantum field theory

Reshetikhin and Turaev proved that the invariant Z_k is part of a $2+1$-dimensional topological quantum field theory (Z_k, V_k):



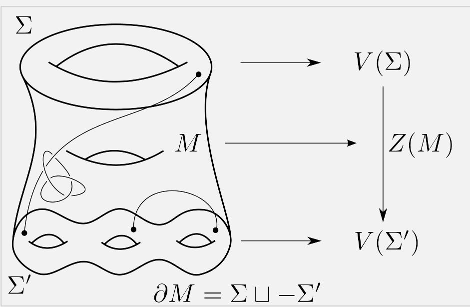
Topological quantum field theory

Reshetikhin and Turaev proved that the invariant Z_k is part of a $2+1$-dimensional topological quantum field theory (Z_k, V_k):



Topological quantum field theory

Reshetikhin and Turaev proved that the invariant Z_k is part of a $2+1$-dimensional topological quantum field theory (Z_k, V_k):



Quantum representations

The data (Z, V) satisfies a number of axioms.

Example

Let $\varphi : \Sigma \to \Sigma$ be a diffeomorphism and consider the *mapping cylinder*

$$C_\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$

Then $Z(C_\varphi) : V(\Sigma) \to V(\Sigma)$ depends on φ only up to isotopy. Define the (projective) *quantum representations*

$$\rho : \mathrm{MCG}(\Sigma) \to \mathrm{PGL}(V(\Sigma))$$

by $\rho([\varphi]) = Z(C_\varphi)$. Furthermore, $Z(C_\varphi) = V(\varphi)$.

Constructing quantum representations

Several equivalent approaches to the construction of quantum $\mathrm{SU}(N)$-representations $(V_{N,k}, \rho_{N,k})$ exist:

- Categorical/combinatorial through modular functors: obtained from representation theory of $U_q(\mathfrak{sl}_N)$ (with $q = \exp(\frac{2\pi i}{k+N})$), the skein theory of the Kauffman bracket/HOMFLYPT polynomial ...
- Conformal field theory: the monodromy of the WZW connection in the sheaf of conformal blocks.
- Geometric quantization of moduli spaces of flat connections/bundles: the monodromy of the Hitchin connection (no marked points).

These approaches are equivalent: Laszlo, Andersen–Ueno, ...

A simple example

Example

Let $f = \mathrm{id} \in \mathrm{MCG}(\Sigma_g)$, $G = \mathrm{SU}(2)$. Then

$$\begin{aligned}\operatorname{tr}\rho_{2,k}(\mathrm{id}) &= \dim V_{2,k}(\Sigma_g)\\ &= \left(\frac{k+2}{2}\right)^{g-1}\sum_{j=1}^{k+1}\left(\sin^2\frac{j\pi}{k+2}\right)^{1-g} \in \mathbb{N}.\end{aligned}$$

This is the *Verlinde formula*. For example,

$$\begin{aligned}\dim V_k(\Sigma_0) &= 1,\\ \dim V_k(\Sigma_1) &= k+1,\\ \dim V_k(\Sigma_2) &= \frac{1}{6}(k+1)(k+2)(k+3).\end{aligned}$$

The genus 0 case

- Let $\Sigma = \mathbb{C} \cup \{\infty\}$ be a genus zero surface with marked points $\{1, \dots, n, \infty\}$ labelled by Young diagrams $\{\square, \dots, \square, \lambda^\star\}$, where λ has at most 2 rows (at most 1 if $N = 2$), and $\star$ denotes the dual diagram.
- Let $V_{N,k}^\lambda$ denote the vector space associated by any of the modular functors to Σ.
- The MCG of Σ (preserving marked points + labels) naturally contains B_n.
- Let $\rho_{N,k}^\lambda : B_n \to \mathrm{GL}(V_{N,k}^\lambda)$ denote the restriction of the quantum representation to this B_n.

Theorem

For $k > n$, $\rho_{N,k}^\lambda$ is equivalent to the diagram representation $\eta_A^{n,d}$ from Jens Kristian's talk with $q = A^4 = \exp(2\pi i/(N+k))$, $d \leftrightarrow \lambda$.

Isotopy invariant dynamics

What dynamical information do mapping classes contain?

Theorem (Nielsen–Thurston)

Let Σ be a surface (possibly punctured but with no boundary). A mapping class $\varphi \in \mathrm{MCG}(\Sigma)$ is either

- *finite order,*
- *infinite order but has a power preserving the homotopy class of an essential simple closed curve (φ is reducible), or*
- *pseudo-Anosov: there are transverse measured singular foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ of Σ, $x > 1$ and a diffeo. f, $[f] = \varphi$, s.t.*

  $$f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, x^{-1}\mu^s), \quad f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, x\mu^u).$$

  Here, x is called the stretch factor of φ.

For surfaces with boundary, replace boundaries by punctures.

Visualizing pseudo-Anosov braids

Source: Mark A. Stremler

- Left: Initial position.
- Center: Stirring by finite order braid.
- Right: Stirring by pseudo-Anosov braid.

The NT classification vs. quantum reps

Are the quantum reps $\rho_{N,k}^\lambda$ sensitive to the trichotomy?

Conjecture (Andersen–Masbaum–Ueno '06)

Consider a general genus g surface Σ with n marked points. Assume $2g + n > 2$, and let $\varphi \in \mathrm{MCG}(\Sigma)$ be a pseudo-Anosov. Then there exists k_0 s.t. $\rho_{N,k}(\varphi)$ has infinite order for $k > k_0$.

Question (Andersen–Masbaum–Ueno '06)

Do $\rho_{N,k}$ determine stretch factors of pseudo-Anosovs?

AMU: These are true for a sphere with four marked points.

Generalizing AMU

Theorem (Egsgaard, SFJ)

The AMU conjecture holds true for all $\rho_{N,k}^\lambda$ for homological pseudo-Anosovs $\varphi \in B_n$: those with only odd-pronged singularities in the marked points and even-pronged singularities in the other interior points. Furthermore, stretch factors may be determined from k-limits of eigenvalues of $\rho_{N,k}^\lambda$ for these pseudo-Anosovs.

Main steps in proof

- Recall that $\rho_{N,k}^\lambda \cong \eta_A^{n,d}$ for $A^4 = q = \exp(2\pi i/(k+N))$.
- The order of $\eta_A^{n,d}(\varphi)$ at a primitive root of unity depends only on the order of the root.
- It suffices to show that the spectral radius of $\eta_A^{n,d}(\varphi)$ is greater than 1 for an $A \in \mathrm{U}(1)$: Every $z \in \mathrm{U}(1)$ may be approximated by *primitive* n'th roots of unity z_n (Iwaniec).

Generalizing AMU (continued)

Proof (continued).

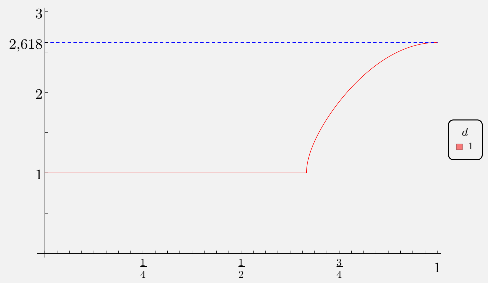
- Main result: $\eta_{\exp(-\pi i/4)}^{n,d}$ is essentially an exterior power of the lifted action on homology of the ramified double cover.
- The pseudo-Anosov φ lifts to a pseudo-Anosov $\tilde{\varphi}$ on the covering surface with the same stretch factor.
- The foliations of $\tilde{\varphi}$ have consistently orientable leaves. The stretch factor of a pseudo-Anosov with this property is the spectral radius of its action on homology.
- For exterior powers of homology, we need to ensure that eigenvectors lie in the image of morphism of representations.
- For odd n this is possible by the explicit description of the representation.
- For even n, use induction on d and a known decomposition $\eta_A^{n+1,d+1}|_{B_n} \cong \eta_A^{n,d} \oplus \eta_A^{n,d+2}$.

$\square$

Examples: Plots of spectral radii

For $\varphi \in B_n$, consider the functions $\mathrm{sr}_d(\varphi) : [0,1] \to \mathbb{R}_+$

$$\mathrm{sr}_d(\varphi)(x) = \text{spectral radius of } \eta_A^{n,d}(\varphi) \text{ at } q = A^4 = \exp(\pi i x).$$

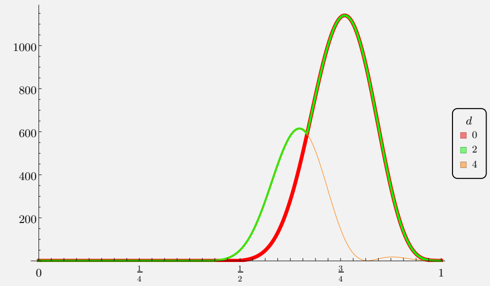


The pseudo-Anosov $\sigma_1\sigma_2^{-1} \in B_3$ (dashed line = stretch factor).

Examples: Plots of spectral radii

For $\varphi \in B_n$, consider the functions $\mathrm{sr}_d(\varphi) : [0,1] \to \mathbb{R}_+$

$$\mathrm{sr}_d(\varphi)(x) = \text{spectral radius of } \eta_A^{n,d}(\varphi) \text{ at } q = A^4 = \exp(\pi i x).$$

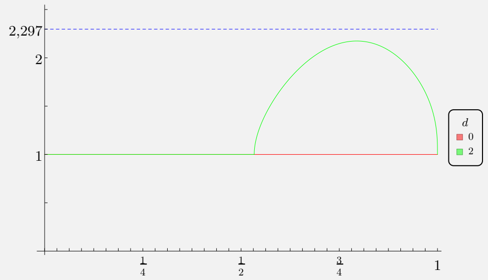


A pseudo-Anosov in B_6 acting trivially on homology of double cover (Brown).

Examples: Plots of spectral radii

For $\varphi \in B_n$, consider the functions $\mathrm{sr}_d(\varphi) : [0,1] \to \mathbb{R}_+$

$$\mathrm{sr}_d(\varphi)(x) = \text{spectral radius of } \eta_A^{n,d}(\varphi) \text{ at } q = A^4 = \exp(\pi i x).$$

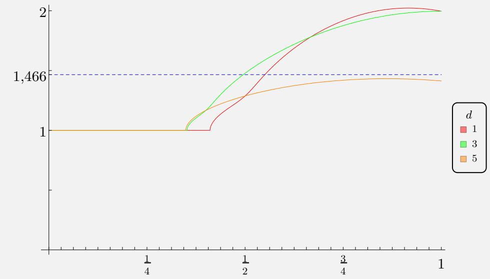


Small stretch factor, $\sigma_1\sigma_2\sigma_3^{-1} \in B_4$ (non-homological, Lanneau–Thiffeault).

Examples: Plots of spectral radii

For $\varphi \in B_n$, consider the functions $\mathrm{sr}_d(\varphi) : [0,1] \to \mathbb{R}_+$

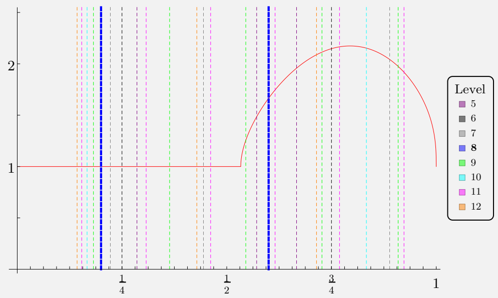
$$\mathrm{sr}_d(\varphi)(x) = \text{spectral radius of } \eta_A^{n,d}(\varphi) \text{ at } q = A^4 = \exp(\pi i x).$$



Small stretch factor, $\sigma_2^{-2}(\sigma_1 \cdots \sigma_6)^2 \in B_7$ (non-homological).

Concrete levels

We can read off at *which* levels, orders become infinite.



Plot for $d = 0$, $\sigma_1\sigma_2\sigma_3^{-1} \in B_6$; bold line is for SU(2) level $k = 8$.

Concrete levels

Theorem (Masbaum, '99)

Quantum SU(2)*-representations have elements of infinite order for all levels* k*, except perhaps for* $k = 1, 2, 4, 8$.

Theorem (Laszlo–Pauly–Sorger, '13)

The quantum SU(2)*-representations of the sphere with four marked points has finite image for* $k = 1, 2, 4, 8$.

Proposition

Quantum representations have infinite order elements at level $k = 8$ *as well.*

Examples of homological pseudo-Anosovs

Example

In B_3, all pseudo-Anosovs are homological. This way, we recover the result of Andersen–Masbaum–Ueno for the sphere with four marked points.

Example (Penner)

- Assume n is even, and let $\sigma_1, \dots, \sigma_{n-1}$ be the standard generators of B_n.
- Take any word φ in the generators where the signs of powers correspond to the parity of the index.
- For example: $n = 6$, $\varphi = \sigma_1^2\sigma_2^{-4}\sigma_3^5\sigma_4^{-10}\sigma_5^{60}$.
- Suppose that each generator appears at least once in the word. Then φ is a homological pseudo-Anosov.