Quantum invariants of torus bundles and their asymptotics

Barcelona 2012

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Centre for Quantum Geometry of Moduli Spaces

April 27th 2012

Notation

- Let $G = \mathrm{SU}(N)$, and let M be an (oriented connected framed) closed 3-manifold.
- Let $\mathcal{A}$ be the space of connections in $G \times M \to M$, and let $\mathcal{G}$ be the group of gauge transformations.
- Define the Chern–Simons functional $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

- For $g \in \mathcal{G}$, we have $\mathrm{CS}(g^* A) - \mathrm{CS}(A) \in \mathbb{Z}$, and we can consider

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The Chern–Simons partition function

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One can construct a topological invariant Z_k of 3-manifolds, called the quantum invariant, which behaves under gluing (or surgery) the way Z_k^{phys} is supposed to do.

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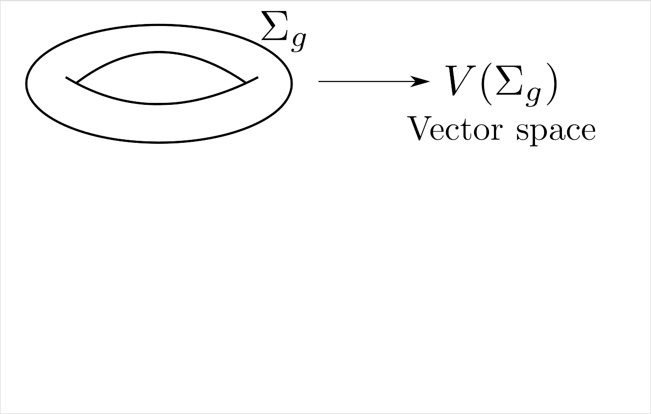

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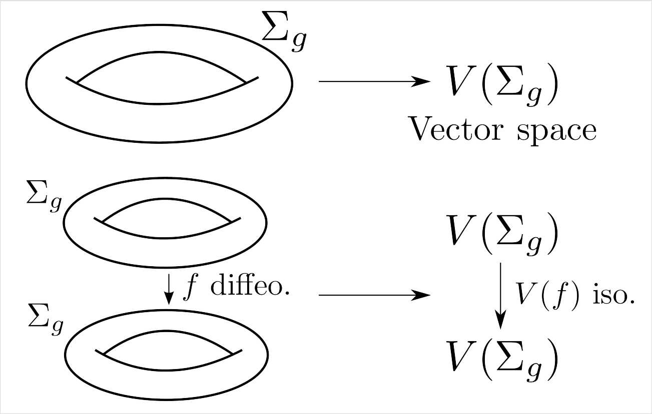

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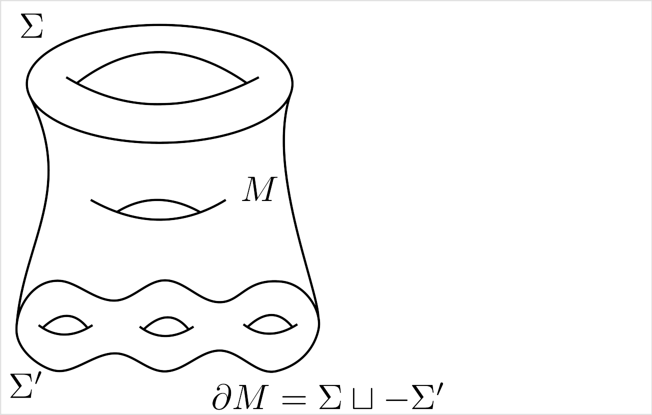

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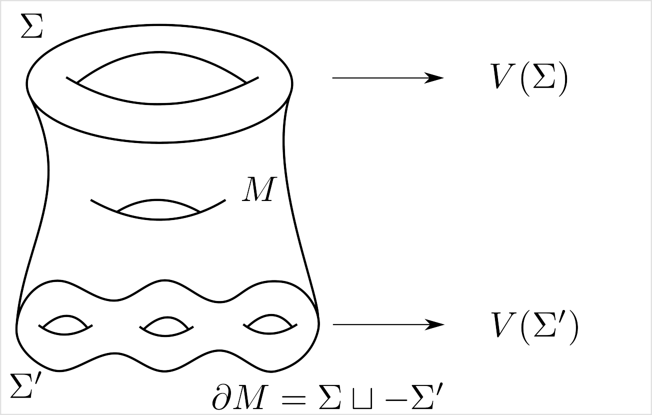

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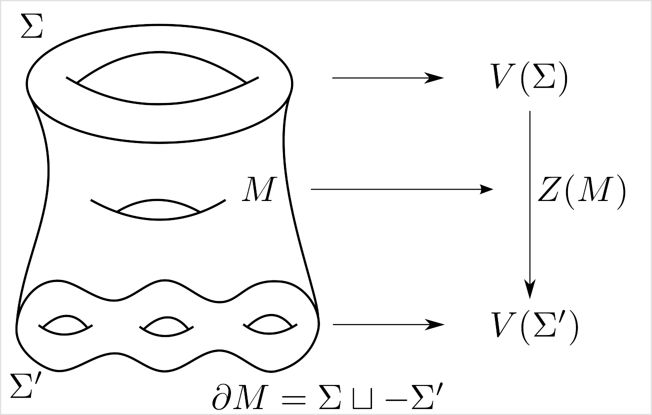

Quantum representations

The data (Z_k, V_k) satisfies a number of axioms.

Example

Let $\varphi : \Sigma \to \Sigma$ be a diffeomorphism and consider the *mapping cylinder* and the *mapping torus*

$$M_\varphi = \Sigma \times [0, \tfrac{1}{2}] \cup_\varphi \Sigma \times [\tfrac{1}{2}, 1]$$
$$T_\varphi = \Sigma \times [0, 1]/((x, 0) \sim (\varphi(x), 1)).$$

Then $Z_k(M_\varphi) : V_k(\Sigma) \to V_k(\Sigma)$ depend on φ only up to isotopy. Define the *quantum representations* $\rho_k : \mathrm{MCG}(\Sigma) \to \mathrm{Aut}(V_k(\Sigma))$ by $\rho_k([\varphi]) = Z_k(M_\varphi)$. Furthermore, $Z_k(M_\varphi) = V_k(\varphi)$ and $Z_k(T_\varphi) = \operatorname{tr} Z_k(M_\varphi) = \operatorname{tr} \rho_k([\varphi])$.

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A Dehn twist

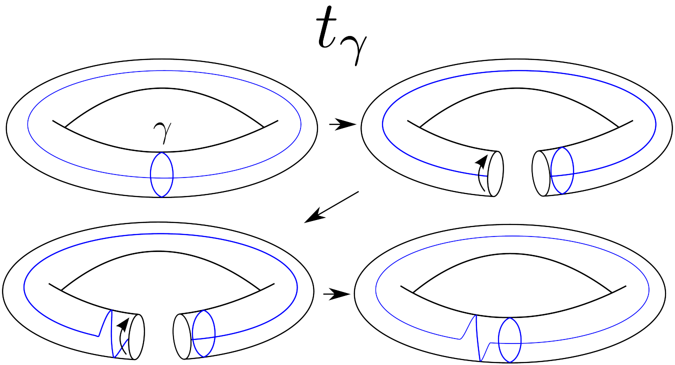

Figure: The Dehn twist t_γ about a curve γ.

The Dehn–Lickorish theorem

Theorem (Dehn–Lickorish)

The mapping class group $\mathrm{MCG}(\Sigma)$ *is generated by a certain finite set of Dehn twists about curves in* Σ*.*

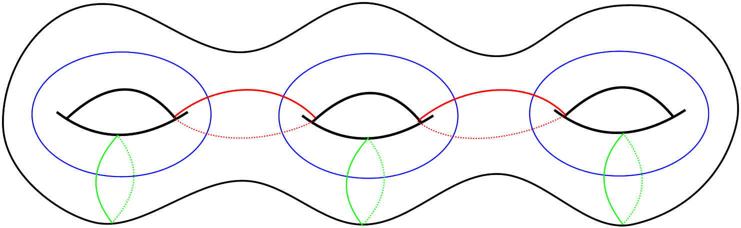

Figure: The Dehn–Lickorish generators in a genus 3 surface.

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Example

Let $f = \mathrm{id} \in \mathrm{MCG}(\Sigma_g)$, $G = \mathrm{SU}(2)$. Then

$$Z_k(T_{\mathrm{id}}) = Z_k(\Sigma_g \times S^1) = \operatorname{tr} \rho_k(\mathrm{id}) = \dim V_k(\Sigma_g)$$
$$= \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g} \in \mathbb{N}.$$

This is the *Verlinde formula*. For example,

$$\dim V_k(S^2) = 1,$$
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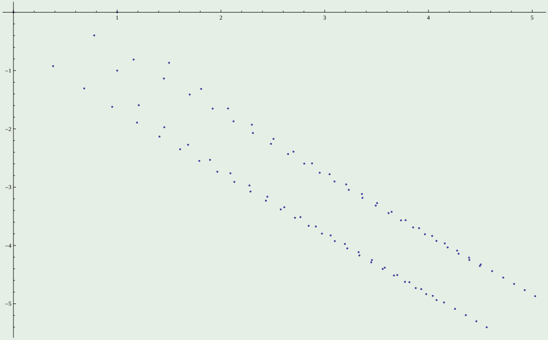

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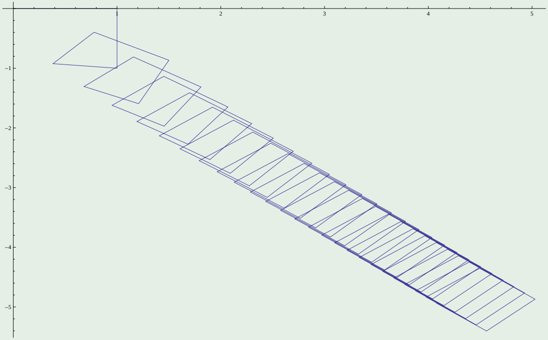

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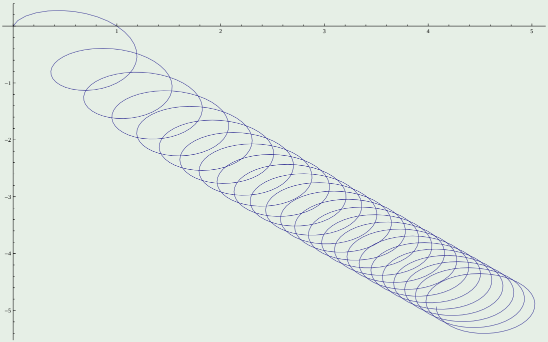

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Asymptotic expansion conjecture

Recall that the partition function looked like

$$Z_k^{\text{phys}}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \,\mathrm{CS}(A)} \mathcal{D}A.$$

Let $\mathcal{M}$ be the moduli space of flat connections on a 3-manifold M, and let $0 = c_0, c_1, \ldots, c_n$ be the values of CS on $\mathcal{M}$.

Conjecture (The asymptotic expansion conjecture)

There exist $d_j \in \frac{1}{2}\mathbb{Z}$, $b_j \in \mathbb{C}$, $a_j^l \in \mathbb{C}$ for $j = 0, \ldots, n$, $l \in \mathbb{N}_0$ such that $Z_k(M)$ has the asymptotic expansion

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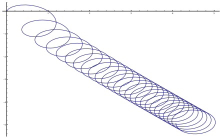
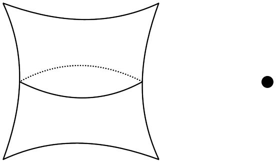
The example revisited



$$Z_{k-2}(T_{t_\gamma}) = e^{\frac{\pi i}{2k}} \left(\sqrt{k/2}\, e^{-\pi i/4} e^{2\pi i k 0} - \frac{e^{2\pi i k 3/4}}{2} - \frac{1}{2} \right).$$

$\mathcal{M}(T_{t_\gamma})$:



$c_0 = 0$ $\qquad$ $c_1 = \frac{3}{4}$

$d_0 = \frac{1}{2}$ $\qquad$ $d_1 = 0$

Theorems

Theorem (Jeffrey, '92)

The AEC holds for every mapping torus T_f of a torus diffeomorphism $f \in \mathrm{MCG}(S^1 \times S^1) \cong \mathrm{SL}(2, \mathbb{Z})$ with $|\mathrm{tr}(f)| > 2$.

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Summary of torus bundles

$f \in \mathrm{SL}(2,\mathbb{Z})$	$\{c_j\}$ of T_f
$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$	$\{0\}$
$\begin{pmatrix} \pm 1 & -b \\ 0 & \pm 1 \end{pmatrix}$, $b \neq 0$ even	$\{\frac{j^2}{b} \mid j = 0, \ldots, \frac{\lvert b\rvert}{2}\}$
$\begin{pmatrix} \pm 1 & -b \\ 0 & \pm 1 \end{pmatrix}$, b odd	$\{\frac{j^2}{b} \mid j = 0, \ldots, \frac{\lvert b\rvert - 1}{2}\} \cup \{-\frac{b}{4}\}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\lvert a+d\rvert \neq 2$	$\left\{ \frac{-c\gamma^2 + (a-d)\gamma\beta + b\beta^2}{d+a\pm 2} \;\middle\vert\; \begin{array}{l} 0 \leq \beta < c, \\ 0 < \gamma \leq \lvert a+d\pm 2\rvert \end{array} \right\}$

Table: Summary of phases and growth rates of quantum invariants of torus bundles.

Pretty pictures

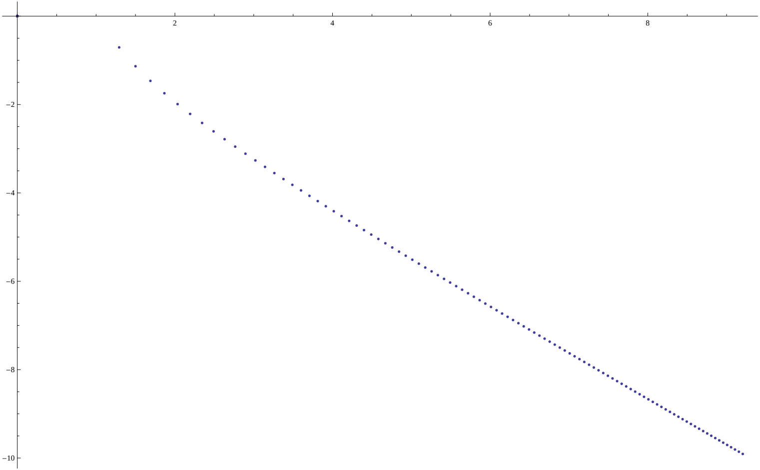

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 2,\ G = \mathrm{SU}(2)$.

Pretty pictures

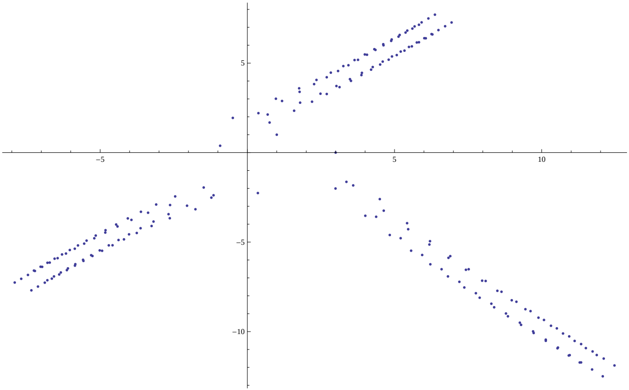

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 3,\ G = \mathrm{SU}(2)$.

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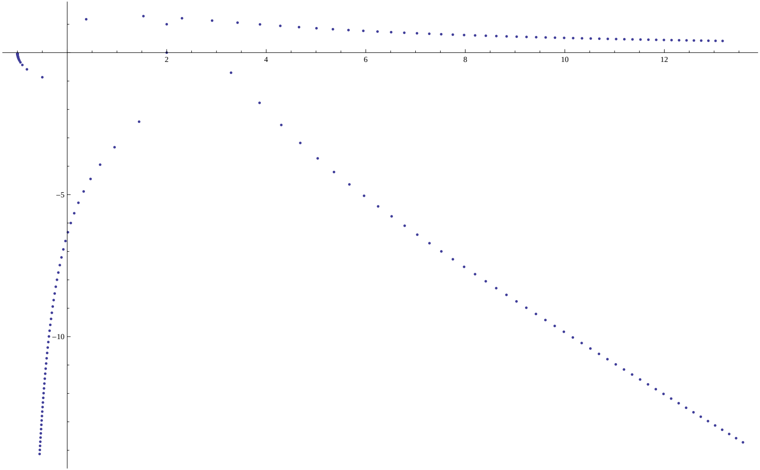

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 4,\ G = \mathrm{SU}(2)$.

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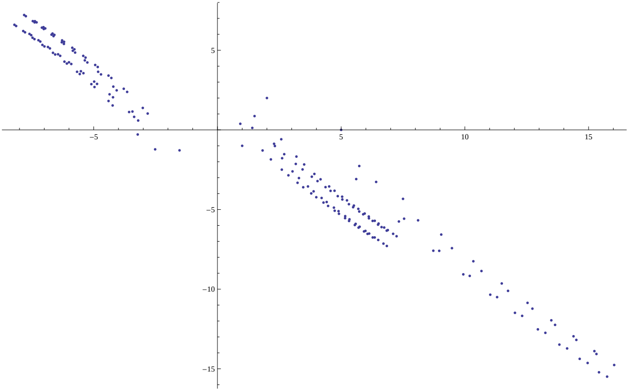

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 5, G = \mathrm{SU}(2)$.

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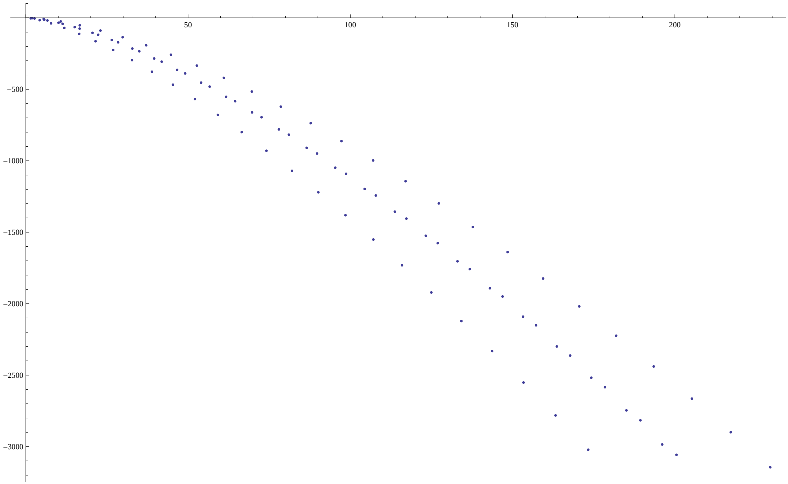

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 2, m = 1,\ G = \mathrm{SU}(2)$.

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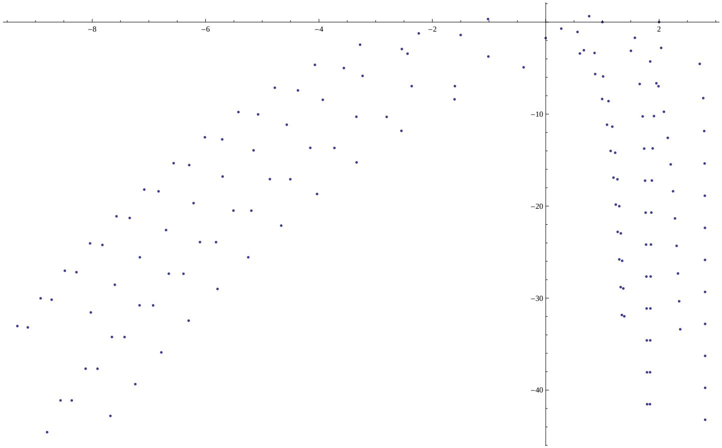

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 1,\ G = \mathrm{SU}(3)$.

Pretty pictures

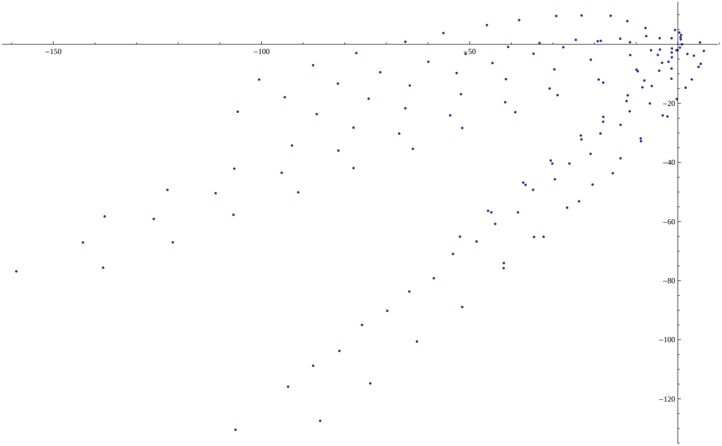

Figure: Plots of $Z_k(T_{t_\gamma^m})$ for $g = 1, m = 1,\ G = \mathrm{SU}(4)$.

Mapping tori with links

- Assume that M contains a framed link L, and choose for every component L_i of L a finite dimensional representation R_i of $G = \mathrm{SU}(N)$.
- Consider

$$Z_k^{\mathrm{phys}}(M, L, R) = \int_{\mathcal{A}_P/\mathcal{G}_P} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) \exp(2\pi i k \, \mathrm{CS}(A)) \mathcal{D}A.$$

- Again, there is a corresponding mathematical invariant $Z_k(M, L)$, components of L labelled by elements of $\Lambda = P_k(\mathfrak{sl}(N))$.

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Thanks ...

... for listening!