Qualifying exam

Integral lattices in TQFT

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Notation

- Let $p \geq 5$ be an odd integer, and let $d = (p-1)/2$.
- Let ζ_p denote a primitive p'th root of unity, and let

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\zeta_p] & \text{if } p \equiv -1 \pmod 4 \\ \mathbb{Z}[\zeta_{4p}] & \text{if } p \equiv 1 \pmod 4 \end{cases}$$

- Put $A = -\zeta_p^{d+1} \in \mathcal{O}$, $h = 1 - \zeta_p \in \mathcal{O}$, and define $\mathcal{D} \in \mathcal{O}$ by

$$\mathcal{D}^2 = \frac{-p}{(\zeta_p - \zeta_p^{-1})^2}.$$

Note: $h^{p-1} \sim p$ so $\mathcal{O}[\frac{1}{p}] = \mathcal{O}[\frac{1}{h}]$. Also, $\mathcal{D} \sim h^{d-1}$, and $A^2 = \zeta_p$.

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The BHMV SO(3)-TQFT

- The quantum invariant $\langle \cdot \rangle_p$ associates to every closed (framed) 3-manifold M containing a banded link L an element $\langle (M, L) \rangle_p \in \mathcal{O}[\frac{1}{p}]$.
- For M compact with boundary ∂M having coloured structure $l(\partial M)$, and let $K(M, l(\partial M))$ be the relative Kauffman $\mathcal{O}[\frac{1}{p}]$-module of skeins in M arising as the expansion of coloured graphs in M meeting $l(\partial M)$.
- For a coloured surface (Σ, l), and 3-manifolds (M, L), (M', L'), $\partial M = \partial M' = \Sigma$, we define a bilinear form $\langle \cdot, \cdot \rangle_{M,M'} : K(M, l) \times K(M', l) \to \mathcal{O}[\frac{1}{p}]$ by

$$\langle L, L' \rangle_{M,M'} = \langle M \cup_\Sigma -M', L \cup_l -L' \rangle_p.$$

- To (Σ, l) we associate a $\mathcal{O}[\frac{1}{p}]$-module $V_p(\Sigma)$ such that for M, $\partial M = \Sigma$, there is a surjective map $K(M, l) \to V_p(\Sigma)$ whose kernel is the left radical of this form.

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Integrality of the invariant

Theorem (Murakami '93, Masbaum–Roberts '97)

Let $p \geq 5$ be a prime, and let $I_p(M) = \mathcal{D}\langle M\rangle_p$. Then $I_p(M) \in \mathcal{O}$.

Question: Can we make I_p into an "integral" TQFT when p is prime?

Answer: More or less.

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Integral TQFT

Let (Σ, l) be given, and let p be prime.

Definition

Let $\mathcal{S}_p(\Sigma)$ be the $\mathcal{O}$-submodule of $V_p(\Sigma)$ generated by all vectors of the form $[(M, L)] \in V_p(\Sigma)$, where every component of M meets Σ.

Note that $\mathcal{S}_p(\Sigma)$ carries a natural $\mathcal{O}$-valued Hermitian form

$$(x, y)_\Sigma = \mathcal{D}^{b_0(\Sigma)} \langle x, y \rangle_\Sigma .$$

Theorem (Gilmer '04)

$(\mathcal{S}_p, Z_p)$ defines a functor from the category of targeted bordisms to the category of free finitely generated $\mathcal{O}$-modules.

Note: This functor does not satisfy the tensor product axiom.
Main goal: Describe an explicit basis of the lattice $\mathcal{S}_p(\Sigma)$ and $\mathcal{S}_p^\sharp(\Sigma)$.

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Applications

Definition

The cut number $c(M)$ of a closed oriented connected 3-manifold M is the maximal number of closed oriented surfaces that can be embedded in M without disconnecting it.

Let $\mathfrak{o}(M, L)$ be the highest power of h dividing $I_p(M, L)$ in $\mathcal{O}$.

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$$c(M) \leq \frac{\mathfrak{o}(M, L)}{d - 1}.$$

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Sketch of proof.

- Find a set of $c(M)+1$ surfaces Σ_i, such that no subset of c surfaces disconnect M, splitting M as $M = Y \cup Y'$.
- Let P be a cobordism in M from $\Sigma = \sqcup_i \Sigma_i$ to itself, constructed using c curves connecting the Σ_i.
- Describe P as surgery on $\Sigma \times I$ along these c curves.
- Calculate $I_p(M)$ as the union of Y, Y' and P using fusion rules, and keeping track of the powers of h arising, using the basis of $\mathcal{S}_p(\Sigma)$.

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Applications

Definition

Let N be a connected compact 3-manifold. Let $\mathcal{J}_p(N)$ be the ideal in $\mathcal{O}$ spanned by all the $I_p(M)$, where M is a closed, connected, and oriented 3-manifold containing N.

Proposition

If N_1 embeds in N_2, then $\mathcal{J}_p(N_1) \subseteq \mathcal{J}_p(N_2)$.

Theorem

Let $\partial N = \Sigma$. Then $\mathcal{J}_p(N)$ is generated over $\mathcal{O}$ by the inner products $([N], \mathfrak{b})_\Sigma$, where $\mathfrak{b}$ varies over the basis of $\mathcal{S}_p(\Sigma)$.

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Lollipop trees

Definition

Let (Σ, l) be a coloured surface of genus g bounding a handlebody H_g, and let s be the number of components in l. A 1-3-valent graph G in H_g meeting (Σ, l) as usual is called a lollipop tree if

1. G consists of exactly g loop edges.
2. If $s > 0$, then there is an edge, called the *trunk* such that removing the interior of it, the graph splits into two trees – one containing all loop edges, and one containing the s univalent vertices.

Proposition

The vectors in $V_p(\Sigma)$ associated to p-admissible colourings of a lollipop tree, where loop edges are coloured by elements of $[0, d-1]$, form a basis of $V_p(\Sigma)$.

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Example of a lollipop tree

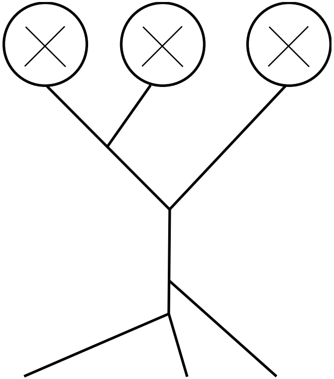

Figure: A lollipop tree and its colouring

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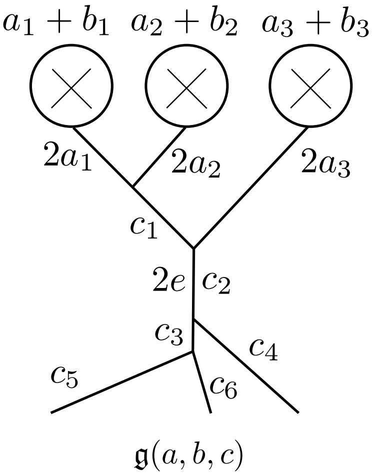

Figure: A lollipop tree and its colouring

The basis of $\mathcal{S}_p^\sharp(\Sigma)$

Similarly, define

$$\mathfrak{b}^\sharp(a,b,c) = h^{-\sum_i b_i - \lceil \frac{1}{2}(e+\sum_i a_i)\rceil}(2+z_1)^{b_1}\cdots(2+z_g)^{b_g}\mathfrak{g}(a,0,c).$$

Note that $\mathfrak{b}^\sharp(a,b,c)$ is an h-rescaling of $\mathfrak{b}(a,b,c)$.

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The tensor product axiom

- Let Σ, Σ' be closed surfaces. The natural map
$$V_p(\Sigma) \otimes V_p(\Sigma') \to V_p(\Sigma \sqcup \Sigma')$$
$$[(M, L)] \otimes [(M', L')] \mapsto [(M \sqcup M', L \sqcup L')]$$
restricts to $\mathcal{S}_p(\Sigma) \otimes \mathcal{S}_p(\Sigma') \to \mathcal{S}_p(\Sigma \sqcup \Sigma')$.
- Question: Is this an isomorphism?
- If the surfaces are uncoloured, it is.
- Put $\varepsilon(\mathfrak{b}) = 1$ if $e = d - 1$ and $\sum_i a_i - e$ is odd, and put $\varepsilon(\mathfrak{b}) = 0$ otherwise.

Theorem

Let $\Sigma_1, \ldots, \Sigma_n$ be connected with graph-like bases $\mathcal{B}_i$ of $\mathcal{S}_p(\Sigma_i)$. Then
$$\mathcal{B} = \{h^{-\lfloor \frac{1}{2} \sum_i \varepsilon(\mathfrak{b}_i) \rfloor} \mathfrak{b}_1 \otimes \cdots \otimes \mathfrak{b}_n \mid \mathfrak{b}_i \in \mathcal{B}_i\}$$
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A 3-manifold M, $\partial M = \Sigma$, *represents a partition* P of $\pi_0(\Sigma)$, if P coincides with the partition given by the fibres of $\pi_0(\Sigma) \to \pi_0(M)$.

A v-graph in M, $\partial M = \Sigma$, is a coloured graph together with link components coloured by v.

Proposition

Choose any collection $\mathcal{M}$ of 3-manifolds presenting all partitions of $\pi_0(\Sigma)$. Then $\mathcal{S}_p(\Sigma)$ is generated over $\mathcal{O}$ by v-graphs in $M \in \mathcal{M}$.

Proof of Theorem.

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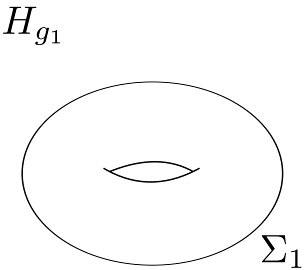

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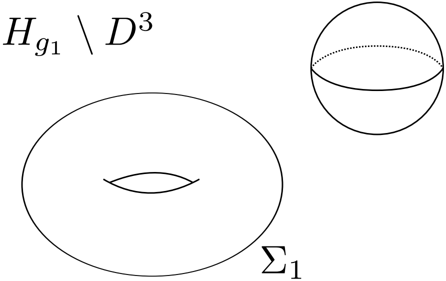

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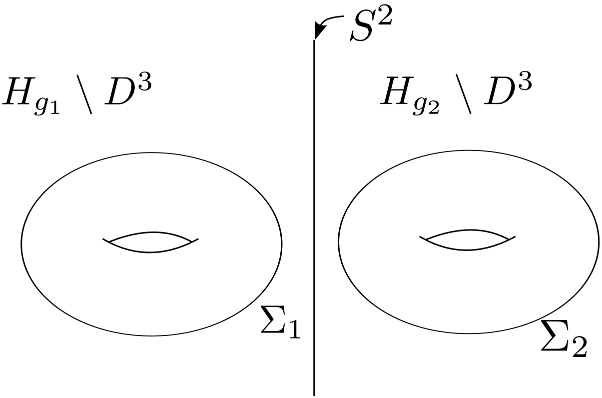

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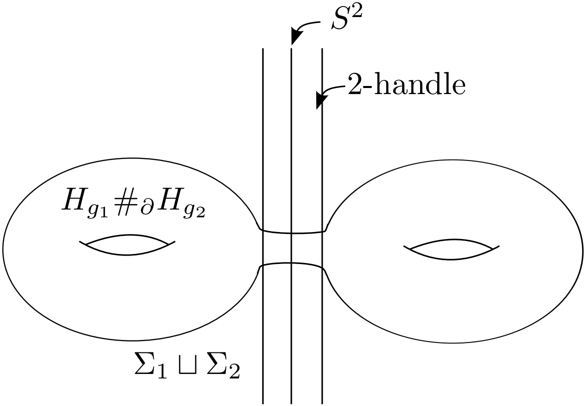

Finite torsion modules for $p \equiv -1 \pmod 4$

- Assume that $p \equiv -1 \pmod 4$. Then $\mathcal{O}/h\mathcal{O} \cong \mathbb{F}_p$.
- The hermitian form $(\cdot,\cdot)_\Sigma$ induces an $\mathbb{F}_p$-valued form $(\cdot,\cdot)_\Sigma$ on the $\mathbb{F}_p$-vector space $\mathcal{S}_p(\Sigma)/h\mathcal{S}_p(\Sigma)$.

Proposition

The form $(\cdot,\cdot)_\Sigma$ on $\mathcal{S}_p(\Sigma)/h\mathcal{S}_p(\Sigma)$ is symmetric, and the (ordinary) mapping class group of Σ acts on $\mathcal{S}_p(\Sigma)/h\mathcal{S}_p(\Sigma)$ preserving the form.

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Finite torsion modules for $p \equiv 1 \pmod 4$

- Assume that $p \equiv 1 \pmod 4$. Let $\mathcal{O}^+ = \mathbb{Z}[\zeta_p]$. Then $\mathcal{O}^+/h\mathcal{O}^+ \cong \mathbb{F}_p$.
- There are $\mathcal{O}^+$-modules $\mathcal{S}_p^+(\Sigma)$ such that $\mathcal{S}_p^+(\Sigma) \otimes_{\mathcal{O}^+} \mathcal{O} = \mathcal{S}_p(\Sigma)$, and there is a natural (skew-)hermitian form $(\cdot,\cdot)_\Sigma^+$ on $\mathcal{S}_p^+(\Sigma)$.

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The form $(\cdot,\cdot)_\Sigma^+$ on $\mathcal{S}_p^+(\Sigma)/h\mathcal{S}_p^+(\Sigma)$ is $(-1)^g$-symmetric, and the (ordinary) mapping class group of Σ acts on $\mathcal{S}_p^+(\Sigma)/h\mathcal{S}_p^+(\Sigma)$ preserving the form.

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