Research statement

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My research revolves around the mathematical field that has come to be known as *quantum topology*. Still only loosely defined, this field generally involves the mathematical study of Witten's quantum Chern–Simons theory; a quantum field theory whose relation with knot theory [25] has led to a number of developments in topology and geometry over the last 20 years. In this research statement I will discuss the mathematics arising from Chern–Simons theory, give an overview of my previous research and list several open problems, some of which I am already studying and several that I hope to study in the future. To me, one of the most interesting open problems in Chern–Simons theory and indeed in mathematical physics in general is making mathematical sense of the a priori ill-defined path integrals arising. Likewise, to me, one of the most interesting aspects – as far as applications of Chern–Simons theory are concerned – is its central role in the construction of the theoretical framework of topological quantum computation, cf. [14], [15], and [13].

1. Introduction

Let in the following G be a simply connected compact Lie group and let M be a closed oriented 3-manifold containing a framed link L. Let $P \to M$ be a principal G-bundle, let $\mathcal{A}_P$ denote the space of all connections in P, let $\mathcal{G}_P$ denote the space of gauge transformations of P acting on $\mathcal{A}_P$. Recall that the Chern–Simons action defines a map

$$\mathrm{CS} : \mathcal{A}_P/\mathcal{G}_P \to \mathbb{R}/\mathbb{Z}.$$

Let now $k \in \mathbb{Z}_{>0}$ be a *level* and choose for every component L_i of L a finite-dimensional representation R_i of G. Let $\mathrm{hol}_A(L_i) \in G$ denote the holonomy of a connection A about a component L_i of L. We then define the (physical) Chern–Simons partition function

$$Z^{\mathrm{phys}}_{k,G}(M, L) = \int_{\mathcal{A}_P/\mathcal{G}_P} \prod_i \mathrm{tr}(R_i(\mathrm{hol}_A(L_i))) \exp(2\pi i k\, \mathrm{CS}(A)) \mathcal{D}A. \tag{1}$$

Witten argued in [25] that this expression defines a topological invariant. Much of quantum topology is inspired by the interpretation of the above integral, which a priori makes no sense from a mathematical point of view as the measure $\mathcal{D}A$ is not defined. Attemps to make out of it a rigorous definition quickly arose for a number of different groups G, using the representation theory of quantum groups in [21], [22] and [23] (see also [24]) and using the skein theory of the Kauffman bracket and HOMFLY polynomial, [10], [9], [11], [8]. Let $Z_k(M, L) = Z_{k,G}(M, L)$ denote the mathematically defined invariant, which we refer to as the *quantum invariant*.

One of the main properties of the quantum invariants is their behaviour under cutting and gluing along surfaces embedded in the 3-manifold under scrutiny. More precisely, the invariants extend to invariants of manifolds with boundary and define a TQFT, [25], [7]: a functor from the category of 3-dimensional 2-framed cobordisms between surfaces to the category of complex vector spaces. Concretely, to an oriented surface Σ, we associate a vector space $V_k(\Sigma)$ depending on the level k, and to a compact oriented 3-manifold M with boundary $\partial M = \Sigma_1 \sqcup \overline{\Sigma_2}$, we associate a linear map $Z_k(M, L) : V_k(\Sigma_1) \to V_k(\Sigma_2)$. In particular, for a homeomorphism $f : \Sigma \to \Sigma$, the quantum invariant of the mapping cylinder of f, viewed as a cobordism C_f from Σ to itself, defines a (projective) linear isomorphism $\rho_k(f) = Z_k(C_f) : V_k(\Sigma) \to V_k(\Sigma)$. This descends to a projective representation $\rho_k : \mathrm{MCG}(\Sigma) \to \mathbb{P}V_k(\Sigma)$ is called the *level k quantum representation of the mapping class group* $\mathrm{MCG}(\Sigma)$. The link L may intersect the surface Σ in a number of marked points labelled by the additional data of the representations R_i; the mapping classes in question will be assumed to preserve this data, and the above described functor will depend non-trivially on it as well; we exclude this dependence from the notation to simplify the exposition in this research statement.

By the properties of TQFTs, for a mapping torus M_f of $f : \Sigma \to \Sigma$, the quantum invariant is the character of the quantum representation, $Z_k(M_f) = \operatorname{tr} \rho_k(f)$, and similarly, knowing a Heegaard decomposition of a 3-manifold M breaks down the study of $Z_k(M)$ to that of the quantum representations.

Now, the quantum representations admit a number of different constructions, all of which are known to be equivalent: the vector spaces $V_k(\Sigma)$ can be defined as certain Hom-modules of quantum groups and in the setup of skein theory, the TQFTs instead admit a combinatorial description allowing for algorithmic calculations [20]. For now, we focus on a more geometric construction: a complex structure on the surface Σ induces a complex structure σ on the corresponding moduli space $\mathcal{M}(\Sigma)$ of flat G-connections on Σ whose behaviour near marked points are determined by the R_i, and one here takes $V_k(\Sigma)$ to be the space of holomorphic sections $H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}_\sigma^k)$ of the k'th power of the Chern–Simons line bundle $\mathcal{L} \to \mathcal{M}(\Sigma)$ with respect to this complex structure; the dependence on the complex structure is encoded in the Hitchin connection ∇ (see [16]) in the bundle over the Teichmüller space of Σ whose fiber over a complex structure σ is $H^0(\mathcal{M}(\Sigma)_\sigma, \mathcal{L}_\sigma^k)$.

2. Asymptotic analysis of the quantum invariants

The rigorous definition of $Z_k(M)$ relies a surgery description of M, and so calculations have generally been done for manifolds with particularly simple surgery description, and the precise geometric content of Z_k has largely remained an open question; one I hope to explore. This, for instance, is built into the well-known Volume Conjecture, [19], stating that a particular evaluation of the Jones polynomial of a knot in a very precise way determines the volume of its complement in S^3.

Another open question[1] in the field is the so-called *Asymptotic Expansion Conjecture* (AEC): From its definition, it is hard to relate the quantum invariant Z_k to its physical heritage Z_k^{phys} and doing so exactly is impossible as only one is rigourously defined. However, via the method of stationary phase, one heuristically obtains an expansion for (1) for large values of the level k, which may then be compared to similar expansions for the mathematical invariant Z_k. Doing so, one ends up at the following conjecture, where we consider the invariants $Z_{k,G}$ arising when $G = \mathrm{SU}(N)$ and let $\mathcal{M}(M)$ denote the moduli space of flat $\mathrm{SU}(N)$-connections on M.

Conjecture 2.1 (Asymptotic expansion conjecture (AEC))**.** *Let M be a closed oriented 3-manifold, let $r = k + N$, and let $\{c_0 = 0, \dots, c_m\}$ be the values of the Chern–Simons action on $\mathcal{M}(M)$. Then there exist $d_j \in \frac{1}{2}\mathbb{Z}$, $b_j \in \mathbb{C}$, and $a_j^l \in \mathbb{C}$ for $j = 0, \dots, m$, $l = 1, 2, \dots$ such that*

$$Z_k(M) \sim_{k\to\infty} \sum_{j=0}^{m} e^{2\pi i r c_j} r^{d_j} b_j \left(1 + \sum_{l=1}^{\infty} a_j^l r^{-l/2}\right),$$

where $\sim_{k\to\infty}$ denotes an asymptotic expansion in the Poincaré sense (see [5]).

It is known that an expansion as in this conjecture is unique and that the constants d_j, b_j, a_j^l therefore must be topological invariants of $Z_k(M)$, and one might go on to formulate conjectures for geometric interpretations of these. For more details, see [1, Sect. 7,2].

The study of this conjecture for particular mapping tori of torus homeomorphisms was carried out by Andersen and myself in [5] (see also [18], [17]), tying the story for torus bundles, and it

[1] According to MathOverflow, it is currently a close race:
`https://mathoverflow.net/questions/104326/motivation-and-unsolved-problems-of-tqft`

has been studied for a large number of different families of 3-manifolds by various authors (see [5, Sect. 1.2] for a recent survey).

In [18, Sect. 5.5] we outline the relevant analysis to generalize the methods to [5] to the larger family of 3-manifolds consisting of mapping tori of Dehn twists in higher genus surfaces.

Another interesting question is concerned with the asymptotics of the observables of Witten's theory, namely what happens when the manifold is allowed to contain a link L (compare again with the Volume Conjecture); a study of this is the content of joint work with Andersen, Himpel, McLellan, Martens, and myself [4]. More precisely, we set up the framework necessary for using moduli space techniques to determine the asymptotics in the case of a surface with marked points. We then study a version of the AEC in the case of a finite order mapping torus; for these, many of the technical problems involving Kähler quantization of the relevant moduli spaces become redundant, but actually understanding the geometric quantization in this case is itself an interesting problem whose solution would shed light on the general AEC.

Finally, we note that some steps towards understanding the AEC for general mapping tori has been carried out in [18, Sect. 6.2.1], the main result relating to mapping tori of homeomorphisms whose action on moduli space has isolated fixed points; we hope to expand this result to a general study describing the asymptotics in terms of fixed point data.

3. The genus 0 case and surface dynamics

Of course, the study of the mapping class groups of surfaces is central in low-dimensional topology, and in joint work with Jens Kristian Egsgaard, we have considered to which extent the quantum representations can recover the properties of surface homeomorphisms; it is for example well-known that the collection of quantum representations, $\bigoplus_k \rho_k$ is faithful [2] (for particular marked point data), and a preprint by Andersen [3] shows that they can also be used to recover the Nielsen–Thurston classification of mapping classes, but it would be interesting to see exactly how much information one is able to recover. For instance, in [5] we show that for torus homeomorphisms, by taking particular subsequences of the ρ_k, one recovers in a precise way the topological entropies of Anosov homeomorphisms. This should be seen in the context of the so-called AMU conjecture.

Conjecture 3.1 ([6])**.** *Let Σ be a surface with $\chi(\Sigma)$, and let φ be a pseudo-Anosov mapping class. Then there exists $k_0 \in \mathbb{Z}_{>0}$ such that $\rho_k(\varphi)$ has infinite order for $k > k_0$.*

As mentioned above, it is not clear if the quantum invariants are sensitive to the geometry of the 3-manifolds, and just as the volume conjecture can be taken as a statement that they are indeed, the AMU conjecture gives a similar statement for the quantum representations. Concretely, the authors of [6] augment their conjecture by asking if it is possible to extract the entropy of a pseudo-Anosov from its quantum representation.

The authors prove the conjecture in the case of a sphere with five marked points labelled by certain $\mathrm{SU}(N)$-representations. The resulting vector spaces $V_k(S^2)$ are 2-dimensional and in the context of quantum computation, they are exactly the spaces representing single qubits. In this context, the quantum representations ρ_k define the appropriate set of quantum gates, and roughly speaking, the modular properties of the underlying TQFT define a way to represent a general collection of qubits by picking out certain subspaces of the vector spaces obtained by adding marked points.

In [12] we show that the methods of [6] generalize to spheres with any number of marked points, labelled by a set of $\mathrm{SU}(N)$-representations appropriate for quantum computation, and obtain as a result a proof of the AMU conjecture for pseudo-Anosovs whose singularity data is of a particular type. For the future, we hope to expand our results to general pseudo-Anosovs on spheres. Once again, there is good reason to expect moduli space techniques to provide the necessary tools for this [18].

Finally, remark that in the genus 0 case, the AMU conjecture is closely related to the open question of faithfulness of the Jones representation which in turn is directly related to the open question of whether or not the Jones polynomial is an unknot detector. It is therefore natural for me to hope to shed light on these questions in the future as well.

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