INTRODUCTION TO TQFT, TIFR, MUMBAI, MARCH 2014

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These are notes for a lecture I gave at TIFR, Mumbai; I wrote them to keep track of what I wanted to say more than anything, so please take them for what they are. Comments are very welcome and may be sent to s@fuglede.dk.

1. Witten's quantum Chern–Simons theory

These first two lectures are really the story of quantum topology; I will go all the way back to its origin: Witten's paper on Chern–Simons theory and the Jones polynomial (1989).

Let M be a closed, connected, and oriented 3-manifold. Let $G = \mathrm{SU}(N)$ (or in general, a simple, simply-connected, and connected Lie group), and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{A} = \mathcal{A}(M)$ be the space of connections in the trivial G-bundle $G \times M \to M$ over M, i.e. $\mathcal{A} \cong \Omega^1(M; \mathfrak{g})$; concretely, we can think of an element of $\mathcal{A}$ as a skew-hermitian matrix of 1-forms that has trace 0. Let $\mathcal{G}$ be the space of gauge transformations in this bundle, acting on $\mathcal{A}$ by pullback, i.e. $\mathcal{G} \cong C^\infty(M, G)$. Writing it like this, an element $g \in \mathcal{G}$ acts on a connection A by

$$g^*A = g^{-1}Ag + g^{-1}dg$$

Now, define the Chern–Simons action $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(dA \wedge A + \tfrac{2}{3} A \wedge A \wedge A).$$

Here, the wedge product of two matrices of 1-forms should be interpreted as the combination of matrix multiplication and wedge product. A calculation shows that CS defines a map $\mathrm{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$.

Witten's idea was to construct from this a topological invariant of the 3-manifold M. More precisely, let $k \in \mathbb{N}$, and put

$$Z_k(M) = \int_{\mathcal{A}/\mathcal{G}} \exp(2\pi i k\, \mathrm{CS}(A))\, \mathcal{D}A.$$

Here's a problem though: there's no known measure $\mathcal{D}A$ which makes sense here, but by using physics arguments, one is still able to manipulate the integral (typically called a path integral).

Assume now that $\partial M = \Sigma \neq \emptyset$, let $A \in \mathcal{A}(\Sigma)/\mathcal{G}(\Sigma)$, let $\mathcal{A}_A$ be the connections in $\mathcal{A}(M)$ that restrict to A [drawing], and let $\mathcal{G}_\partial$ be the gauge transformations that restrict to the identity on the boundary. Then we can define

$$Z_k(M)(A) = \int_{\mathcal{A}_A/\mathcal{G}_\partial} \exp(2\pi i k\, \mathrm{CS}(A'))\, \mathcal{D}A'.$$

Here's the kicker: as k grows large, the main contributions to this integral come from *flat* connections, and the (moduli) space of flat connections on a surface up to gauge is finite-dimensional (this is also a moduli space of bundles). Then, $Z_k(M)$ is to be interpreted as a holomorphic section of the k'th power of some line bundle over this moduli space, and there will be a finite-dimensional space of such sections. This is a special feature for Chern–Simons theory among quantum field theories which allows us to use rigorous mathematical arguments in manipulations of the path integral.

2. Atiyah's axioms for TQFT

Let's try to axiomatize this, following Atiyah's "Topological quantum field theory": A $(d+1)$-dimensional TQFT (Z, V) over a field Λ associates to a closed oriented d-manifold Σ a finite-dimensional vector space $V(\Sigma)$ over Λ, and to an oriented $(d+1)$-manifold M an element $Z(M) \in V(\partial M)$, such that

- If $f : \Sigma \to \Sigma'$ is an orientation preserving diffeomorphism, then we have a linear isomorphism $V(f) : V(\Sigma) \to V(\Sigma')$, such that $V(f \circ g) = V(f) \circ V(g)$. If f extends to a diffeomorphism $f : M \to M'$ with $\partial M = \Sigma$, $\partial M' = \Sigma'$, then $V(f)(Z(M)) = Z(M')$.
- $V(\Sigma \text{ with opposite orientation}) = V(\Sigma)^*$.
- $V(\emptyset) = \Lambda$, $Z(\emptyset) = 1$, $Z(\Sigma \times [0,1]) = \mathrm{id} \in V(\Sigma) \otimes V(\Sigma)^* = \mathrm{Aut}(V(\Sigma))$.
- Gluing: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$, and if $M = M_1 \cup_{\Sigma_3} M_2$ s.t. $\partial M_1 = \Sigma_1 \cup \Sigma_3$ and $\partial M_2 = \Sigma_2 \cup -\Sigma_3$, then $Z(M) = \langle Z(M_1), Z(M_2)\rangle$, in the dual pairing from $V(\Sigma_3)$ [drawing].

Example 1. If M is closed, $Z(M) \in \mathbb{C}$ is a topological invariant of M.

If $M = \Sigma \times [0,1]/(x,0) \sim (f(x),1)$ is a mapping torus of a diffeomorphism $f : \Sigma \to \Sigma$ [drawing], then one can show from the axioms that $Z(M_f) = \operatorname{tr} V(f)$, so in this case, it suffices to understand the two-dimensional part of the theory. In particular, if $f = \mathrm{id}$, then $Z(M_f) = Z(M \times S^1) = \dim V(\Sigma)$.

One can also show from the axioms that $V(f)$ depends only on f up to isotopy. Thus there is a map $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\text{isotopy} \to \mathrm{Aut}(V(\Sigma))$ which is typically called the *quantum representation* of $\mathrm{MCG}(\Sigma)$.

In fact, every closed oriented 3-manifold M has a Heegaard decomposition $M = H \cup_f -H$, where H is a handlebody, and $f : \Sigma \to \Sigma$ some diffeomorphism. It then follows that

$$Z(M) = \langle Z(H), V(f)(Z(H))\rangle,$$

so in order to understand the invariant $Z(M)$, one really only needs to understand the functor V, and the invariant Z on a single 3-manifold, the handlebody.

3. 3-dimensional TQFT

Let us now turn back to the case of interest: Witten's 3-dimensional quantum field theory, which supposedly fits into the framework of TQFT. Rigorous mathematical constructions of these that emulate Witten's construction in various ways quickly arose, the first of which were Reshetikhin and Turaev's construction using the representations theory of quantum groups. In modern language, the 2-dimensional part of the $(2+1)$-dimensional TQFT is referred to as a modular functor and by the above discussion, it often suffices to understand these. Notable constructions include the following:

- The representation theory of the quantum group $U_q(\mathfrak{sl}_N)$ at a $(k+N)$'th root of unity, $q = \exp(2\pi i/(k+N))$ (Daniel will perhaps elaborate on this); here, V is obtained as certain $U_q(\mathfrak{sl}_N)$-modules.
- The skein theory of the Jones/HOMFLYPT polynomial (due to Blanchet, Habegger, Masbaum, and Vogel); here, V is obtained as a quotient of a module of links (see below).
- Conformal field theory; here V is the space of vacua/conformal blocks.
- Geometric quantization of the moduli space $\mathcal{M}$ of flat connections on the surface; here $V = H^0(\mathcal{M}, \mathcal{L}^k)$ for a prequantum line bundle $\mathcal{L} \to \mathcal{M}$ (Niels?).

It is worth emphasizing that these are all so different that they allow for techniques to be drawn in from any of their underlying fields; moreover, the various constructions are understood to be equivalent[1], so that one may use any such techniques in parallel. In particular, each approach has its own advantages: for instance, the first two have a highly combinatorial flavor – enough so that computer implementation is readily available – while the fourth is well suited for studying large k asymptotics.

3.1. **Skein theory.** Let's be concrete and actually construct the TQFT for SU(2), following a construction of Roberts of the skein theoretical TQFT: Define $A = i\exp(\frac{\pi i}{2(k+2)}) \in \mathbb{C}$, and for an oriented compact 3-manifold M, let $K(M)$ be the complex vector space spanned by isotopy classes of banded links (inclusions $\bigsqcup_{j=1}^n S^1 \times I \to M$) in M modulo the local relation [drawing of Kauffman bracket].

[1]The connection between the fourth and the first three is still a bit unclear when the surface contains marked points; something I have not been talking about so far

Since for a link in S^3, all links may be resolved to a number of unknots, one finds that $K(S^3) = \mathbb{C}$. Now, let H and H' be standard handlebodies with $\partial H = \partial H' = \Sigma$. Then we can view $S^3 = H \cup H'$ [drawing]. There then is a natural pairing

$$\langle \cdot, \cdot \rangle : K(H) \times K(H') \to K(S^3) = \mathbb{C}$$

given by taking the union of links [drawing]. By taking the left kernel of $K(H)$ and right kernel of $K(H')$ with respect to this pairing, we obtain a non-degenerate pairing

$$\langle \cdot, \cdot \rangle : V(\Sigma) \times V'(\Sigma) \to K(S^3) = \mathbb{C}.$$

Here we already have our vector space (which turns out to be finite-dimensional!), and $Z(H) \in V(\Sigma)$ corresponds simply to the empty link in H. We now just need to define the action of the mapping class group of Σ. It is a theorem due to Lickorish that this group is generated by a finite set $T \cup T' \subseteq \mathrm{MCG}(\Sigma)$, where T consists of diffeomorphisms (Dehn twists) that extend over H and similarly T' consists of diffeomorphisms that extend over H'. For $f \in T$, we can therefore simply let f act on $K(H)$, which then in turn defines an automorphism of $V(\Sigma)$. Likewise, for $f \in T'$, f acts naturally on $V'(\Sigma)$, so we can use the pairing to define $V(f) \in \mathrm{Aut}(V(\Sigma))$ by

$$\langle V(f)x, y \rangle = \langle x, f^{-1}.y \rangle.$$

Finally, if $f \in \mathrm{MCG}(\Sigma)$ is a word in $f = f_1 \circ \cdots \circ f_n$ with $f_i \in T \cup T'$, we simply define

$$V(f) = V(f_1) \circ \cdots \circ V(f_n). \tag{1}$$

We are now done if we can show that this is actually well-defined, which turns out to almost be the case.

Theorem 2 (Roberts)**.** *The $V(f)$ of (1) depends only on the decomposition in $T \cup T'$ up to a k-dependent root of unity.*

Let us finally note that it turns out that there is no way to get around this root of unity (called the anomaly of the TQFT), but there are several concrete descriptions of it that one uses instead.

3.2. **Quantum invariants from surgery.** In the above, we appealed to Heegaard decompositions to cheat our way to quantum invariants $Z_k(M)$. Whereas reasonable, this is not quite how it was done originally, neither by Reshetikhin and Turaev, nor by Blanchet, Habegger, Masbaum, and Vogel. Instead, they defined the invariants as surgery invariants. Previously we have not talked about *why* the invariants constructed actually have anything to do with Witten's path integral. In short, Witten noticed that under surgery, his invariants would behave in a particular way, which may then be emulated and used as a building block for a mathematical construction.

Let us briefly describe how this construction goes: (integral) surgery associates to a banded link in S^3 a 3-manifold, which is obtained by removing a tubular neighbourhood of the link, and gluing in a new torus according to the banded structure (consider one boundary component to be the core of the solid torus removed and glue the meridinal curve from the new torus to the other boundary component of the link, twisting everything about a bit).

Theorem 3 (Lickorish–Wallace, Kirby, Fenn–Rourke)**.** *Using surgery, there is a one-to-one correspondence between* 3*-manifolds up to homeomorphism and banded links in S^3 up to so-called Kirby moves.*

The strategy is now the following: one tries to define an invariant of banded links in S^3 which is invariant under Kirby moves. When this is the case, the above theorem may be then directly applied to obtain a topological invariant of 3-manifolds.

For skein theory, this invariant of links is defined as follows: the coloured Jones polynomial J_m, $m \in \mathbb{N}$ associates to a link L a Laurent polynomial $J_{m,L}(q)$. Then, for every k, the average of the first k coloured Jones polynomials, $J_{1,L}(q), \ldots, J_{k,L}(q)$, does not change under Kirby moves if the polynomials are evaluated at $q = \exp(2\pi i/(k+2))$.