Master Class on TQFTs

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CRM, Bellaterra, 2012

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Disclaimer

These are notes from a master class given by Jørgen Ellegaard Andersen, Gregor Masbaum, Christian Pauly, Benjamin Himpel, Hans Christian Herbig, and Richard Wentworth in April 2012, during the event *Master Class and Workshop on Topological Quantum Field Theories* in Bellaterra and Barcelona. They have been written and TeX'ed during the lecture and have not been proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturers. Also, note that these notes contain none of the drawings drawn during the master class, and in particular, this makes the parts on skein theory hardly accessible to those that did not particate in the class. That being said, feel very free to send any comments and/or corrections to fuglede@qgm.au.dk.

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1 April 19th, 2012

1.1 Andersen I

The master class is about *topological quantum field theories* (TQFT). More specifically, it is about the Reshetikhin–Turaev TQFT, which depends on a quantum group, where we will be specifically talking about $U_q(\mathfrak{sl}(N,\mathbb{C}))$ for $q = \exp(2\pi i/(K+N))$.

The history of things are as follows: Things started in skein theory; the first player was Jones, defining the Jones polynomial. He was studying subfactors obtaining a representation of braid groups and certain traces on these that gave rise to a knot polynomial. Atiyah posted a challenge for the mathematical community, asking for a 3-dimensional interpretation of the Jones polynomial. In comes Witten who considered quantum Chern–Simons theory; as of yet, this is not mathematically rigourisly defined, but what Witten showed was that it contained the Jones polynomial. Around the same time, Atiyah, Segal, and Witten proposed a set of axioms that a topological quantum field theory should satisfy. Reshetikhin and Turaev saw this and using the theory of representations of quantum groups were able to define something satisfying these axioms. Witten also saw that quantum Chern-Simons theory is related to conformal field theory: What the TQFT associates to a 2-manifold is a finite dimensional vector space, and somehow this vector space should be the same that people in conformal field theory were talking about. A lot of people should be mentioned in these developments. Some are TUY, who developed mathematically the relevant part of conformal field theory. There was subsequently a development by Blanchet, Habegger, Masbaum and Vogel who used skein theory to construct the theory of $N = 2$, and Blanchet further constructed the theory for general N using skein theory and Turaev's modular category construction. Another approach to all of this is geometric quantization of moduli spaces, discussed also in Witten's paper. This approach was studied extensively in the beginning, but somehow a full construction of the TQFT using this does not exist as of yet.

Thus there are four different approaches to TQFT: The quantum group picture and skein theory share a lot of common features, being very combinatorial. On the other hand, conformal field theory (CFT) and geometric quantization (GQ) involve a lot of geometry and are somewhat harder to work out. A lot of people, including Laszlo, have worked out isomorphisms between CFT and GQ. This master class will be about constructing an isomorphism between the CFT picture and the skein theoretical picture.

We turn now to the two-dimensional part of the TQFT, which completely determines the TQFT. This two-dimensional part is called a *modular functor*:

Definition 1.1.1. A *surface* is a two-dimensional closed manifold.

A *pointed surface* (Σ, P, V) is a surface Σ and a finite subset of points $P \subseteq \Sigma$, where V is a projective tangent vector at every point in P, i.e. for $p \in P$ elements of $(T_p\Sigma \setminus \{0\})/\mathbb{R}_+$. A *morphism of pointed surfaces* $f : (\Sigma_1, P_1, V_1) \to (\Sigma_2, P_2, V_2)$ is the isotopy class of a diffeomorphism which takes P_1 to P_2 and V_1 to V_2, and isotopies are in this class of diffeomorphisms.

Definition 1.1.2. Suppose Σ is a connected oriented surface. Then a Lagrangian subspace of $H_1(\Sigma,\mathbb{Z})$ is a maximal isotropic subspace of $H_1(\Sigma,\mathbb{Z})$ with respect to $(\ ,\ ) : H_1(\Sigma,\mathbb{Z}) \times H_1(\Sigma,\mathbb{Z}) \to \mathbb{Z}$. If $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_n$, we consider only Lagrangians of the form $L = \bigoplus L_i$, where $L_i \subseteq H_1(\Sigma;\mathbb{Z})$ is Lagrangian.

Definition 1.1.3. A *marked surface* (Σ, P, V, L) is a pointed surface (Σ, P, V) together with a Lagrangian subspace L of $H_1(\Sigma,\mathbb{Z})$. A *morphism of marked surfaces* is a pair $(f, n) : (\Sigma_1, P_1, V_1, L_1) \to (\Sigma_2, P_2, V_2, L_2)$, where $f : (\Sigma_1, P_1, V_1) \to (\Sigma_2, P_2, V_2)$ is a morphism of pointed surfaces, and $n \in \mathbb{Z}$ is simply an integer. The composition of $(f_1, n_1) : (\Sigma_1, P_1, V_1, L_1) \to (\Sigma_2, P_2, V_2, L_2)$ and $(f_2, n_2) : (\Sigma_2, P_2, V_2, L_2) \to (\Sigma_3, P_3, V_3, L_3)$ is

$$(f_2, n_2) \circ (f_1, n_1) = (f_2 \circ f_1, n_1 + n_2 - \sigma((f_2 \circ f_1)_* L_1, (f_2)_* L_2, L_3)),$$

where σ is the Wall signature cocycle, which will be defined later.

Definition 1.1.4. A *label set* is a finite set of *labels* Λ (in CFT the set of basic elementary particles) with an involution $\hat{}$ with an element $1 \in \Lambda$ with $\hat{1} = 1$.

Definition 1.1.5. The category of Λ-labelled marked surfaces has objects marked surfaces whose marked points are labelled by elements of the label set, and morphisms are the obvious things (DIY; i.e. Do it yourself, Ignasi may or may not have a wife).

The marked surfaces form the category $\mathcal{C}_{\text{LMS}}$ on which the modular functor will be defined. The target category is the category $\mathcal{C}_{\text{VS}}$ of vector spaces with linear maps. We turn now to the axioms that a modular category will have to satisfy.

We have the following operations on Λ-labelled surfaces:

- Reversing orientation (reverse orientation on the surface and apply $\hat{}$ to all labels).
- Disjoint union.
- Gluing (of non-labelled surfaces): Consider $(\Sigma, \{p_+, p_-\} \sqcup P, \{v_+, v_-\} \sqcup V, L)$ (Jørgen draws a picture) and form the following labelled surface: Let $\tilde{\Sigma} = \{p_+, p_-\} \sqcup P(T_{p_+}\Sigma) \sqcup P(T_{p_-}\Sigma)$ be the surface with oriented boundary formed by blowing up Σ about two points, using the projective tangent vectors to give orientations (another picture). Choose $c : P(T_{p_-}\Sigma) \to P(T_{p_+}\Sigma)$ be an orientation reversing map with $c(v_-) = v_+$ and let Σ_c be the result of gluing $\tilde{\Sigma}$ using c. This can be understood through a normalization process (another picture), that is we have a normalization map n on Σ such that $n(\Sigma)$ is a particular quotient $q(\Sigma_c)$. Using these natural maps, we let $L_c = q_*^{-1}(n_*(L))$. Define the *glued surface* to be (Σ_c, P, V, L_c). It is easy to see that homotopic c will give diffeomorphic resulting glued surfaces.

We will simply write (Σ, λ) for labelled surfaces.

Definition 1.1.6. A *modular functor* is a functor $V : \mathcal{C}_{\text{LMS}} \to \mathcal{C}_{\text{VS}}$ is a functor satisfying the following axioms:

- A. Disjount union: Under disjount union,
$$V((\Sigma_1, \lambda_1) \sqcup (\Sigma_2, \lambda_2)) = V(\Sigma_1, \lambda_1) \otimes V(\Sigma_2, \lambda_2).$$
- A. Gluing: Under gluing,
$$V(\Sigma_c, \lambda) \cong \bigoplus_{\mu \in \Lambda} V(\Sigma, \mu, \hat{\mu}, \lambda).$$
- A. Empty: $V(\emptyset) = \mathbb{C}$.
- A. Sphere with one point: We have
$$V((S^2, p, v, \{0\}), \lambda) = \begin{cases} \mathbb{C} & \lambda = 1 \\ \{0\} & \lambda \neq 1 \end{cases}.$$
- A. Sphere with two points: We have
$$V((S^2, \{p_1, p_2\}, \{v_1, v_2\}, \{0\}), \lambda, \mu) = \begin{cases} \mathbb{C} & \hat{\lambda} = \mu \\ \{0\} & \hat{\lambda} \neq \mu \end{cases}.$$

Some extra properties a modular functor can have include *duality*: I.e. non-degenerate pairings
$$(\ ,\ )_{\Sigma,\lambda} : V(\Sigma, \lambda) \otimes V(\overline{\Sigma}, \hat{\lambda}) \to \mathbb{C},$$
respecting the dualities. Similarly, one could consider Hermitian modular functors (DIY).

We leave out duality as an axiom, since what we will construct on the conformal field theory side will not have a duality. The skein theory modular functor does, though, and using the isomorphism to be constructed, we can simply transfer the duality from here to the conformal field theory.

1.2 Pauly I

We will now turn to question of how to construct the modular functor using CFT, as described by Tsuchiya, Ueno, and Yamada (TUY, 1989), i.e. the functor $\mathcal{C}_{\text{LMS}} \to \mathcal{C}_{\text{VS}}$, where now objects Σ are Riemann surface/a complex projective curve.

In order to do so, we need to review some representation theory of (affine) Lie algebras. Denote by $\mathfrak{g}$ a simple complex Lie algebra. These are classified by Dynkin diagrams. A *representation* is a Lie algebra morphism $\mathfrak{g} \to \text{End}(V_\lambda)$, or, V_λ is an irreducible $\mathfrak{g}$-module. We review some properties of these. We decompose $\mathfrak{g}$ in a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. There is a distinguished vector $v_\lambda \in V_\lambda$ such that $n_+ v_\lambda = 0$, and $H v_\lambda = \lambda(H) v_\lambda$ for any $H \in \mathfrak{h}$. Here $\lambda \in \mathfrak{h}^*$ and v_λ is called the *highest weight vector*.

We define the set of *dominant weights*

$$P_+(\mathfrak{g}) = \{\lambda = \sum_{i=1}^{\rho} \alpha_i \omega_i \mid \alpha_i \in \mathbb{N}\},$$

where the ω_i are the fundamental weights of $\mathfrak{g}$, and $\rho = \dim_{\mathbb{C}}(\mathfrak{h}) = \text{rk}(\mathfrak{g})$.

Example 1.2.1. The ones we will be interested in are $\mathfrak{sl}(r)$, corresponding to the Dynkin diagram A_{r-1}. Here $V_{\omega_i} = \wedge^i \mathbb{C}^r$. There is an involution on $P_+(\mathfrak{sl}(r))$, that can be described as follows: We have a natural pairing $\wedge^i \mathbb{C}^r \cong (\wedge^{r-i}\mathbb{C}^r)^* = V^*_{\omega_{r-1}}$. The map $V_\lambda \mapsto (V_\lambda)^* = V_{\hat{\lambda}}$ thus gives an involution on the set of dominant weights. We can also write $\hat{\lambda} = -w_0(\lambda) \in P_+(\mathfrak{g})$, where w_0 is the longest element of the Weyl group $W(\mathfrak{g})$. In our particular case, $\hat{\omega_i} = \omega_{r-i}$.

For two root vectors $[X_\alpha, X_\beta] = \text{scalar} \cdot X_{\alpha+\beta}$. These scalars can be put into a matrix called the Cartan matrix. Properties of these Cartan matrices can be generalized to generalized Cartan matrices, as done by Kac and Moody, leading to other kinds of Lie algebras, not necessarily finite dimensional. They found a family $\hat{\mathfrak{g}}$, closely related to the simple $\mathfrak{g}$, and can be realized as $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}(\!(t)\!) \oplus \mathbb{C}c (\oplus \mathbb{C}d)$, where $\mathbb{C}(\!(t)\!)$ is the field of Laurent series in one variable, and the factor $\mathbb{C}d$ is not really needed. The Lie bracket on $\hat{\mathfrak{g}}$ is given by

$$[X \otimes f(t), Y \otimes g(t)] = [X, Y] \otimes f(t)g(t) + c \cdot (X \mid Y) \operatorname{Res}_{t=0}(g(t)\, df(t)),$$

where $(\,|\,)$ is the Cartan–Killing form on $\mathfrak{g}$. So for example, if $f(t) = t^n$, $g(t) = t^m$, then

$$[X \otimes t^n, Y \otimes t^m] = [X, Y] \otimes t^{n+m} + c(X \mid Y)\delta_{n+m,0} n.$$

The main result by Kac and Moody is that these infinite dimensional Lie algebras are classified by extended Dynkin diagrams.

Simlar to the triangular decomposition, we have a decomposition

$$\begin{aligned}\hat{\mathfrak{g}} &= \mathfrak{g} \otimes t\mathbb{C}[\![t]\!] \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}c \\ &=: \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \hat{\mathfrak{g}}_- \oplus \mathbb{C}c.\end{aligned}$$

An exercise is that these 4 subspaces are Lie subalgebras. We turn now to the representation theory of $\hat{\mathfrak{g}}$, as worked by Kac: Given $l \in \mathbb{N}$ called a *level* and $\lambda \in P_+(\mathfrak{g})$, there exists a $\hat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ or $\mathcal{H}_{\lambda,l}$, unique up to isomorphism, with the following properties:

1. $V_\lambda = \{v \in \mathcal{H}_\lambda \mid \hat{\mathfrak{g}}_+ v = 0\} \subseteq \mathcal{H}_\lambda$
2. c acts as $l \cdot \text{Id}_{\mathcal{H}_\lambda}$.
3. $\mathcal{H}_\lambda$ is generated by V_λ over $\hat{\mathfrak{g}}_-$ with one relation

$$X_\theta(-1)^{l-(\theta|\lambda)+1} v_\lambda = 0,$$

where θ is the maximal root of $\mathfrak{g} \in \mathfrak{h}^*$, normalized with $(\theta \mid \theta) = 2$, and $X_\theta \in \mathfrak{g}$, and we use the notation $X(n) = X \otimes t^n \in \hat{g}$.

So we have that

$$\mathcal{H}_\lambda \cong U(\hat{\mathfrak{g}}_-)V_\lambda / U(\hat{\mathfrak{g}})\dot{X}_\theta(-1)^{l-(\theta|\lambda)+1}v_\lambda.$$

We introduce now

$$(\Lambda =)P_l(\mathfrak{g}) = \{\lambda \in P_+(\mathfrak{g}) \mid (\lambda \mid \theta) \leq l\},$$

which cuts out a finite set of dominant weights, preserved by the involution $\lambda \mapsto \hat{\lambda}$.

Example 1.2.2. For $\mathfrak{g} = \mathfrak{sl}(r)$,

$$P_l(\mathfrak{sl}(r)) = \{\lambda = \sum \alpha_i \omega_i \mid 0 \leq \alpha_i, \sum \alpha_i \leq l\}.$$

Remark 1.2.3. We have a filtration on $\mathcal{H}_\lambda$, induced by the filtration $F^pU(\hat{\mathfrak{g}}_-)$, $p \in \mathbb{N}$ on the universal enveloping algebra $U(\hat{\mathfrak{g}}_-)$, given by those elements that can be generated by at most p elements. So we have $F^pU(\hat{\mathfrak{g}}_-) \twoheadrightarrow F^p(\mathcal{H}_\lambda) \subseteq \mathcal{H}_\lambda$, and we put

$$\mathcal{H}_\lambda(p) = F^p(\mathcal{H}_\lambda)/F^{p+1}(\mathcal{H}_\lambda) \hookrightarrow \mathcal{H}_\lambda,$$

which will appear as eigenspace for L_0-Virasoro operators, as will be explained later.

An important fact is that in the filtration $\mathcal{H}_\lambda = F^0\mathcal{H}_\lambda \supset F^1\mathcal{H}_\lambda \supset \dots$, we have the $\mathfrak{g}$-module $V_\lambda \cong \mathcal{H}_\lambda(0) \subseteq \mathcal{H}_\lambda$, and $\mathcal{H} - \lambda = \bigoplus_{p=0}^\infty \mathcal{H}_\lambda(p)$.

Remark 1.2.4. The representation theory of $\hat{\mathfrak{g}}$ is independent the coordinate t. That is, if we do a change of coordinates $u(t) = u = t +$ higher order terms, this induces an automorphism $\mathbb{C}((t)) \to \mathbb{C}((t))$ by $t \mapsto u(t)$, and it induces an Lie algebra endomorphism $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$ by $X \otimes f(t) \mapsto X \otimes f \circ u$, $c \mapsto c$. We denote this endomorphism by γ_u. For a representation π_λ, Now the composition

$$\hat{\mathfrak{g}} \xrightarrow{\gamma_u} \hat{\mathfrak{g}} \xrightarrow{\pi_\lambda} \mathrm{End}(\mathcal{H}_\lambda)$$

is another representation. Now $\gamma_u(c) = c$, $\gamma_u(\hat{\mathfrak{g}}_+) = \hat{\mathfrak{g}}_+$ and again $\hat{\mathfrak{g}}_+\dot{V}_\lambda = 0$, so the representation $\pi_\lambda \circ \gamma_u$ is isomorphic to the old representation. In other words, there exists a canonical isomorphism $\Gamma_U \in \mathrm{GL}(\mathcal{H}_\lambda)$ (up to a scalar) such that the diagram

$$\begin{array}{ccc} \hat{\mathfrak{g}} & \xrightarrow{\pi_\lambda} & \mathrm{End}(\mathcal{H}_\lambda) \\ \downarrow{\scriptstyle \gamma_u} & & \downarrow{\scriptstyle \text{conj. with } \Gamma_u} \\ \hat{\mathfrak{g}} & \xrightarrow{\pi_\lambda} & \mathrm{End}(\mathcal{H}_\lambda). \end{array}$$

commutes. Thus what we will be considering when discussing conformal blocks is not only functions in t, but we will allow functions with poles.

Fix $g, n \in \mathbb{N}$ and $(C, \vec{p} = (p_1, \dots, p_n))$ is a stable connected Deligne–Mumford curve (in $\overline{\mathcal{M}_{g,n}} \supset \mathcal{M}_{g,n}$) over $\mathbb{C}$ with arithmetic genus $g = \dim H^1(C, \mathcal{O}_C)$ and n marked points. Here, *stable* means that we allow singularities but only ordinary double points (drawing here) and that $|\mathrm{Aut}(c)| < \infty$ (meaning e.g. that for $\mathbb{P}^1$ there must be at least three marked points).

Fix also a level $l \in \mathbb{N}$, and $\mathfrak{g}$ a simple Lie algebra, and a labelling $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in P_l(\mathfrak{g}) = \Lambda$. Put $\mathcal{H}_\lambda = \bigotimes_{i=1}^n \mathcal{H}_{\lambda_i}$ (depending on l). We introduce the ring $\mathcal{O}(C \setminus \vec{p})$ of rational functions on C that can have poles of arbitrary order at $\vec{p}$ and are regular on $C \setminus \vec{p}$. For all i, we choose coordinates z_i at p_i. For each i, and $f \in \mathcal{O}(C \setminus \vec{p})$, we write $f = \sum_{n \geq -n_0}^\infty a_{n,i} z_i^n$, and for $X \in \mathfrak{g}$, we define

$$X[f_i] = \sum a_{n,i} X \otimes t^n \in \hat{\mathfrak{g}}.$$

There is now an action of $\mathfrak{g} \otimes \mathcal{O}(C \setminus \hat{\mathfrak{g}})$ on $\mathcal{H}_{\vec{\lambda}}$: We write $v = v_1 \otimes \cdots \otimes v_n$ for $v \in \mathcal{H}_{\vec{\lambda}}$ and define

$$X \otimes f(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^n v_1 \otimes \cdots \otimes X[f_i]v_i \otimes \cdots \otimes v_n.$$

Definition 1.2.5. The *space of covacua* is

$$V_{l,\vec{\lambda}}(C,\hat{p}) = \mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(C\setminus\vec{p})\mathcal{H}_{\vec{\lambda}},$$

and the *space of vacua* or *conformal block* $V^*_{\vec{\lambda},l}(C,\vec{p})$ is the dual of the space of covacua. I.e. elements of $\mathrm{Hom}_{\mathfrak{g}(C\setminus\vec{p})}(\mathcal{H}_{\vec{\lambda}},\mathbb{C})$, where the subscript denotes invariance under $\mathfrak{g}(C\setminus\vec{p})$. This invariance is referred to as *gauge invariance*.

Remark 1.2.6. We make a digression, explaining the link with moduli spaces. Let $\mathfrak{g}=\mathfrak{sl}(r)$ (the following also work for other Lie algebras). There is a theorem due to Weil, Witten, Faltings, and a lot of other people: If $\vec{p}=p_1$ and let $\lambda_1 = 0\in\Lambda$, the distinguished label that is the trivial representation, then there is a canonical isomorphism

$$V^*_{\vec{\lambda},l}(C,p)\cong H^0(\mathrm{SU}_C(r),\mathcal{L}^{\otimes l}),$$

where $\mathrm{SU}_C(r)$ is the moduli space of semi-stable rank r vector bundles with determinant $\mathcal{O}_C$ over C, and $\mathcal{L}$ is the determinant line bundle over $\mathrm{SU}_C(r)$.

1.3 Masbaum I

We turn now to the combinatorial construct of modular functors. We start with a bit of history, forming a part of the big plan.

Reshetikhin and Turaev defined using $U_q\mathfrak{sl}_2$, for q a root of unity, a TQFT. This means in particular that there is a modular functor $V(\Sigma)$ and a 3-manifold invariant $Z(M^3)\in V(\partial M)$. In fact in the picture we have in mind here, the 3-manifold invariant will be used to construct the modular functor. Masbaum together with Blanchet, Habegger and Vogel used skein theory of the Kauffman bracket (a particular version of the Jones polynomial) to construct a TQFT. Turaev invented the notion of modular category and explained how from such a thing, one can always get a TQFT. In the litterature, one finds that $U_q\mathfrak{sl}_n$ at q a root of unity gives a modular category, by combining a lot of results. Similarly, one can take the skein theory of the HOMFLYPT polynomial, as described by Blanchet. In these lectures, we describe how one gets a modular category following Blanchet, and also discuss the 3-manifold invariants.

A *link* in a 3-manifold is just a collection of embedded circles in S^3 (drawing). We will always be talking about *framed links*/knots: For K a knot, the framing is an orientation together with a non-zero normal vector field up to scaling by $\mathbb{R}_+$, giving a parallel of the knot, and considered up to isotopy. For a fixed knot K, then framings up to isotopy correspond *non-canonically* to $\{\pm1\}\times\mathbb{Z}$, the $\{\pm1\}$ corresponding to the orientation, and the $\mathbb{Z}$ to the isotopy class of the normal vector field. One can change this number by 1 by giving the knot a positive twist (drawing). In pictures we can always indicate a framing just by the orientation; any framed knot/link has a diagram in the plane with framing encoded by (the drawing that is not included here ...)

A *tangle* is a link with boundary. We want to define two versions of a tangle category $\mathcal{T}$. The first version is the following: Objects are finite sequences $(\varepsilon_1,\dots,\varepsilon_p)$, $\varepsilon_i=\pm1$. Morphisms between objects are framed tangles up to isotopy with orientations of the components corresponding to the signs determined by the objects (drawing), and the framing should be standard at the boundary points (another drawing). Composition of morphisms is given by stacking tangles. In fact the category is a *monoidal category* (or a *tensor category*), i.e. for objects $\mathrm{ob}_1,\mathrm{ob}_2\in\mathrm{Ob}(\mathcal{T})$, we can define $\mathrm{ob}_1\otimes\mathrm{ob}_2\in\mathrm{Ob}(\mathcal{T})$ and similarly for morphisms by concatenation/juxtaposition for objects, and placing diagrams next to each other. This tensor product is strictly associative, i.e.

$$(\mathrm{ob}_1\otimes\mathrm{ob}_2)\otimes\mathrm{ob}_3=\mathrm{ob}_1\otimes(\mathrm{ob}_2\otimes\mathrm{ob}_3).$$

The identity/unit object $\mathbf{1}$ for the tensor product is the empty sequence.

A *ribbon category* $\mathcal{C}$ is a tensor category with identity object $\mathbf{1}$ together with a chosen set of morphisms satisfying some axioms:

- For every object V, we have an operator $\theta_V \in \mathrm{End}_{\mathcal{C}}(V)$ called a *twist*. In the tangle category, this corresponds to the straight line between objects, given a full positive twist (drawing).
- For objects V, W, we have $c_{V,W} : V \otimes W \to W \otimes V$ called a *braiding*. In the tangle category, this is the tangle of a crossing (drawing). Between the twist and braiding, we have the Yang–Baxter equation (or the Reidemester 3 move or the braid relation) (drawing).
- For every object V, we have an object V^* called the *dual* and morphisms $b : \mathbf{1} \to V \otimes V^*$ and $e : V^* \otimes V \to \mathbf{1}$ called *coevaluation* and *evaluation* respectively. In the tangle category, duality is given by cups and caps (drawing) and $(\varepsilon_1, \dots, \varepsilon_p)^* = (-\varepsilon_p, \dots, -\varepsilon_1)$.

The axioms are exactly those that hold in the tangle category. The ribbon category we consider will be constructed out of tangles and thus satisfy all of these axioms automatically.

Example 1.3.1. Another trivial example of a ribbon category is the category Vect of finite dimensional $\mathbb{C}$-vector spaces with $\mathbf{1} = \mathbb{C}$, $\theta_V = \mathrm{id}_V$, and $C_{V,W}(x \otimes y) = y \otimes x$, and V^* the dual of V. Then for $1 \in \mathbb{C}$, $b(1) = \sum e_i \otimes e_i^*$.

For the quantum group $U_q\mathfrak{g}$, i.e. the q-deformation of $U\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$, a simple Lie algebra, the category of $U_q\mathfrak{g}$-modules is a ribbon category. E.g. the braiding $c_{V,W}$ is constructed using the$R$-matrix. We will construct via the Homflypt polynomial a ribbon category equivalent to what one obtains from quantum groups.

In a ribbon category, we can talk about the *quantum dimension* of objects, $\dim X \in \mathrm{End}(\mathbf{1})$ for an object X by something like $e \circ c \circ (\theta^{\pm} \otimes 1) \circ b \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$.

We have the following universal property of ribbon categories: If $\mathcal{C}$ is a ribbon category, T a tangle, $T \in \mathrm{Hom}_{\mathcal{T}}(\mathrm{ob}_1, \mathrm{ob}_2)$, such that every component of T has been labelled by some object of $\mathcal{C}$ (drawing), then we can associate to T a morphism $Z(T)$ of $\mathcal{C}$.

One can prove, using this formalism that in a ribbon category $(V^*)^* \cong V$, canonically.

Next time we will define a modular category, which is essentially a ribbon category with a collection of certain simple objects, corresponding to the set Λ. Then, from a modular category, one can then construct a TQFT.

1.4 Pauly II

Recall that we discussed the conformal block, $V^*_{l,\vec{\lambda}}(C, \vec{p})$. This morning we described the connection with algebraic geometry.

Example 1.4.1. Consider the case genus $= 0$, $C = \mathbb{P}^1$. We had a filtration

$$\mathcal{H}_\lambda \supset \mathcal{H}_\lambda(0) = V_\lambda,$$

V_λ a $\mathfrak{g}$-module. We have an inclusion $V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots V_{\lambda_n} \xhookrightarrow{i} \mathcal{H}_{\vec{\lambda}}$. We use the notation $\langle\psi| \in V^*_{l,\vec{\lambda}}(C, \vec{p})$ and $|\varphi\rangle \in V_{l,\vec{\lambda}}(C, \vec{p})$, and write $\langle\psi|\varphi\rangle$ for the pairing. Then gauge invariance means $\langle\psi|X[f]|\varphi\rangle = 0$ for all X, f, φ, or $\langle\psi|X[f] = 0$. The inclusion i induces an injective linear map $V^*_{l,\vec{\lambda}}(\mathbb{P}^1, \vec{p}) \hookrightarrow V^*_{\vec{\lambda}}$. Consider the constant map $f = 1$. Then gauge invariance is $\langle\psi|X \otimes 1 = 0$, so in fact the conformal block is a $\mathfrak{g}$-invariant subset of $V^*_{\vec{\lambda}}$. As a corollary of this, $\dim V^*_{l,\vec{\lambda}}(\mathbb{P}^1, \vec{p}) < \infty$.

Theorem 1.4.2 (TUY). *Suppose $p_i = t_i \in \mathbb{C} \subseteq \mathbb{P}^1$. Then we have a linear map*

$$V_{\vec{\lambda}} \to V_{\vec{\lambda}}, \quad v_1 \otimes \cdots \otimes v_n \mapsto \sum_{i=1}^{n} t_i v_1 \otimes \cdots \otimes X_\theta v_i \otimes \cdots \otimes v_n,$$

satisfies

$$V^*_{l,\vec{\lambda}}(\mathbb{P}^1, \vec{p}) = \{\varphi \in (V^*_{\vec{\lambda}}) \mid \varphi|_{\mathrm{Im}(T^{l+1})} \equiv 0\}.$$

Example 1.4.3. Consider $\mathfrak{sl}(2)$. Then $V_{\omega_1} = \mathbb{C}^2$ and

$$\Lambda = P_l(\mathfrak{sl}(2)) = \{0, \omega_1, \ldots, l\omega_1\} = \{0, 1, \ldots, l\}.$$

Consider $n = 3$, $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda^3$. Then

$$\dim V_{\vec{\lambda},l}(\mathbb{P}^1, \vec{p}) = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ even, } |\lambda_1 - \lambda_2| \leq \lambda_3 \leq \lambda_1 + \lambda_2,\ \lambda_1 + \lambda_2 + \lambda_3 \leq 2l \\ 0 & \text{otherwise.} \end{cases}$$

The first two equations are simply the Clebsch–Gordan condition, i.e. the condition that $V_{\lambda_3} \hookrightarrow V_{\lambda_1} \otimes V_{\lambda_2}$. The last equation is a quantum add-on.

Corollary 1.4.4. *If* $n = 1$,

$$\dim V_{\lambda,l}(\mathbb{P}^1, p) = \begin{cases} 0 & \lambda \neq 0 \\ 1 & \lambda = 0 \end{cases}.$$

If $n = 2$,

$$\dim V_{(\lambda_1,\lambda_2),l}(\mathbb{P}^1, (p_1, p_2)) = \begin{cases} 0 & \lambda_2 \neq \hat{\lambda_1} \\ 1 & \lambda_2 = \hat{\lambda_1}. \end{cases}$$

Consider now any curve C of any genus g. The results of TUY are the following:

1. $V_{\vec{\lambda},l}(C, \vec{p}) < \infty$.

2. There is a locally free sheaf over $\overline{\mathcal{M}_{g,n}}$, denoted $\mathcal{V}^*_{\vec{\lambda},l}(\mathcal{T}, \vec{\sigma})$, such that the fiber of this is $V_{\vec{\lambda},l}(C, \vec{p})$ over $(C, \vec{p}) \in \overline{\mathcal{M}_{g,n}}$, and it has a projectively flat connection. This implies that $\dim_{\vec{\lambda},l}(C, \vec{p})$ does not depend on C nor $\vec{p}$.

3. Propagation of vacua: For all $q \in C \setminus \vec{p}$, we have canonical isomorphisms

$$V^*_{l,\vec{\lambda}} \leftarrow V^*_{l,\vec{\lambda}\cup 0}(C, \vec{p} \cup q).$$

   This isomorphism can be understood as follows: We have $\mathcal{H}_0(0) = V_0 = \mathbb{C} \ni v_0$, and map $\mathcal{H}_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$ mapping $c \mapsto v \otimes v_0$, and one needs to check what happens to the gauge invariance condition under this map.

4. Factorization (gluing): Suppose C is a nodal curve with node $n \in C$ and the partial desingularization $\tilde{C} \xrightarrow{\pi} C$ over the node n, so $\pi^{-1}(n) = \{a, b\}$. Then there is an isomorphism

$$\bigoplus_{\mu \in \Lambda} V^*_{l,\vec{\lambda}\cup\mu\cup\hat{\mu}}(\tilde{C}, \vec{p} \cup \{a, b\}) \to V^*_{l,\vec{\lambda}}(C, \vec{p}).$$

   given by describing linear maps $\mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_\mu \otimes \mathcal{H}_{\hat{\mu}}$, mapping $v \mapsto v \otimes \mathrm{id}_\mu$, where $\mathrm{id}_\mu \in V_\mu \otimes V^*_\mu \subseteq \mathcal{H}_\mu \otimes \mathcal{H}_{\hat{\mu}}$.

Sketch of proof of finite dimensionality. Consider only the case $n = 1$. The first observation is that $\langle \hat{\mathfrak{g}}_+, \mathfrak{g}(C \setminus \{p\}) \rangle \subseteq \hat{\mathfrak{g}}$ has finite codimension. Here $\mathfrak{g}(C \setminus p)$ are elements that can be written $X[f]$, where $f = \alpha/z^n + \cdots$, $n \in \mathbb{N}$, where we have chosen coordinates z at p.

We can choose elements $e_i \in \hat{\mathfrak{g}}$ such that

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g}(C \setminus p) \oplus \bigoplus_{i=1}^{n} \mathbb{C}e_i$$

and such that the e_i are locally finite, i.e. that $\dim\langle e_i^k \cdot u\rangle_{k\in\mathbb{N}} < \infty$ for all $u \in \mathcal{H}_\lambda$.

The last step is to use the PBW theorem

$$U(\hat{g}) = U(\mathfrak{g}(C \setminus p)) \otimes \bigotimes_{m_i\in\mathbb{N}} e_1^{m_1} \cdots e_n^{m_n} \otimes U(\hat{\mathfrak{g}}_+).$$

If we write $\mathcal{H}_\lambda = U(\hat{g})V_\lambda$, we can express

$$\mathcal{H}_\lambda = U(\mathfrak{g}(C \setminus p))L,$$

where L is some finite dimensional vector space, $L \twoheadrightarrow \mathcal{H}_\lambda/\mathfrak{g}(C \setminus p)$, so the dimension of the conformal block is finite dimensional. □

Tomorrow we will discuss the projectively flat connection described above.

In the last part of this lecture, we make a small digression on the Verlinde formula not relevant for the rest of the lectures. The Verlinde formla is the formula which gives the dimension of the conformal blocks. The aim is to compute $\dim V_{l,\vec{\lambda}}(C, \vec{p}) = N_g(\lambda_1 + \cdots + \lambda_m)$, where here $\lambda_i \in \Lambda$ and we consider the formal sums of such. To this purpose it is practical to consider the *fusion ring*

$$\mathcal{R}_l(\mathfrak{g}) = \mathbb{Z}^{|\Lambda|} = \{\sum_{\lambda\in\Lambda} a_\lambda \lambda, a_\lambda \in \mathbb{Z}\}.$$

Factorization says that

$$N_g(\lambda_1 + \cdots + \lambda_n) = \sum_{\mu\in\Lambda} N_{g-1}(\lambda_1 + \cdots + \lambda_n + \mu + \hat{\mu}) = \sum_{\mu_1,\ldots,\mu_g\in\Lambda_+} N_0(\lambda_1 + \cdots + \lambda_n + \mu_1 + \hat{\mu_1} + \cdots + \mu_n + \hat{\mu_n}).$$

The next step is to note the following properties of N_0:

$$N_0(0) = 1, \quad N_0(\lambda) = 0, \lambda \neq 0$$
$$N_0(\hat{\lambda}) = N_0(\lambda),$$
$$N_0(x + y) = \sum N_0(x + \lambda)N_0(y + \hat{\lambda}).$$

The idea, due to Verlinde, is to define a multiplication on $\mathcal{R}_l(\mathfrak{g})$,

$$\lambda \cdot \mu = \sum_{\nu\in\Lambda} N_0(\lambda + \mu + \hat{\nu})\nu.$$

Factorization implies associativity of this product and defines a ring structure on the fusion ring. By induction on n,

$$N_0(\lambda_1 + \cdots + < lambda_n) = t(\lambda_1 \cdots \lambda_n),$$

where $t \in R_l(\mathfrak{g}) \to \mathbb{Z}$ is $\sum n_\lambda \lambda \mapsto n_0$. Thus

$$N_g(\lambda_1 + \cdots + \lambda_n) = t(\lambda_1 \cdots \lambda_n \omega^g),$$

where $\omega = \sum_{\mu\in\Lambda} \mu\hat{\mu} \in R_l(\mathfrak{g})$. Also, $t(x\omega) = \mathrm{Tr}(x)$, where x is viewed as the operator of multiplication in the ring. One obtains the following: If $\widehat{R_l(\mathfrak{g})}$ is the character of $R_l(\mathfrak{g})$, one has

$$N_l(\lambda_1 + \cdots + \lambda_n) = \sum_{\chi\in\widehat{R_l(\mathfrak{g})}} \chi(\ldots)$$

Remark 1.4.5. Now, let $\mathfrak{g} = \mathfrak{sl}(2)$. Let Γ be a trivalent graph with Betti number $b_1(\Gamma) = g$ and n leaves. A labelling og Γ is a map f from the edges of Γ to Λ. It is called admissible if for every vertex p of Γ with adjacent edges e_1, e_2, e_3 we have $(f(e_1), f(e_2), f(e_3)) \in \Lambda^3$ satisfy the quantum Clebsch–Gordan conditions. Then $N_g(\lambda_1 + \cdots + \lambda_n)$ is the number of admissible labellings of Γ. (Pauly draws the case $g = 2$.)

2 April 20th, 2012

2.1 Masbaum II

Last time we discussed the framed tangled category and considered particular elementary tangles: an evaluation, a coevaluation, a braiding, and a twist. We saw that any was composed of these and that we have a universal ribbon category.

We turn now to modular categories. We will not give the precise definition right now, but they are certain ribbon categories with extra properties that are very similar to those categories of representations of quantum groups. In such we have much more structure; we have direct sums, subrepresentations and so on, and somehow we have to make this out of tangles.

The next big thing in the construction is skein theory. Let R be a commutative ring, containing some elements a, s, v. Let $R\mathcal{T}$ be the R-linear tangle category with the same objects as $\mathcal{T}$ but with $\mathrm{End}_{R\mathcal{T}}$ between two objects being R-linear combinations of elements of $\mathrm{End}_{\mathcal{T}}$. We consider the HOMFLYPT skein relations,

$$a^{-1} \cdot \text{positive crossing} - a \cdot \text{negative crossing} - (s - s^{-1}) \cdot \text{two upwards strands} = R_1 \in \mathrm{End}_{R\mathcal{T}}(+,+),$$
$$\text{strand with positive twist} - av^{-1} \cdot \text{upwards strand} = R_2 \in \mathrm{End}_{R\mathcal{T}}(+)$$
$$\text{unknot} - \frac{v^{-1} - v}{s - s^{-1}} = R_3 \in \mathrm{End}_{R\mathcal{T}}(\emptyset),$$

where $\emptyset = \mathbf{1}$. We define the *Homflypt category* $\mathcal{HT}$ with the same objects as $\mathcal{T}$ with

$$\mathrm{Hom}_{\mathcal{HT}}(\mathrm{ob}_1, \mathrm{ob}_2) = \mathrm{Hom}_{\mathcal{RT}}(\mathrm{ob}_1, \mathrm{ob}_2)/\mathcal{I}(\mathrm{ob}_1, \mathrm{ob}_2),$$

where $\mathcal{I}$ is the tensor ideal generated by R_1, R_2, R_3 (drawing), i.e. we consider all linear combinations of tangles up to skein relations.

Theorem 2.1.1 (HOMFLYPT). *The set $\mathrm{End}_{\mathcal{HT}}(\emptyset)$ of R-linear combinations of framed links in S^3 up to isotopy, modulo the Homflypt relations, is isomorphic to R. The isomorphism is called the Homflypt polynomial.*

The $\mathrm{SL}(n)$-specialization will be the following:

$$a = q^{-1/(2N)}, \quad v = q^{-N/2}, \quad s = q^{1/2},$$

and R consists of rational functions in $a = q^{-1/(2N)}$ or q generic (i.e. not a root of unity). Later we will do the $\mathrm{SL}(N)_K$-specialization, where q will be a primitive root of unity of order $K + N$, really $q = \exp(2\pi i/(K + N))$. The Jones polynomial is the special case $N = 2$. We could from the Homflypt relations make the specialization $v = a$ to obtain something depending not on the framing, but that is not we do. In the $\mathrm{SL}(N)$-specialization, the Homflypt polynomial of a link is a Laurent polynomial in $q^{-1/(2N)}$.

We will use the following notation: N and K will be global variables, and n a local variables. Corresponding to N and K in Christian's talks are r and l.

We now want to describe $\mathrm{End}_{\mathcal{HT}}(++\cdots+)$ (with n +'es). This is actually the Hecke algebra H_n, that is, the R-algebra generated by $\sigma_1, \dots, \sigma_{n-1}$, where σ_i is n upwards strands with a crossing between the i'th and $i+1$'st, together with the relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad \text{if } |i-j| \geq 2,$$
$$a^{-1}\sigma_1 - a\sigma_1^{-1} = (s - s^{-1})1.$$

Note here that the σ_i are all conjugate. If $a = s = 1$, this is the group algebra $R\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group. The plan now is to construct certain idempotents in H_n, corresponding to the label set Λ. For this, we recall the Schur–Weyl duality. Recall that in Christian's talk,

Λ consisted of some of the $\mathrm{SL}(N)$-representations, so how do we get those? If V is the standard representation, then we have

$$S^n V = \mathrm{Im}(f_n) \subseteq V^{\otimes n},$$

where $\mathfrak{S}_n$ acts on $V^{\otimes n}$, and

$$f_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \pi,$$

e.g. $2f_2 = 1 + (1\,2)$. Another representation is

$$\Lambda^k V = \mathrm{Im}(g_n),$$

where

$$g_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}\,\pi \pi$$

Now, the point is that one can get all representations by combining these in a smart way, governed by Young diagrams. A *Young diagram* λ is a partition of $n = |\lambda|$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0)$ with $\sum \lambda_i = n$ (drawing). We say that V is the representation associated to a Young diagram with a single box, $V_\square$, $S^n V$ is the one associated to a line of boxes, and $\Lambda^k V$ the one associated to a column of boxes. The slogan now is that we want to put the symmetrizers f on the rows and the anti-symmetrizers g on the columns. We want to write $n! f_n = \sum_\pi \pi = f(1, 2, \ldots, n)$, so for the Young diagram $\lambda = (4, 3, 1, 1)$, we want to put numbers in the boxes of the diagram and write

$$\begin{aligned} f_\lambda &= f(1,2,3,4) f(5,6,7) f(8) f(9), \\ g_\lambda &= g(1,5,8,9) g(2,6) g(3,7) g(4). \end{aligned}$$

If we now take the *Young idempotents* $y_\lambda = \frac{1}{\mathrm{hl}\lambda} f_\lambda g_\lambda$, then $y_\lambda \in \mathbb{C}\mathfrak{S}_n$ is a projector, i.e. $y_\lambda^2 = y_\lambda$. Here

$$\mathrm{hl}(\lambda) = \prod_{\text{cells } c \text{ of } \lambda} \mathrm{hl}(c),$$

where the hook length of a cell (i, j) is $\lambda_i - i + \check{\lambda}_j - j + 1$.

Theorem 2.1.2 (Schur–Weyl duality)**.** *There is a bijective correspondence of Youngdiagrams with $N-1$ rows and irreducible representations of* $\mathrm{SL}(N)$*, i.e.* $P_+(\mathfrak{sl}(N))$*, sending* λ *to* $V_\lambda = \mathrm{Im}(y_\lambda) \subset V_\square^n$ *with* $n = |\lambda|$.

So for instance, $\Lambda^N V$ corresponded to a single column with N rows which is then the trivial representation $V_\emptyset$. In general, the Young diagram $(\lambda_1 \geq \lambda_2 \geq \cdots)$ gets mapped to $(\lambda_1 - \lambda_2)\omega_1 + \cdots (\lambda_{N-1} - \lambda_N)\omega_{N-1}$, where ω_i are the fundamental weights, e.g. $\Lambda^k V$ corresponds to ω_k. In particular, the length of λ_N does not matter.

Remark 2.1.3. V_λ and y_λ also make sense when λ has N rows but we can shave off a column with N boxes (drawing of example). If λ has more than n rows, $y_\lambda = 0$.

Fact 2.1.4. *The dual $V_\lambda^* = V_{\lambda^*}$, where λ^* is obtained from λ by filling in a rectangle of N rows containing λ, removing λ and rotating the result. This follows from $\omega_i^* = \omega_{N-i}$.*

We now want to deform y_λ into $\tilde{y_\lambda}$, an idempotent in H_n (in the $\mathrm{SL}(N)$-specialization). We define the *quantum symmetrizer*

$$[n]! \tilde{f_n} = s^{-n(n-1)/2} \sum_{\pi \in \mathfrak{S}_n} (as^{-1})^{-l(\pi)} \omega_\pi,$$

where ω_π is the positive permutation braid associated to π (drawing): If $\omega_\pi \mapsto \pi$ underthe map $B_n \to \mathfrak{S}_n$, and π is written as a word in transpositions $\tau_{i,i+1}$, then ω_π is the same word with σ_i in place of $\tau_{i,i+1}$. Here $l(\pi)$ is the minimum number of transpositions $\tau_{i,i+1}$ needed to write π, such that $(-1)^{l(\pi)} = \mathrm{sgn}(\pi)$, and

$$[N] = \frac{s^n - s^{-n}}{s - s^{-1}}, \quad [n]|_{s=1} = n, \quad [n]! = [1][2]\cdots[n].$$

There is then a similar expression for the quantum anti-symmetrizer $\tilde{g}_n$, which we do not have time to write. We then define

$$\tilde{y}_\lambda = \frac{1}{[\mathrm{hl}(\lambda)]}\tilde{f}_\lambda \tilde{g}_\lambda \left(\prod[\lambda_i]!\right) \tilde{f}_\lambda \left(\prod[\check{\lambda}_i]!\right) \tilde{g}_\lambda.$$

(We might have to check the number of factorials here – DIY.)

The idea in the construction of y_λ is pretty simple, graphically (drawing).

2.2 Pauly III

Today we will describe the TUY connection onthe bundle of conformal blocks. Also, we will talk about the sewing procedure, giving a central ingredient in the connection between the theories. We start discussing the Virasoro algebra: Consider $\mathbb{C}(\!(z)\!)\frac{d}{dz}$, the vector fields on a formal disk $D^* = \mathrm{Spec}(\mathbb{C}(\!(z)\!))$. We have a Lie bracket on this space of vector fields, and consider the universal central extension

$$0 \to \mathbb{C}c \to \mathrm{Vir} \to \mathbb{C}(\!(z)\!)\frac{d}{dz} \to 0.$$

The Lie bracket is

$$[f(z)\frac{d}{dz}, g(z)\frac{d}{dz}] = f'g - fg')\frac{d}{dz} - \frac{1}{12}\mathrm{Res}_{z=0}(fg'''\,dz)\cdot c.$$

We have generators $l_n = -z^{n+1}\frac{d}{dz} \in \mathbb{C}(\!(z)\!)\frac{d}{dz}$, $n \in \mathbb{Z}$ with

$$[l_n, l_m] = (n-m)l_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}c.$$

Remark 2.2.1. If C is a smooth curve, and T_C is the tangent sheaf of C with z coordinates at a point $p \in C$. For $n \geq 1$, we can consider the Laurent development

$$0 \to \mathcal{O}_C(-p) \to \mathcal{O}(np) \to \bigoplus_{i=0}^{n}\mathbb{C}z^{-i} \to 0.$$

Tensoring with the tangent sheaf,

$$0 \to T_C(-p) \to T_C(np) \to \bigoplus_{i=0}^{n}\mathbb{C}z^{-1}\frac{d}{dz} \to 0.$$

If $n \gg 0$ (by Riemann–Roch), we have $H^1(C, T_C(np)) = 0$. Thus taking the long exact sequence on cohomology,

$$\cdots \to \bigoplus_{i=0}^{n}\mathbb{C}z^{-1}\frac{d}{dz} \to H^1(T_C(-p)) \to 0.$$

This says that $H^1(T_C(-p))$ parametrizes infinitesimal deformations of C. So if for example we take a family of curves $\mathcal{C} \xrightarrow{\pi} \mathcal{B}$, such that $C \mapsto b \in \mathcal{B}$, then the Kodaira–Spencer map $T_b\mathcal{B} \to H^1(T_C(-p))$. A connection will be a way of associating to an infinitesimal deformation a linear operator acting on the conformal block. Thus we proceed by taking an infinitesimal deformation and lift it to a vector field on the curve, and we get an action on the conformal block via what is called the energy-momentum tensor, which we will describe now.

We first discuss currents and correlation functions.

Lemma 2.2.2 (Residue pairing)**.** *Consider again a curve with n points, and let $\tilde{p} = \sum_{i=1}^{n} p_i \in$ $\mathrm{Div}(C)$, and t_i coordinates at $p_i \in C$. Consider the direct limits*

$$\mathcal{O}(C \setminus \vec{p}) = \lim_{n\to\infty} H^0(C, \mathcal{O}(n\vec{p})) \hookrightarrow \bigoplus_{i=1}^{n} \mathbb{C}(\!(t_i)\!),$$

$$H^0(C, \Omega^1(*p)) = \lim_{n\to\infty} H^0(C, \Omega^1(n\vec{p})) \hookrightarrow \bigoplus_{i=1}^{n} \mathbb{C}(\!(t_i)\!)\, dt_i.$$

Under the residue pairing

$$\bigoplus_{i=1}^{n} \mathbb{C}(\!t_i)\!) \times \bigoplus_{i=1}^{n} \mathbb{C}(\!(t_i)\!)\, dt_i \to \mathbb{C},$$

$$(\vec{f}, \vec{g}\, dt_i) \mapsto \sum_{i=1} \mathrm{Res}_{t_i=0}(f_i g_i\, dt_i),$$

the spaces $\mathcal{O}(C \setminus \vec{p})$ and $H^0(C, \Omega^1(*p))$ are annihilators *to each other.*

One way to see this is by the residue theorem.

Definition 2.2.3. Let $X \in \mathfrak{g}$ and put $X(m) = X \otimes t^m \in \hat{\mathfrak{g}}$. We define the *current* to be the formal sum

$$X(z) = \sum_{m\in\mathbb{Z}} X(m) z^{-m-1},$$

viewing $X(m) \in \mathrm{End}(\mathcal{H}_\lambda)$, via the representations $\hat{\mathfrak{g}} \to \mathrm{End}(\mathcal{H}_{\vec{\lambda}})$.

We want to see these as operators on some algebra via correlation functions.

Definition 2.2.4 (/Theorem)**.** Let $\langle\psi| \in \mathcal{V}^*_{l,\vec{\lambda}}(C, \vec{p})$, $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (P_l(\mathfrak{g}))^n$ and $|\varphi\rangle \in \mathcal{H}_{\vec{\lambda}}$. The expression $\langle\psi|X(z)|\varphi\rangle\, dz$ defines an element in $H^0(C, \Omega^1_C(*\vec{p}))$, referred to as a *correlation function.*

Proof. Choose a coordinate t_j at the point p_j, $j = 1, \ldots, n$. For $z = t_j$,

$$\omega_j = \langle\psi| \sum_{m\in\mathbb{Z}} X(m)|\varphi\rangle t_j^{-m_j-1}\, dt_j \in \mathbb{C}(\!(t_j)\!)\, dt_j.$$

In order to make sense of this, we need to make sure that there is only a finitely many negative powers in the sum. The observation is that for $m \gg 0$, $X(m)|\varphi\rangle = 0$, and so $X(m)\mathcal{H}_\lambda(d) \subseteq$ $\mathcal{H}_\lambda(d - m)$. We obtain

$$\vec{\omega} = (\omega_1, \ldots, \omega_n) \in \bigoplus \mathbb{C}(\!(t_j)\!)\, dt_j.$$

We want to show that $\vec{\omega}$ comes from a global 1-form. We need to check that

$$\sum_{j=1}^{n} \mathrm{Res}_{t_j=0}(\omega_j f_j) = 0$$

for all $f \in \mathcal{O}(C \setminus \vec{p})$. This is implied by the gauge condition which says that

$$\langle\psi|X[f]|\varphi\rangle = 0,$$

looking at the Taylor expansion of the left hand side in the last equation. □

More generally, for 2 correlation currents $X(z)$ and $Y(w)$, we consider $F = \langle\psi|X(z)Y(w)|\varphi\rangle$ and is

$$F \in H^0(C \times C, \Omega^1_C \boxtimes \Omega^1_C(*\Delta + *\vec{p})),$$

where $C \times C$ has coordinates (z, w), and $\Delta \subset C \times C$ is the diagonal, and $\boxtimes$ means that we have poles at Δ, $C \times \{p_i\}$ and $\{p_i\} \times C$. We have an expansion

$$F = \langle\psi|X(z)Y(m)|\varphi\rangle = \frac{l(X \mid Y)}{(z-w)^2}\langle\psi|\varphi\rangle + \frac{1}{(z-w)}\langle\psi|[X,Y](z)|\varphi\rangle + \text{holo. terms}$$

We use the following notation: Choose an orthonormal basis $(J^a)_{a=1,\dots,\dim\mathfrak{g}}$ of $\mathfrak{g}$ with $(J^a \mid J^b) = \delta_{a,b}$. We write g^* for the dual Coxeter number (for $\mathfrak{g} = \mathfrak{sl}(r)$, we have $g^* = r$). We introduce the central charge

$$c_v = \frac{l \dim \mathfrak{g}}{l + g^*} \in \mathbb{Q}.$$

Definition 2.2.5. The *energy-momentum tensor* is

$$T(z) = \lim_{z\to w}\left(\frac{1}{2(g^*+l)}\sum_{a=1}^{\dim\mathfrak{g}} J^a(z)J^a(w) - \frac{c_v}{2}\frac{1}{(z-w)^2}\right),$$

where here the subtraction takes care of the order 2 pole, and the simple poles vanish as the commutator does.

There is another way of describing the non-singular part of the current:

Definition 2.2.6. For $X, Y \in \mathfrak{g}$, $n, m \in \mathbb{Z}$, we define the *normal ordering*,

$$:X(n)Y(m): := \begin{cases} X(n)Y(m) & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & m = n, \\ Y(m)X(n) & n > m. \end{cases}$$

In this notation, we can write the energy-momentum tensor

$$T(z) = \frac{1}{2(g^*+l)}\sum_{a=1}^{\dim\mathfrak{g}} :J^a(z)J^a(w): = \sum_{n\in\mathbb{Z}} L_n z^{-n-2},$$

where the equality defines L_n, which can also be written as

$$L_n = \frac{1}{2(g^*+l)}\sum_{a=1}^{\dim\mathfrak{g}}\sum_{m\in\mathbb{Z}} :J^a(m)J^a(n-m): .$$

Now $:J^a(n)J^a(n-m):$ is well-defined as an element in $\mathrm{End}(\mathcal{H}_\lambda)$: For $v \in \mathcal{H}_\lambda$ and $m \gg 0$, the normal ordering becomes

$$J^a(n-m)J^a(m)v = 0,$$

and if $m \ll 0$,

$$J^a(m)J^a(n-m)v = 0.$$

Thus for any $v \in \mathcal{H}_\lambda$, L_n has only a finite number of terms, and so $L_n \in \mathrm{End}(\mathcal{H}_\lambda)$. One can check that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c_v}{12}\delta_{n+m,0} l \cdot \mathrm{id}$$

in $\mathrm{End}(\mathcal{H}_\lambda)$. Thus the association $\mathrm{Vir} \ni l_n \to L_n$ defines a representation $\mathrm{Vir} \to \mathrm{End}(\mathcal{H}_\lambda)$ first described by Segal–Sugawara.

2.3 Masbaum III

To a Young diagram of n boxes, we associated $\tilde{y}_\lambda \in H_n = \mathrm{End}_{\mathcal{HT}}(+\cdots+)$. If we put $q = 1$, we just get an idempotent $y_\lambda \in \mathbb{C}\mathfrak{S}_n$. To λ we can also associate a representation W_λ of H_n (as a left H_n-module) by $W_\lambda = H_n\tilde{y_\lambda} \subseteq H_n$. Let $d_\lambda = \dim W_\lambda$. In fact, if q is not a root of unity, then as algebras

$$H_n \cong \bigoplus_{\lambda, |\lambda|=n} \mathrm{Mat}_{d_\lambda}(\mathbb{C}),$$

so the λ also parametrize irreducible representations of $\mathfrak{S}_n$, and as left H_n-modules, the above is isomorphic to

$$\bigoplus_{\lambda, |\lambda|=n} W_\lambda^{d_\lambda}.$$

With respect to this isomorphism,

$$\tilde{y}_\lambda \in 0 \oplus \cdots \oplus \mathrm{Mat}_{d_\lambda}(\mathbb{C}) \oplus 0 \oplus \cdots \oplus 0.$$

In particular, $\tilde{y}_\lambda \tilde{y}_\mu = 0$ if $\lambda \neq \mu$.

If V is a vector space, $p \in \mathrm{End}(V)$, with $p^2 = p$, then $V = \mathrm{Im}(p) \oplus \mathrm{Im}(1-p)$. We would like to be able to do this in our category, but we can not. Therefore we perform the *idempotent completion*: If $\mathcal{C}$ is an R-linear category, R a field, with morphisms $\mathrm{Hom}_{\mathcal{C}}(\mathrm{ob}_1, \mathrm{ob}_2)$ being R-vector spaces, then we make a new category $\mathcal{C}^{ic}$ with objects (ob, p) where $\mathrm{ob} \in \mathrm{Ob}(\mathcal{C})$, and $p \in \mathrm{End}_{\mathrm{ob}}(\mathcal{C})$ with $p^2 = p$ and define

$$\mathrm{Hom}_{\mathcal{C}^{ic}}((\mathrm{ob}_1, p_1), (\mathrm{ob}_2, p_2)) = p_1 \mathrm{Hom}_{\mathcal{C}}(\mathrm{ob}_1, \mathrm{ob}_2) p_2 \subseteq \mathrm{Hom}_{\mathcal{C}}(\mathrm{ob}_1, \mathrm{ob}_2).$$

(Important drawing.) We will write (ob_1, p_1) simply as p_1. Further, we allow direct sums of objects.

Another construction we need is purification: Say $\mathcal{C}$ is an R-linear ribbon category, $R = \mathrm{End}_{\mathcal{C}}(\mathbf{1})$. For V an object of $\mathcal{C}$, we defined $\dim V \in R$. For $f \in \mathrm{End}(V)$, we define $\mathrm{tr}(f)$ by closing up the strands (drawing). For instance, if $V = (++)$, then by the Homflypt relations, $\dim V = [N]^2$. Let $\mathcal{N}$ be the tensor ideal in $\mathcal{C}$ of *negligible morphisms*; a morphism $f : V \to W$ is negligible if for all $g : W \to V$ we have $\mathrm{tr}(g \circ f) = 0$. We define the *purified* category $\mathcal{C}^{\mathrm{pur}}$ to have objects the same as $\mathcal{C}$ and morphisms

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{pur}}}(\mathrm{ob}_1, \mathrm{ob}_2) = \mathrm{Hom}_{\mathcal{C}}(\mathrm{ob}_1, \mathrm{ob}_2)/\mathcal{N}(\mathrm{ob}_1, \mathrm{ob}_2).$$

Definition 2.3.1. An object V is called *simple* if Schur's lemma holds, i.e. $\mathrm{End}_{\mathcal{C}}(V) = R\,\mathrm{Id}$.

The last thing we need is duality. Recall that the object associated to the Young diagram λ with one column and N boxes, then the image of y_λ is $\tilde{g}_N$, and V_λ is the trivial representation. One is then allowed to move strands through strands coloured with $\tilde{g}_N$ (drawing). We want y_{λ^*} to be dual to y_λ (drawings). This can be obtained by formally requiring that the object corresponding to the Young diagram λ_1^N with N rows of length λ_1 be trivial. Formally, we make a new skein theory $\mathcal{H}'\mathcal{T}$, where we allow pictures where N strands enter a $\tilde{g}_N$ or pictures where N strands exit a $\tilde{g}_N$ (drawing). So $\mathrm{Ob}(\mathcal{H}'\mathcal{T}) = \mathrm{Ob}(\mathcal{HT})$ but for morphisms, we allow these morphisms with the new relation that we can connect the $\tilde{g}_N$-boxes any way we like (drawing).

Theorem 2.3.2. *For q not a root of unity, the category $(\mathcal{H}'\mathcal{T})^{ic,\mathrm{pur}}$ is equivalent to the category of $U_q\mathfrak{sl}_N$. It is a semi-simple category, i.e. every object is a direct sum of simple ones, and simple objects are in correspondance with y_λ where λ are Young diagrams with less than or equal to $N-1$ rows.*

Theorem 2.3.3. *Consider $\mathrm{SL}(N)_K$, with $q^{N+K} = 1$ a primitiveroot of unity. Then $(\mathcal{H}'\mathcal{T})^{ic,\mathrm{pur}}$ is a semi-simple ribbon category (with braiding the one we have, and duality constructed as above) with isomorphism classes of simple objects in correspondance with y_λ where λ are Young diagrams with less than or equal to K columns. This is also the set $P_K(\mathfrak{sl}(N))$.*

We discuss now what it is that happens if $q^{N+K}=1$. The key point is that many of the $\tilde{y}_\lambda$ no longer exist because

$$\tilde{y_\lambda} = \frac{1}{[\mathrm{hl}(\lambda)]}\tilde{f}_\lambda \tilde{g}_\lambda,$$

and the hook length may be zero and therefore not invertible, but we can always define $\tilde{y}_\lambda$ can always be defined for the Young diagrams described in the Theorem. Some diagrams not in the set might still have $[\mathrm{hl}(\lambda)] \neq 0$ (for instance, the g_N survive which is important for the duality), but then what happens is that $\dim(\tilde{y}_\lambda)=0$, which implies that y_λ is isomorphic to zero in $(\mathcal{H}'\mathcal{T})^{ic,\mathrm{pur}}$. In fact, if $|\lambda| = n$,

$$\dim(\tilde{y_\lambda}) = \prod_{(i,j)\in\lambda} \frac{[N+j-i]}{[\mathrm{hl}(i,j)]}.$$

2.4 Pauly IV

We introduce some more notation: Suppose $f \in \mathbb{C}(\!(z)\!)$ and $l = l(z)\frac{d}{dz} \in \mathbb{C}(\!(z)\!)\frac{d}{dz}$. Expanding, $f(z) = \sum_{m\geq -m_0} a_m z^m$, and $l(z) = \sum_{m\geq -m_0} b_m z^{m+1}\frac{d}{dz}$. We write

$$\hat{\mathfrak{g}} \ni X[f] = \sum_{m\geq -m_0} a_m X(m) = \mathrm{Res}_{z=0}(X(z)f(z)\,dz)$$
$$T[l] = \sum_{m\geq -m_0} b_m L_m = \mathrm{Res}_{z=0}(T(z)l(z)\,dz).$$

We turn now to the definition of the TUY/WZW connection. First we put things into context. Suppose we have a family of *smooth* curves, $\pi : \mathcal{C} \to \mathcal{B}$ with sections $\sigma_i : \mathcal{B} \to \mathcal{B}$, $i = 1,\dots,m$, $\pi^{-1}(b) = C_b$ with coordinates along $\sigma_i(\mathcal{B}) \subseteq \mathcal{C}$. Zeroes of z_i are $\sigma_i(B)$. Denote by $\Theta_\mathcal{B}$ the tangent sheaf of $\mathcal{B}$. We want to construct $\nabla^{\mathrm{TUY}} : \Theta_\mathcal{B} \to \mathrm{End}_\mathbb{C}(\mathcal{V}^*_{l,\vec{\lambda}}(C,\vec{\sigma}))$. We have the Kodaira–Spencer map $\Theta_\mathcal{B} \to H^1(C, T_C(-\sigma_i))$. Write $S = \bigsqcup_{i=1}^n \sigma_i(\mathcal{B})$. We let $(\pi_*)\Theta'_{\mathcal{C}/\mathcal{B}}(*S)_\pi$ be the vector fields constant along the fibers of π. In terms of local coordinates, if $x \in \mathcal{C}_b \mapsto b$ and z are vertical coordinates, and $u = (u_1,\dots,u_m)$ are horizontal coordinates, a vector field in $\Theta_\mathcal{B}$ can be written $\sum_{i=1}^m b_i(u)\frac{d}{du_i}$, and a vector field constant along the fibers is then $a(z,u)\frac{d}{dz} + \sum_{i=1}^m b_i(u)\frac{d}{du_i}$, the b_i depending not on z. We have an exact sequence of sheaves over $\mathcal{V}$.

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S) \to (\pi_*)\Theta'_{\mathcal{C}/\mathcal{B}}(*S)_\pi \xrightarrow{\theta} \Theta_\mathcal{B} \to 0,$$

where $\pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S)$ are the vertical vector fields of $\mathcal{C}/\mathcal{B}$, and θ is the map forgetting the vertical part of the vector field.

There is an $\mathcal{O}_\mathcal{B}$-linear map, the Laurent expansion at $\sigma_i(\mathcal{B}) \subseteq \mathcal{C}$

$$\pi_*\Theta'_{\mathcal{C}/\mathcal{B}}(*S)_\pi \to \bigoplus_{i=1}^n \mathbb{C}(\!(z_i)\!)\frac{d}{dz_i},$$
$$l \mapsto (l_1(z)\frac{d}{dz},\dots,l_n(z)\frac{d}{dz}).$$

For l a local section of $\pi_*\Theta_{\mathcal{C}/\mathcal{B}})(*S)_\pi$ such that $\theta(l) = X \in \Theta_B$ and for $f \otimes |\psi\rangle \in \mathcal{O}_\mathcal{B} \otimes \mathcal{H}^*_{\vec{\lambda}}$, we define

$$D(l)f\otimes|\psi\rangle := X(f)\otimes|\psi\rangle + \sum_{i=1}^n f\otimes T[l_i]|\psi\rangle,$$

for $X \in \Theta_\mathcal{B}$, where $T[l_i]$ acts on the i'th component of $\mathcal{H}^*_{\vec{\lambda}} = \mathcal{H}_{\lambda_1}\otimes\cdots\otimes\mathcal{H}_{\lambda_n}$.

It has the following properties:

1. $D(l)(\mathcal{V}^*_{l,\vec{\lambda}}(\mathcal{C},\vec{\sigma})) \subseteq \mathcal{V}^*_{l,\vec{\lambda}}(\mathcal{C},\vec{\sigma})$.
2. If $l \in \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*S)$ (i.e. l is a vertical vector field, $\theta(l) = 0$) then $D(l) = a(l)\,\mathrm{Id}$, where $a : \pi_*\Theta_{\mathcal{C}/\mathcal{B}}(*s) \to \mathcal{O}_B$. This implies that $\nabla_X := D(l)$ is well-defined up to homothety, i.e. we get a projective connection.
3. We can calculate the curvature:

$$[D(l), D(m)] = D([l,m]) + \lambda(l,m)\cdot \mathrm{id},$$

for $\lambda \in \mathcal{O}_B$, so ∇_X is projectively flat.

We would like to discuss the proof of the second property, as it will tell us something about how to lift the projective connection to an honest one.

Proof of 2. The idea is thefollowing: We introduce a global bi-differential on $\mathcal{C}/\mathcal{B}$, $\omega \in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \Omega^1_{\mathcal{C}/\mathcal{B}} \boxtimes \Omega^1_{\mathcal{C}/\mathcal{B}}(2\Delta))$. At a curve C, restriction to the diagonal gives

$$H^0(C\times C, \Omega^1_C \boxtimes \Omega^1_C(2\Delta)) \to H^0(C, \mathcal{O}_C) \cong \mathbb{C},$$

which maps $\omega(z,w)\,dz\,dw = \frac{\alpha}{(z-w)^2} + \cdots \mapsto \alpha$. We consider those mapping to 1. The pre-image of 1 is an affine space we call PS for the space of projective structures. Choose $\omega \in \mathrm{PS}$. We have to show that

$$\langle\psi|D(l)|\varphi\rangle = \langle\psi|\sum_{i=1}^n T[l_i]|\varphi\rangle = \sum_{i=1}^n \mathrm{Res}_{z_i=0}\, l_i(z)\langle\psi|T(z)|\varphi\rangle\,dz$$

is some scalar times $\langle\psi|\varphi\rangle$. We find

$$\langle\psi|T(z)|\varphi\rangle = \lim_{w\to z}\left(\frac{1}{g^*+l}\sum_{a=1}^{\dim\mathfrak{g}}\langle\psi|J^a(z)J^a(w)|\varphi\rangle - \frac{c_v}{2}\frac{\langle\psi|\varphi\rangle}{(z-w^2)}\right).$$

(Note from the TeXer: Lots of stuff happening here, so I'll skip the rest of the proof – sorry.) □

Definition 2.4.1. Suppose there exists $\omega \in H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C},\cdots)$ a bi-differential as above. Then we define

$$\nabla^{(\omega)}_X = D(l) - a^\omega(l),$$

where a^ω is the scalar appearing in the proof above. By construction, this is well-defined (i.e. it does not depend on the lift l with $\theta(l) = X$). Its curvature is

$$R_{\nabla^{(\omega)}_X} \in \Omega^2_{\mathcal{B}} \otimes \mathrm{id},$$

i.e. it is projectively flat. It is explicitly given by (where $\theta(m)Y$)

$$R_{\nabla^{(\omega)}}(X,Y) = -a^\omega([l,m]) + X(a^\omega(m)) - Y(a^\omega(l)) - \frac{c}{12}\sum_{j=1}^N \mathrm{Res}(\frac{d^2 l_j}{dz_j} m_j\,dz_j),$$

where

$$a^\omega(l) = -\frac{c}{12}\sum_{j=1}^N \mathrm{Res}_{z_j=0}(l(z_j)S(z_j)\,dz_j).$$

In particular, the curvature is $c/2$ times something.

In the next lectures, we will define a sheaf of abelian vacua, and tensoring with it will give rise to a connection without projective ambiguities.

Observe: If we choose a symplectic basis of $H_1(C,\mathbb{Z}) = \mathbb{Z}^{2g}$ a theorem by Fay implies that there is a unique prime form $E(z,w) \in H^0(C\times C, \mathcal{O}_{C\times C}(\Delta))$ with $E(z,w) = -E(w,z)$, defined in terms of theta functions. Then

$$\frac{\partial^2 \ln E(z,w)}{\partial z \partial w} = \omega(z,w)$$

is a bi-differential. This implies that $\nabla^{(\omega)}$ is the WZW-connection.

Another consequence of having the projective connection is that the dimension of $\dim V_{\vec{\lambda},l}(C,\vec{p})$ does only depend on g, n and $\vec{\lambda}$ for C smooth. In fact TUY shows that this holds true on the boundary of moduli space. This follows from something called sewing: We can decompose $\mathcal{H}_\lambda = \bigoplus_{d=0}^\infty \mathcal{H}_\lambda(d)$, the summands being eigenspaces for the L_0, called the Virasoro operator. A result of Kac is that there is a bilinear form $(\,|\,) : \mathcal{H}_\lambda \times \mathcal{H}_{\hat{\lambda}} \to \mathbb{C}$ such that $(\mathcal{H}_\lambda(d), \mathcal{H}_{\hat{\lambda}}(d')) = 0$ for $d \neq d'$ and that restricted to $\mathcal{H}_\lambda(d) \times \mathcal{H}_{\hat{\lambda}}(d)$ is non-degenerate. This implies that $\mathcal{H}_{\hat{\lambda}}(d) \cong \mathcal{H}_\lambda(d)^*$. This allows us to produce.

Then for every d, $\mathcal{H}_\lambda(d) \otimes \mathcal{H}_{\hat{\lambda}}(d) \ni \gamma(d) = \mathrm{id}_{\mathcal{H}_\lambda(d)}$. For $d = 0$, $\gamma(0) \in \mathcal{H}_\lambda(0) \otimes \mathcal{H}_{\hat{\lambda}}(0)$ is what we considered previously. Consider the normalization $\tilde{C} \to C$ of a nodal curve and suppose we have a family curves $\mathcal{C}_\tau \to C$. One example is $\mathcal{C}_\tau = (xy - \tau z^2)$ which as $\tau \to 0$ gives a nodal curve whose normalization is a union of two curves. Then factorization gave us a map $V^*_{l,\vec{\lambda}\cup\mu\cup\hat{\mu}}(\tilde{C}, \vec{p}\cup\{a,b\}) \to V^*_{l,\vec{\lambda}}(C,\vec{p})$ m apping $v \mapsto v \otimes \gamma(0)$. If we define $v \mapsto \sum_{d=0}^\infty v \otimes \gamma(d)\tau^d$ we get something in $V^*_{l,\vec{\lambda}}(C,\vec{p})[\![\tau]\!]$, i.e. what we get by tensoring with $\mathbb{C}[\![z]\!]$. This space can be viewed as $\widehat{V_{l,\vec{\lambda}}(\mathcal{C},\vec{p})}$, where $\mathcal{C}$ lives over $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[\tau]) \ni \tau$. We can do this construction in families $\mathcal{C}_\tau \to \mathbb{A}^1 \times \mathcal{B}$ with $\mathcal{C} \cong \mathcal{C}_0 \to \{0\} \times B$. We then have a sewing map

$$\mathcal{V}^*_{l,\vec{\lambda}\cup\mu\cup\hat{\mu}}(\mathcal{C}, \vec{p}\cup a \cup b) \to V^*_{l,\vec{\lambda}}(C,\vec{p})[\![\tau]\!] = \widehat{V^*_{l,\vec{\lambda}}(\mathcal{C}_\tau,\vec{p})},$$

which is a vector bundle over $\mathcal{B}\times\mathbb{A}^1$. On either side we have a connection, and the statement is that sewing is projectively flat; i.e. it takes projectively flat sections to projectively flat connections. The underlying topological surfaces here are Σ and Σ_c as described earlier. Say that sewing maps $\psi \to \tilde{\psi}(\tau)$, a formal power series in τ. Then a result from TUY is that

$$\nabla_{\frac{d}{dz}}(\tilde{\psi}) = \lambda\tilde{\psi}.$$

This implies, using theory of differential equations, that $\tilde{\psi}$ converges around 0, so $\tilde{\psi}(\tau)$ around 0.

3 April 23rd, 2012

3.1 Herbig

The topic is abelian conformal field theory and determinant bundles, following [JEA, KU '07]. To relate the word "conformal" to physics, we here consider spin j ($j = 0$) bc ghost systems.

1. Fermionic Fock spaces: We first set up some notation. Let $\mathbb{Z}_h = \{n + \frac{1}{2} \mid n \in \mathbb{Z}\}$, let $p \in \mathbb{Z}$ be a charge, and let $\mathbb{Z}_{h<p} = \{\nu \in \mathbb{Z}_h \mid \nu < p\}$. We are interested in a $\mathbb{C}$-vector space $\mathcal{W}^\dagger$, $\dim_\mathbb{C} \mathcal{W}^\dagger = \infty$, with a descending filtration $\{F^m\mathcal{W}^\dagger\}_{m\in\mathbb{Z}}$, for which we require the following:

1. $\bigcup_m F^m\mathcal{W}^\dagger = \mathcal{W}^\dagger$, $\bigcap_{m\in\infty} F^m\mathcal{W}^\dagger = \{0\}$.
2. $\dim_\mathbb{C} F^m\mathcal{W}^\dagger / F^{m+1}\mathcal{W}^\dagger = 1$.
3. The topological associated to $\{F^m\mathcal{W}^\dagger\}$ is complete.

We consider the following basis: $\{e^\nu\}_{\nu\in\mathbb{Z}_h}$ with $e^{m+1/2}\in F^m\setminus F^{m+1}$ and write

$$F^m\mathcal{W}^\dagger=\{u\in\mathcal{W}^\dagger\mid u=\sum_{\nu\in\mathbb{Z}_h,\nu>m}a_\nu e^\nu\}.$$

There is a non-canonical isomorphism $\mathbb{C}((\xi))\to\mathcal{W}^\dagger$ mapping $\xi^n\mapsto e^{n+1/2}$.

We introduce also a dual basis $\{\overline{e}_\nu\}_{\nu\in\mathbb{Z}_h}$ to $\{e^\nu\}_{\nu\in\mathbb{Z}_h}$ and put

$$\mathcal{W}=\bigoplus_{\nu\in\mathbb{Z}_h}\mathbb{C}\overline{e}_\nu.$$

We have a dual pairing $(\,|\,):\mathcal{W}^\dagger\times\mathcal{W}\to\mathbb{C}$, with $(e^\nu\mid\overline{e}_\mu)=\delta^\nu_\mu$.

Consider an increasing function $\mu:\mathbb{Z}_{h<p}\to\mathbb{Z}$ such that there exists an n_0 with the property that for all $\nu<n_0$, $\mu(\nu)=\nu$. We introduce M a *Maya diagram*, which in a sense is just the graph of μ. More precisely $M=\{\mu(p-\frac{1}{2}),\mu(p-\frac{3}{2}),\dots\}$. Let $\mathcal{M}_p$ be the set of Maya diagrams of charge p. We define a *degree*

$$d(\mu):=\sum_{\nu\in\mathbb{Z}_{h<p}}\mu(\nu)-\nu.$$

We decompose $\mathcal{M}_p$ by degree and write $\mathcal{M}_p=\bigsqcup_{d\geq 0}\mathcal{M}_p^d$. There is a semi-infinite exterior product and corresponding covectors

$$\begin{aligned}|M\rangle&:=\overline{e}_{\mu(p-1/2)}\wedge\overline{e}_{\mu(p-3/2)}\wedge\dots,\\ \langle M|&:=\dots\wedge e^{\mu(p-3/2)}\wedge e^{\mu(p-1/2)}.\end{aligned}$$

The *dual Fock space* of charge p is

$$\mathcal{F}(p)=\bigoplus_{M\in\mathcal{M}_p}\mathbb{C}|M\rangle.$$

The *Fock space* is

$$\mathcal{F}^\dagger(p)=\prod_{M\in\mathcal{M}_p}\mathbb{C}\langle M|.$$

We could also use the degree d as an extra grading on these spaces but do not need it for this talk. What is important is that we have a dual pairing $\langle\,|\,\rangle:\mathcal{F}^\dagger(p)\times\mathcal{F}(p)\to\mathbb{C}$. We write $\mathcal{F}=\bigoplus_{p\in\mathbb{Z}}\mathcal{F}(p)$, $\mathcal{F}^\dagger=\bigoplus_{p\in\mathbb{Z}}\mathcal{F}^\dagger(p)$, and the bracket $\langle\,|\,\rangle$ extends to these spaces. We have *fermionic operators* $\psi_\nu:=i(e^\nu)$ (where i is contraction, obeying the Leibniz rule), so $\psi_\nu;\mathcal{F}(p)\to\mathcal{F}(p-1)$. Similarly, $\overline{\psi}_\nu:=\overline{e}_{-\nu}\wedge:\mathcal{F}(p)\to\mathcal{F}(p+1)$ by wedging. Dualizing, we introduce $\langle u|\psi_\nu v\rangle=\langle u\psi_\nu|v\rangle$ and $\langle u|\overline{\psi}_\nu v\rangle=\langle u\overline{\psi}_\nu|v\rangle$.

We have preferred infinite vectors,

$$\begin{aligned}|p\rangle&:=\overline{e}_{p-1/2}\wedge\overline{e}_{p-3/2}\wedge\dots,\\ \langle p|&:=\cdots e^{p-3/2}\wedge e^{p-1/2}.\end{aligned}$$

the Maya diagrams without gaps. We have $\psi_\nu|0\rangle=0$ iff $\nu>0$ iff $\overline{\psi}_\nu|0\rangle=0$. Similarly, $\langle 0|\psi_\nu=0$ iff $\nu<0$ iff $\langle 0|\overline{\psi}_\nu=0$. We have a commutation relation

$$[\psi_\nu,\overline{\psi}_\nu]_+=\psi_\nu\overline{\psi}_\mu+\overline{\psi}_\mu\psi_\nu=\delta_{\nu+\mu,0},$$

so in physics language, these are conjugate to each other. If $M\in\mathcal{M}_p$, there are half integers $\mu_1<\dots<\mu_r<0$ and $\nu_1<\dots<\nu_s<0$ such that $r-s=p$ and

$$|M\rangle:=\pm\overline{\psi_{\mu_1}}\cdots\overline{\psi_{\mu_r}}\psi_{\nu_1}\cdots\psi_{\nu_s}|0\rangle.$$

Now things get more sophisticated. Let us assume that we have two fermionic operators A and B. We define their *normal ordering*

$$: A_\nu B_\mu : = \begin{cases} -B_\mu A_\nu & \text{if } \mu < 0, \nu > 0, \\ A_\nu B_\mu & \text{otherwise.} \end{cases}$$

We introduce now *field operators*

$$\psi(z) = \sum_{\mu \in \mathbb{Z}_h} \psi_\mu z^{-\mu-1/2},$$
$$\overline{\psi}(z) = \sum_{\mu \in \mathbb{Z}_h} \overline{\psi}_\mu z^{-\mu-1/2}.$$

We also introduce the *current*

$$J(z) = : \overline{\psi}(z)\psi(z) : =: \sum n \in \mathbb{Z} J_n z^{-n-1}$$

and the *energy momentum tensor*

$$T^{(j)} := : (1-j)\frac{d\psi}{dz}(z)\overline{\psi}(z) - j\psi(z)\frac{d\overline{\psi}}{dz}(z) : =: \sum_{n\in\mathbb{Z}} L_n^{(j)} z^{-n-2},$$

where the $L_n^{(j)}$ are called *Virasoro generators*. Here j is the *spin*. The operators $J_n, L_n^{(j)}$ act on $\mathcal{F}(p)$ and $\mathcal{F}^\dagger(p)$ (which is not obvious). They satisfy the commutation relations

$$[J_n, J_m] = n\delta_{n+m,0},$$
$$[L_n^{(j)}, L_m^{(j)}] = (n-m)L_{n+m}^j - \frac{1}{6}(6j^2 - 6j + 1)(n^3 - n)\delta_{n+m,0},$$
$$[L_n^{(j)}, J_m] = -mJ_{n+m} - \frac{1}{2}(2j-1)(n^2+n)\delta_{n+m,0},$$
$$[J_n, \psi(z)] = -z^n\psi(z),$$
$$[J_n, \overline{\psi}(z)] = z^n\overline{\psi}(z)$$
$$[L_n^{(j)}, \psi(z)] = z^n(z\frac{d}{dz} + j(n+1))\psi(z)$$
$$[L_n^{(j)}, \overline{\psi}(z)] = z^n(z\frac{d}{dz} + (1-j)(n+1))\overline{\psi}(z).$$

Usually in physics, $j = 1/2$. From now on, we assume $j = 0$ (and use the index j for other things).

2. Ghost vacua: Consider a semi-stable N-pointed curve $\mathcal{X} = (C; Q_1, \ldots, Q_N; \xi_1, \ldots, \xi_N)$, where the ξ_j are formal coordinates at the points $Q_1, \ldots, Q_N$. Define

$$\mathcal{F}_N = \bigoplus_{p_1,\ldots,p_N\in\mathbb{Z}} \mathcal{F}(p_1) \otimes \cdots \otimes \mathcal{F}(p_N),$$
$$\mathcal{F}_N^\dagger = \bigoplus_{p_1,\ldots,p_N} \mathcal{F}^\dagger(p_!)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}^\dagger(p_N),$$

completing the tensor product. For $\omega \in H^0(C, \omega_C(*\sum Q_j))$ a 1-form holomorphic away from the Q_j, we expand $\omega_j = \left(\sum_{n\geq n_0} a_n\xi_j^n\right) d\xi_j$ and make out of it an operator

$$\psi[\omega_j] : = \mathrm{Res}_{\xi_j=0}(\psi(\xi_j)\omega_j),$$
$$\psi[\omega] : = (\psi[\omega_1], \ldots, \psi[\omega_N]).$$

Analogously, for $f \in H^0(C, \mathcal{O}_C(*\sum Q_j))$, we can define $\overline{\psi}[f]$ by

$$\overline{\psi}[f_j] = \mathrm{Res}_{\xi_j=0}(\overline{\psi}(\xi_j)f_j(\xi_j)\,d\xi_j).$$

Then we let $\psi[\omega]$ and $\overline{\psi}[f]$ act on $\mathcal{F}_N$ and $\mathcal{F}_N^\dagger$, for example

$$\overline{\psi}[f]|u_1 \otimes \cdots \times u_N\rangle = \sum_j \pm|u_1 \otimes \cdots \otimes \overline{\psi}[f_j]u_j \otimes \cdots \otimes u_N\rangle,$$

where there is an obvious Koszul sign that we do not specify.

Definition 3.1.1. We say that $\langle\varphi| \in \mathcal{F}_N^\dagger$ belongs to the *space of ghost vacua* $\mathcal{V}_{ab}^\dagger(\mathcal{X})$ if, similarly to the non-abelian case, we have

$$\langle\varphi|\psi[\omega] = 0 = \langle\varphi|\overline{\psi}[f],$$

for all f, ω as above. We refer to these equations as the gauge condition.

Dually, we introduce $\mathcal{F}_{ab} \subseteq \mathcal{F}_N$ spanned by $\psi[\omega]\mathcal{F}_N$ and $\overline{\psi}[f]F_N$. Factorizing, we refer to space of coinvariants $\mathcal{V}_{ab} = \mathcal{F}_N/\mathcal{F}_{ab}$ as the space of *ghost covacua*.

We discuss the following theorem later:

Theorem 3.1.2. $\mathcal{V}_{ab}^\dagger$ *is* 1*-dimensional.*

If we consider $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $\mathcal{X} = (\mathbb{P}^1, 0, z)$, one can show $\mathcal{V}_{ab}^\dagger(\mathbb{P}^1, 0, z) = \mathbb{C}\langle -1|$. For a curve $\mathcal{X} = (C, Q, \xi)$ with one puncture (C is smooth of genus g), there is a way to construct a generator $\langle\omega(\mathcal{X})| \in \mathcal{V}_{ab}^\dagger \cap F^\dagger(g-1)$ for ω a particular 1-form with prescribed singularities (using Riemann–Roch).

The construction of the space of ghost vacua shows that $\mathcal{V}_{ab}^\dagger(\mathcal{X})$ is canonically isomorphic to the determinant of the canonical bundle, $\det(H^0(C, \omega_C))$. We have operations as in the non-abelian case such as propagation of vacua, partial desinguralization, and compatibility with disjoint union.

3. Sheaf of ghost vacua: Consider $F = (\mathcal{C} \xrightarrow{\pi} \mathcal{B}, s_1, \ldots, s_N, \xi_1, \ldots, \xi_N)$ a family of N-pointed semistable curves. Let Σ be the locus of double points, let $\pi(\Sigma) =: D \subseteq \mathcal{B}$, where D is a divisor of normal crossings. Let $S_j = s_j(B)$ and $S = \sum S_j$. Then we can make analogues of the Fock space,

$$\mathcal{F}_N(\mathcal{B}) = \mathcal{F}_N \otimes_{\mathbb{C}} \mathcal{O}_B,\ \mathcal{F}_N^\dagger(\mathcal{B}) = \mathcal{O}_B \otimes \mathcal{F}_N^\dagger.$$

We also have $\mathcal{V}_{ab}^\dagger(F) \subseteq \mathcal{F}_N^\dagger(B)$, whose elements satisfy

$$\langle\varphi|\psi[\omega_j] = 0 = \langle 0|\overline{\psi}[f]$$

for all $\omega \in \pi_*(\omega_{C/B}(*S))$, $f \in \pi_*(\mathcal{O}_{\mathcal{C}}(*S))$. One defines $\mathcal{V}_{ab}(\mathcal{F})$ analogously.

Proposition 3.1.3. $\mathcal{V}_{ab}^\dagger(F)$ *and* $\mathcal{V}_{ab}(F)$ *are coherent* $\mathcal{O}_B$*-modules.*

The following is easiest when $D = \emptyset$.

Theorem 3.1.4. $\mathcal{V}_{ab}^\dagger(F)$ *and* $\mathcal{V}_{ab}(F)$ *are locally free.*

4. Projectively flat connections: There is a short exact sequence of Lie algebroids (where in Pauly's talks $D = \emptyset$), constructed using the Kodaira–Spencer map,

$$0 \to \pi_*\Theta_{\mathcal{C}/\mathcal{B}} \to \mathcal{L}(F) \xrightarrow{\vartheta} \Theta_B(-\log D) \to 0,$$

where $\mathcal{L}(F) \subseteq \bigoplus_{j=1}^N \mathcal{O}_B(\xi_j))\frac{d}{d\xi_j}$, and we write $\vec{l} = (l_1, \ldots, l_j)$. We also have a bracket $[\,,\,]_d$ on $\mathcal{L}(F)$. Also in analogy with the non-abelian case, we have

$$D(\vec{l})F \otimes |u\rangle = \vartheta(\vec{l})F \otimes |u\rangle - F(\sum l_j(T[l_j])|u\rangle),$$

operating on $\mathcal{F}_N(\mathcal{B})$, preserving $\mathcal{F}_{ab}(\mathcal{B})$. We have a pairing $\langle\,|\,\rangle : \mathcal{V}_{ab}^\dagger(\mathcal{F}) \times \mathcal{V}_{ab}(\mathcal{F}) \to \mathbb{C}$ such that

$$D(\vec{l})\langle\psi|u\rangle + \langle\psi|D(\vec{l})|u\rangle = \vartheta(\vec{l})\langle\psi|u\rangle.$$

The point is that we can now lift $X \in \Theta_{\mathcal{B}}(-\log D)$ to $\vec{l} \in \mathcal{L}(F)$ and use D to define a connection. There exists a b_ω constructed in analogy with the a_ω in the non-abelian case, and a connection

$$\nabla_X^{(\omega)}(\langle\psi|) = D(\vec{l})\langle\psi| + \frac{1}{6}b_\omega(\vec{l})\langle\psi|$$

whose curvature $R(X,Y)$ relates to the non-abelian $R^g(X,Y)$ by

$$R^g(X,Y) = \frac{c_v}{2}R(X,Y)\mathrm{id},$$

and $a_\omega = \frac{c_v}{2}b_\omega$.

3.2 Masbaum IV

This is the last lecture on the combinatorial construction of the TQFT from the HOMFLYPT polynomial. We begin with a review:

We considered (isotopy classes of) framed tangles satisfying skein relations in $q = \exp(2\pi i/(K+N))$ where N and K are the global variables, K the fixed level, and N from $\mathrm{SL}(N)$. We further allowed framed tangles with "coupons" (i.e. the rectangles in previous drawings) and considered a relation on coupons coloured by the idempotent $\tilde{g}_N \in \mathrm{End}_{\mathcal{HT}}(+\cdots+)$, giving the duality morphisms in the idempotent completed category (a bunch of drawings), forcing the object $((-,\ldots,-),y_\lambda)$ to be isomorphic to $((+,\ldots,+),y_{\lambda^*})$, and one can draw the isomorphism explicitly (drawing).

Theorem 3.2.1. *The category $(\mathcal{H}'\mathcal{T})^{ic,pur}$ is a semisimple ribbon category over $R = \mathbb{C}$ ($q = \exp(2\pi i/(K+N))$) with simple objects up to isomorphism in bijection with the set Λ of Young diagrams with less than or equal to K columns and $N-1$ rows, with duality $\hat{\lambda} = \lambda^*$*

Recall that an object is called simple if $\mathrm{End}_{\mathcal{C}}(V) = R\cdot\mathrm{Id}_V$, and a category is called *semisimple* if every object is a direct sum of simple ones. In our language this is possible because we carried out the purification process where we divided out the negligible morphisms. In our category $\mathcal{H}'\mathcal{T}$, what we were arguing was that we can define y_λ for λ "too big", but with quantum dimension 0. This means that the identity morphism of this object is negligible, and so the object is isomorphic to object corresponding to the complementary idempotent $1-y_\lambda$. This means that for any object ob in $\mathcal{H}'\mathcal{T}$, its identity morphism factors as a sum over Λ of things like $\psi_\lambda \circ y_\lambda \circ \varphi_\lambda$ for $\lambda \in \Lambda$. This is referred to by Turaev as the domination category (implied by semisimpleness).

Definition 3.2.2 (Modular category (Turaev))**.** A ribbon category over a field R with finitely many simple objects $(V_\lambda)_{\lambda\in\Lambda}$ up to isomorphism with $\mathbf{1} \in \Lambda$, $\hat{\mathbf{1}} = \mathbf{1}$ is called a *modular category* if:

- The $(V_\lambda)_{\lambda\in\Lambda}$ dominate the category (in the above sense).
- The S-matrix $(S_{\mu\nu})_{\mu,\nu\in\Lambda}$ is invertible (see below).

In any ribbon cateogry $\mathcal{C}$, we have a framed link invariant with link components labelled by objects of $\mathcal{C}$, and we let $S_{\mu\nu}$ be the invariant of the oriented framed Hopf link with components coloured by V_μ, V_ν.

Theorem 3.2.3 (Turaev)**.** *For any modular category, one can construct the full package of a TQFT, a modular functor, 3-manifold invariants, representations of mapping class groups.*

Theorem 3.2.4 (Blanchet)**.** *The category $\mathcal{C}_{N,K} = (\mathcal{H}'\mathcal{T})^{ic,pur}$ is a modular category in the sense of Turaev, with $q = \exp(2\pi i/(N+K))$.*

We will now construct a modular functor from this particular modular category (and refer to Turaev for how to do it in general). So now we have an $R = \mathbb{C}$-valued invariant of framed links with components labelled by objects of $\mathcal{C}_{N,K}$.

We extend this to an invariant I of triples $(M^3, n, \mathcal{L})$, where M^3 is a closed oriented compact 3-manifold, $n \in \mathbb{Z}$, and $\mathcal{L}$ is a framed link in M with components labelled by objects of $\mathcal{C}_{N,K}$. We already have this in the case $M = S^3$, $n = 0$. One does the extension using surgery presentation by links in S^3; this needs the n and the invertibility of the S-matrix. The reason it needs the n is the following: The pair (M, n) is called (following Walker's/Turaev's way of dealing with the framing anomaly, corresponding in CFT to the connection only being projective) an *extended* 3-*maniofld*. There is a notion of gluing extended 3-manifolds *extended surfaces*, i.e. pairs (Σ, L), where Σ is a closed surface and $L \subseteq H_1(\Sigma; \mathbb{R})$ is a Lagrangian (for the intersection form): If we have 3-manifolds M_1 and M_2 with $\partial M_1 = \Sigma$, $\partial M_2 = -\Sigma$ (opposite orientation), then we can form a closed manifold $M = M_1 \cup_\Sigma (-M_2)$ and the n changes under gluing as follows: If we have (M_1, n_1), (M_2, n_2) glued along (Σ, L), then we define (M, n) by

$$n = n_1 - n_2 + \text{Maslov index}(L_{M_1}, L, L_{M_2}),$$

where L_{M_1} and L_{M_2} are the kernels of the inclusions $H_1(\sigma, \mathbb{R}) \to H_1(M_i, \mathbb{R})$, $i = 1, 2$.

Now we could also image M containing a link $\mathcal{L}$ that somehow pierces the surface Σ; i.e. we view the link as coming from tangles in M_1,M_2. What we then do is generalize our notion of extended manifold to manifolds with $\mathcal{C}_{N,K}$-labelled tangles. The surfaces now contain framed points labelled with objects of $\mathcal{C}_{N,K}$, so extended surfaces are now $(\Sigma, P, \vec{\lambda}, L)$, where P is the set of framed points and $\vec{\lambda}$ is the set of objects of $\mathcal{C}_{N,K}$.

In the following description of the "universal construction", we need the following simplification, which holds for the invariant I arising from our category: Namely that it conjugates under reversal of orientation of both the manifold and link. We can then define our modular functor $V(\Sigma, P, \vec{\lambda}, L)$ as follows:

Let $\mathcal{V}(\Sigma, P, \vec{\lambda}, L)$ be the span of 3-manifolds $(M, n, \mathcal{L})$, where $\partial M = \Sigma$, $\partial \mathcal{L} = (P, \vec{\lambda})$, $\mathcal{L}$ a tangle in M, everything viewed up to diffeomorphism releative to the boundary. This is an infinite-dimensional vector space with an hermitian (by the fact about orientation reversal) form

$$\langle (M_1, n_1, \mathcal{L}_2), (M_2, n_2, \mathcal{L}_2) \rangle = I(M_1 \cup -M_2, n_1 + n_2 - \text{Maslov index}(L_{M_1}, L, L_{M_2}), \mathcal{L}_1 \cup -\mathcal{L}_2).$$

We define now $V(\Sigma, P, \vec{\lambda}, L) = \mathcal{V}(\Sigma, P, \vec{\lambda}, L)/\text{left kernel of } \langle\, , \,\rangle$ (note that here the left kernel is the right kernel beacuse of the Hermitian-ness). Note here that the L is really used: It is a fact that this invariant I multiplies by some number $\kappa \neq 0$ if n is increased by 1. This fact implies that $V(\Sigma, P\vec{\lambda}, L)$ is independent of L up to isomorphism; the L only serves to make the construction functorial.

Theorem 3.2.5. *With this approach,* $\dim V(\Sigma, P, \vec{\lambda}, L) < \infty$ *given by the Verlinde formula. Furthermore, (surpressing dependence on everything), we have a canonical isomorphism* $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \sqcup \Sigma_2)$ *(for this one again uses the invertibility of S-matrix) – recall in Pauly's construction this was part of the condition but here it is something that needs to be proven.*

Masbaum ends by explaining the gluing/factorization in this picture by drawing.

3.3 Himpel

This talk will cover sewing, ghost vacua under change of formal coordinates, construction of a preferred holomorphic section of $\mathcal{V}^\dagger_{ab}(\mathcal{F})$.

We start off with a nodal curve $\mathcal{X} = (C, Q_1, \dots, Q_N, \xi_1, \dots, \xi_N)$, with a desinguralization $\tilde{\mathcal{X}} = (\tilde{C}, Q_1, \dots, Q_N, P_+, P_-, \dots)$ (drawing), such that the nodal curve sits over the point 0 of the disc parametrizing the curves (with coordinates q).

Theorem 3.3.1. *For all* $\langle \Phi | \in \mathcal{V}^\dagger_{ab}(\tilde{X})$*, there exists a unique flat section* $\langle \tilde{\Phi} |$ *of* $\mathcal{V}^\dagger_{ab}(\mathcal{F})$ *such that*

$$i^*_{+,-}(\langle \Phi |) \langle \tilde{\Phi}(0) |,$$

where $i_{+,-} : \mathcal{F}_N \hookrightarrow \mathcal{F}_{N+2}$, $i^*_{+,-} : \mathcal{F}^\dagger_{N+2} \to \mathcal{F}_N$ *gives an isomorphism* $\mathcal{V}^\dagger_{ab}(\tilde{\mathcal{X}}) \overset{\cong}{\to} \mathcal{V}^\dagger_{ab}(\mathcal{X})$ *mentioned in Hans-Christian's lecture, which maps*

$$|u\rangle \mapsto (|0\rangle \otimes |-1\rangle - |-1\rangle \otimes |0\rangle) \otimes |u\rangle$$

The idea of the proof is to define $\tilde{\Phi}(q)$ as a formal power series in q (as a sum over charge p and degree d) in terms of bases of $\mathcal{F}_d(p)$ and a dual basis with respect to some bilinear pairing such that $\langle\tilde{\Phi}(0)| = i^*_{+,-}\langle\Phi|$, and then to show that $\tilde{\Phi}(q)$ formally satisfies a gauge condition, and that for a certain $\tilde{l} = H^0(\tilde{C}, \Theta_C(*\sum_{j=1}^N Q_j))$ such that there are coordinate neighbourhoods $U_\pm$ of $P_\pm$ with local coordinates z, w with center $P_\pm$ and $\tilde{l}_{U_+} = \frac{1}{2}z\frac{d}{dz}$, $\tilde{l}_{U_-} = \frac{1}{2}w\frac{d}{dw}$ and compute

$$q\frac{d}{dq}(\langle\tilde{\Phi}|u\rangle) - \sum_{j=1}^{N}\langle\tilde{\Phi}(q)|l_j(T[l_j(\xi_j)])|u\rangle - \frac{1}{6}b_\omega(\vec{l})\langle\tilde{\Phi}(q)|u\rangle = 0,$$

where b_ω is a bidifferential of the form $\frac{dv\,du}{(v-u)}$ + holomorphic terms. and $l_j(\xi_j)$ is a Laurent expansion of $\tilde{l}$ with respect to ξ_j. This differential equation is of Fuchsian type, which implies that $\tilde{\Phi}(q)$ converges.

We turn now to the proof of the fact that $\dim_{\mathbb{C}} \mathcal{V}^\dagger_{ab}(\mathcal{X}) = 1$. The first thing one has to realize is that $d(g) := \dim \mathcal{V}^\dagger_{ab}(\mathcal{X})$ only depends on C, if C is nonsingular (by coherence of the sheaf and propagation). If C has one double point, we get a family $\mathcal{F}(q)$. Suppose $\mathcal{F}(q) = g$, $q \neq 0$. By coherence, the dimension is upper semi-continuous, $\dim \mathcal{V}^\dagger_{ab}(\mathcal{F}(0)) \geq \dim(\mathcal{V}^\dagger(q))$. By the theorem,

$$\dim \mathcal{V}^\dagger_{ab}(\mathcal{F}(0)) \leq \dim \mathcal{V}^\dagger_{ab}(\mathcal{F}(q)),$$

and by partial desingularization/normalization,

$$d(g-1) = \dim \mathcal{V}^\dagger_{ab}(\tilde{\mathcal{X}}) = \dim \mathcal{V}^\dagger_{ab}(\mathcal{F}(0)) = \dim \mathcal{V}^\dagger_{ab}(\mathcal{F}(q)) = d(g),$$

and by propagation, $\dim \mathcal{V}^\dagger_{ab}(\mathcal{X})$ is independent of $\mathcal{X}$, and by example, the dimension is 1.

Corollary 3.3.2. *The connection $\nabla^{(\omega)}$ extends holomorphically over the nodal curves. Furthermore, $\mathcal{V}^\dagger_{ab}(\mathcal{F})$ and $\mathcal{V}_{ab}(\mathcal{F})$ are invertible $\mathcal{O}_{\mathcal{B}}$-modules.*

Theorem 3.3.3. *If $\mathcal{F} = (\pi : C \to B = E \times D; s_1, \ldots, s_N, \xi_1, \ldots, \xi_N)$, where E is a small polydisc with coordinates $(u_1, \ldots, u_m)$, D a disc, such that $\hat{\mathcal{F}} = (\pi : \hat{C} \to E, s_1, \ldots, s_N, \xi_1, \ldots, \xi_N)$ is a family of nodal curves and $\tilde{\mathcal{F}} = (\tilde{\pi} : \tilde{S} \to E; s_1, \ldots, s_N, t_+, t_-, \xi_1, \ldots, \xi_N, z, w)$ the family of normalizations, then for $\langle\varphi| \in \mathcal{V}^\dagger_{ab}(\tilde{\mathcal{F}})$, let $\langle\Phi(q)| \in \mathcal{V}^\dagger_{ab}(\mathcal{F})$ be the element obtained by sewing (and a family of bidifferentials $\omega(q)$). If $\langle\tilde{\psi}(q)| \in \mathcal{V}^\dagger_{ab}(\mathcal{F})$ is another element obtained from $\langle\psi| = \nabla^{(\omega(0))}_{\frac{d}{du_i}}\langle\varphi| \in \mathcal{V}^\dagger_{ab}(\tilde{\mathcal{F}})$ by sewing, then*

$$\nabla^{(\omega(q))}_{\frac{d}{du_i}}\langle\tilde{\Phi}(q)| - \langle\tilde{\psi}(q)| = \frac{1}{6}(b_{\omega(q)}(\vec{m}^{(i)}) - b_{\omega(0)})(\vec{m}^{(i)})\langle\tilde{\Phi}(q)|,$$

and $\vartheta(\vec{m}^{(i)}) = \frac{d}{du_i}$, where ϑ was defined in Hans-Christian Herbig's talk.

For the preferred holomorphic section, take $\mathcal{X} = (C; Q; \xi)$ for a smooth curve C, and fix a symplectic basis $\{\alpha_i, \beta_i\}_{i=1,\ldots,g} \subseteq H_1(C, \mathbb{Z})$. Then one can define

$$\langle\omega(\mathcal{X})| \in \mathcal{F}^\dagger(g-1) \cap \mathcal{V}^\dagger_{ab}(\mathcal{X}),$$
$$\langle\omega(\mathcal{X}, \{\alpha, \beta\})| = \cdots e(\omega_{g+2}) \wedge \cdots \wedge e(\omega_1),$$

where $e(\omega_i) = \sum_{n=0}^\infty a_n^{(i)} e^{n+1/2}$, $e(\omega_{g+j}) = e^{-j-1/2} + \sum_{n=-j}^\infty a_n^{(g+j)} e^{n+1/2}$, $\{\omega_1, \ldots, \omega_g\}$ is a normalized basis of holomorphic 1-forms on C, $\omega_{g+n} = \omega_Q^{(n)}$ are meromorphic 1-forms with pole of order $n+1$ at Q and holomorphic elsewhere and satisfy

$$\int_{\alpha_i} \omega_Q^{(n)} = \frac{-2\pi i I_n^i}{n}, \int_{\beta_i} \omega_Q^{(n)} = 0,$$

$1 \leq i \leq g$, and

$$\omega_Q^{(n)} = \left(\frac{1}{\xi^{n+1}} + \sum_{m=1}^{\infty} q_{n,m} \xi^{m-1} \right) d\xi,$$

where I_n^i are given by

$$\omega_i = \left(\sum_{n=1}^{\infty} I_n^i \xi^{n-1} \right) d\xi,$$

$i = 1, \ldots, g$, and $\left(\sum_{n=1}^{\infty} I_n^i \xi^{n-1} \right) = a_n^{(i)}$.

Theorem 3.3.4. *1) Under change of formal coordinates $\mathcal{X}_h = \{C; Q; \eta = h(\xi)\}$,*

$$\langle \omega(\mathcal{X}, \{\alpha, \beta\}) | G[h] = \langle \omega(\mathcal{X}_h, \{\alpha, \beta\}) |,$$

where $G[h]$ has not been defined.

2) If $\{\tilde{\alpha}_i, \tilde{\beta}_i\}$, and $\operatorname{span}\{\beta_1, \ldots, \beta_g\} = \operatorname{span}\{\tilde{\beta}_1, \ldots, \tilde{\beta}_g\}$, then

$$\langle \omega(\mathcal{X}, \{\alpha, \beta\}| = \det U \langle \omega(\mathcal{X}, \{\tilde{\alpha}, \tilde{\beta}\})|,$$

where $(\tilde{\beta}_1, \ldots, \tilde{\beta}_g) = U(\beta_1, \ldots, \beta_g)$.

3) If $\mathcal{X}_1 = (C, P, \xi)$, $\mathcal{X}_2 = (C; Q, \eta)$, consider the propagation $\mathcal{X} = (C; P, Q, \xi, \eta)$, and propagation maps i_1, i_2, so e.g. $i_1 : |0\rangle \mapsto |u\rangle \otimes |0\rangle$. Then

$$i_2^* \circ (i_1^*)^{-1} \langle \omega(\mathcal{X}_1, \{\alpha, \beta\})| = \langle \omega(\mathcal{X}_2, \{\alpha, \beta\})|.$$

4) If π is the normalization map (drawing), then

$$i^*(i_{+,-}^*)^{-1}(\langle \omega(\mathcal{X}, \{\alpha, \beta\})|) = \langle \omega(\hat{X}, \{\hat{\alpha}, \hat{\beta}\})|,$$

where $\{\hat{\alpha}, \hat{\beta}\} = \{\alpha_i, \beta_i\}_{i=1,\ldots,g-1}$.

5) Choose smoothly varying bases $\{\alpha_i(t), \beta_i(t)\}$, $t \in U_b \subseteq \mathcal{B}$ in a neighbourhood of $b \in B$. Then $\langle \omega(\mathcal{X}_t, \{\alpha(t), \beta(t)\})|$ is a holomorphic section.

6) For the section around C_0 with double point from Theorem 3.3.1, choose continuous bases $\{\alpha_i(t), \beta_i(t)\} \subseteq H_1(\pi^{-1}(t), \mathbb{Z})$, $t \in (0,1) \subseteq D$, such that for $t \to 0$, this gives a basis $\{\alpha_1(0), \ldots, \alpha_g(0), \beta_1(0), \ldots, \beta_{g-1}(0$ Let $\mathcal{X}(t) = (\pi^{-1}(t), s_1(t), \ldots, s_N(t), \xi_1, \ldots, ß_N)$, then

$$\langle \omega(\mathcal{X}, \{\alpha(0), \beta(0)\}|) = \lim_{t \to 0} \langle \omega(\mathcal{X}_t, \{\alpha_t, \beta_t\})|.$$

3.4 Wentworth

The idea of conformal symmetry in physics is very wide ranging, in QFT and statistical mechanics. Any quantum field theory starts with a Hilbert space of states. For a CFT, these spaces carry representations of the Virasoro algebra (with a central extension). There are particular fields called primary fields corresponding to highest weight vectors. The fields form an algebra called the operator product epxansion. When there are only finitely many primary fields, one talks about *rational conformal field theory*, in which case one can say a lot. Some examples which are important in statistical mechanics are those with $0 \leq c < 1$ which are called minimal models (discrete series). Other are *WZNW models*, which are the ones that have been discussed here.

Outline of this talk:

1. Few more examples of representation of Vir_c.

2. Local (chiral) fields.

3. Definition of CFT.
4. Ward identities and correlation functions.
5. Conformal blocks

Examples: The free boson: Generators are $\{a_n\}_{n\in\mathbb{Z}}$ with relations

$$[a_m, a_n] = m\delta_{m+n,0}$$

and normal ordering

$$: a_i a_j := \begin{cases} a_i a_j & i \le j \\ a_j a_i & i > j \end{cases}.$$

As we have seen, this carries irreducible representations on Fock space, and one gets representations of Vir_c by

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : a_{-k}a_{n+k} :, .$$

Recall here that Vir_c has generators L_m, $m \in \mathbb{Z}$ with relations

$$[L_m L_n] = (m-n)L_{m+n}\frac{(m^3-m)}{12}\delta_{m+n,0}.$$

Here $c = 1$.

Another example is that of free fermions,

$$\{\psi_m, \psi_n\} = \psi_m\psi_n + \psi_n\psi_m = \delta_{m+n,0}.$$

Here m could be in $\mathbb{Z}$ (*Ramond sector*) or $m \in \mathbb{Z} + 1/2$ (*Neveu–Schwarz sector*). Normal ordering is

$$: \psi_i\psi_j := \begin{cases} \psi_i\psi_j & i \le j \\ -\psi\psi_i & i > j \end{cases}$$

and L_n has a certain expression (that I didn't have time to write down); here $c = 1/2$.

A third example is the b-c ghost system, with two sets of generators $\{b_n, c_n\} = \delta_{m+n,0}$, $\{b_n, b_m\} = \{c_n, c_m\} = 0$, $\lambda \in 1/2\mathbb{Z}$,

$$L_n = \sum_{\gamma\in\mathbb{Z}} (\lambda n - j) : B_j c_{n-j} :,$$
$$c_\lambda = -2(6\lambda^2 - 6\lambda + 1),$$

$c_{-1} = -26$.

The last example here is the Sugawara construction, $c = 2k\dim\mathfrak{g}/(Q+2k)$, k the level.

Now V is a Hilbert space. A field is $\varphi(z) = \sum_{n\in\mathbb{Z}}\varphi^{(n)}z^{-n-1}$, $\varphi^{(n)} \in \mathrm{End}V$, for all $v \in V$, $\varphi^{(n)}v = 0$ for $n \gg 1$. A collection of fields is (mutually) *local* if there exists N s.t. $(z-w)^N[\varphi_1(z), \varphi_2(w)] = 0, z \neq w$.

Definition 3.4.1. A 2-dimensional field theory is a Hilbert space V, a linear space $\mathcal{L}F$ of local fields, a linear map $\Phi : V \to \mathcal{L}F$ (state-field correspondence), a vacuum $|0\rangle \in V$, and a preferred endomorphism $\tau \in \mathrm{End}V$ such tat $\Phi|0\rangle = I$, $\Phi(v)|0\rangle|_{z=0} = v$ and $[\tau, \varphi(z)] = \partial_z\varphi$.

Time ordering or radial ordering is $T(\varphi_1(z_1)\cdots\varphi_N(z_N)) = \varphi_1(z_1)\cdots\varphi_N(z_N)$ when $|z_1| > \cdots > |z_N|$. We will be interested in correlation functions $\langle\varphi_1(z_1)\cdots\varphi_N(z_N)\rangle := \langle 0|T(\varphi_1(z_1)\cdots\varphi_N(z_N))|0\rangle$. Normal ordering of fields is

$$: \varphi_1(z)\varphi_2(w) := \varphi_1^+(z)\varphi_2(w) + \varphi_2(w)\varphi_1^-(z).$$

Theorem 3.4.2. *There exists an operator product expansion (OPE): $\varphi_1, \varphi_2 \in \mathcal{L}F$,*

$$T(\varphi_1(z)\varphi_2(w)) = \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}} + :\varphi_1(z)\varphi_2(w):$$

Dog's Lemma is that $c_j \in \mathcal{L}F$ again. In fact, if everything has an asymptotic state, then $c_j = \Phi(\varphi_1^{(j)}\varphi_2^{(-1)}|0\rangle)$. In the litterature, this is written

$$\varphi_1(z)\varphi_2(w) \sim \sum_{j=0}^{N-1} \frac{c_j(w)}{(z-w)^{j+1}}.$$

The OPEs determine commutation relations and vice versa. Explicitly,

$$[\varphi_1^{(m)}, \varphi_2^{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} c_j^{(m+n+j)}.$$

(Here should be a proof of that)

The ODEs from the examples from the beginning are:

Free boson:

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$
$$a(z)a(w) \sim \frac{1}{(z-w)^2}.$$

For the fermions,

$$\psi(z) = \sum_{n\in\mathbb{Z}+1/2} \psi_n z^{-n-1/2},$$
$$\psi(z)\psi(w) \sim \frac{1}{z-w}.$$

For the b-c ghost system,

$$b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-\lambda}, \quad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-1+\lambda},$$
$$b(z)c(z) \sim \frac{1}{z-w}.$$

For WNZW,

$$J_a(z) = \sum_{n\in\mathbb{Z}} \hat{J_a} z^{-n-1},$$
$$J_a(z)J_b(w) \sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{if_{ab}^c J_c(w)}{z-w}.$$

Define a field $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T}{z-w}.$$

Then the L_n satisfy the Virasoro relations with central charge c. The T is called the energy momentum tensor.

Definition 3.4.3. A *conformal field theory* is a 2-dimensional field theory with local field $T(z) \in \mathcal{L}F$ satisfying the last OPE above and $L_{-1} = \tau$. Normally we should assume also:

1. L_0 is diagonalizable on V with non-negative eigenvalues (positive energy condition)
2. $L_n^* = L_{-n}$ (unitary condition).

In this case V divides up into highest weight representations.

Remark 3.4.4. $T(z)|0\rangle_{z=0} \in V$, so $L_n|0\rangle = 0$ if $n \geq -1$. In particular, $|0\rangle$ is invariant under $\mathfrak{sl}_2 = \text{span}\{L_0, L_{\pm 1}\}$.

Definition 3.4.5. $\varphi \in \mathcal{L}F$ (assume $\varphi(z)|0\rangle|_{z=0}$ regular) is a *primary field* of conformal weight Δ if

$$T(z)\varphi(w) \sim \frac{\Delta}{(z-w)^2}\varphi(w) + \frac{1}{z-w}\partial_w \varphi.$$

The set of primary fields are in 1-1 correpsondence with highest weight vectors, which is equivalent to saying that $\varphi(z)$ is a section of $(T^*C)^\Delta \otimes \text{End}V$.

Proposition 3.4.6. *TFAE,* $\varphi = \sum_{n\in\mathbb{Z}} \varphi^{(n)} z^{-n-1}$, $\varphi^{(n)}|0\rangle$ *if* $n \geq 0$, $\varphi = \Phi(v)$, $v = \varphi^{(-1)}|0\rangle$.

1. φ *is a primary field*
2. $L_m v = 0$, $m > 0$, $L_0 v = \Delta v$
3. $[L_m, \varphi^{(n)}] = ((m+1)\Delta - (m+n+1))\varphi^{(m+n)}$
4. $[L_m, \varphi(z)] = z^m(z\partial_z + (m+1)\Delta)\varphi(z)$.

Theorem 3.4.7 (Conformal Ward identities)**.** φ_i *primary.*

1. $\sum \partial_{z_i} \langle \varphi_1(z_1)\cdots\varphi_N(z_N)\rangle = 0$
2. $\sum (z_i\partial_{z_i} + \Delta_i)\langle \varphi_1(z_1)\cdots\varphi_N(z_N)\rangle = 0$
3. $\sum (z_i^2\partial_{z_i} + 2z_i\Delta_i)\langle \varphi_1(z_1)\cdots\varphi_N(z_N)\rangle = 0$.

Theorem 3.4.8 (2nd Ward identity)**.**

$$\langle T(z)\varphi_1(z_1)\cdots\varphi_N(z_N)\rangle = \sum_{i=1}^{N} (\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i})\langle \varphi_1(z_1)\cdots\varphi_N(z_N)\rangle$$

This implies: If φ_1 are primary fields,

$$\langle \varphi_1(z_1)\varphi_2(z_2)\rangle = \begin{cases} c(\frac{dz_1 dz_2}{(z_1-z_2)^2})^\Delta, & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases}$$

$$\langle \varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3) = c_{123}\prod_{i<j}(\frac{dz_i dz_j}{z_{ij}^2})^{1/2(\Delta_1+\Delta_j-\Delta/2)},$$

$\Delta = \Delta_1 + \Delta_2 + \Delta_3$.

$$\langle \varphi_1(z_1)\ldots\varphi_4(z_4)\rangle = A\frac{z_{12}z_{34}}{z_{13}z_{24}}\prod_{i<j}(\frac{dz_i dz_j}{z_{ij}^2})^{(\Delta_i+\Delta_j)/2-\Delta/6}.$$

$$\tau = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

$$A_{kl}^{ij}(\tau) = \sum_g F_{kl}^{ij}(g,\tau).$$

The latter is the conformal block.

4 April 24th, 2012

4.1 Andersen II

Recall that a modular functor was a functor V from the category of labelled (by Λ a finite label set with an involution $*$ and a $1 \in \Lambda$ with $1^* = 1$) marked surfaces $\mathbf{\Sigma} = (\Sigma, P, V, L)$, P finitely many points in Σ, $V \in \times_{p\in P} P(T_p(\Sigma))$, $L \subseteq H_1(\Sigma, \mathbb{Z})$, to the category of finite dimensional complex vector spaces.

Now we start the definition of this through the geometric setting. A *marked Riemann surface* is $C = (C, Q, V)$, where C is a Riemann surface, $Q \subseteq C$ a finite subset, V non-zero tangent vectors at Q. A morphism $f : C_1 \to C_2$ between such things is a biholomorphism of Riemann surfaces with $f(Q_1) = Q_2$ and $f_*(V_1) = V_2$. A complex structure on a pointed surface (Σ, P) is a marked Riemann surface (C, Q, W), and $\varphi : (\Sigma, P) \to (C, Q)$. If $\tilde{\varphi} : (\Sigma, P) \to (\tilde{C}, \tilde{Q}, \tilde{W})$ is another complex structure, it is called equivalent to the first one, if there is a morphism of marked Riemann surfaces $\Phi : C \to \tilde{C}$, making the resulting diagram commute up to diffeomorphisms that are isotopic to the identity among diffeomorphisms which are the identity to the first order at P. By definition, *Teichmüller space* $\mathcal{T}_{(\Sigma,P)}$ is the set of equivalence classes of complex structures on (Σ, P).

Theorem 4.1.1 (Baer)**.** *Teichmüller space is a finite dimensional complex manifold.*

Note that as defined, it is not contractible, because of our requirement on first order jets above. Let $\mathcal{T}^{(r)}_{(\Sigma,P)} = \mathcal{T}_{(\Sigma,P)}/\mathbb{R}^P_+$ and we have projections $\pi_P : \mathcal{T}_{(\Sigma,P)} \to \times_{p\in P}(T_p(\Sigma) \setminus \{0\})$, and $\pi_P : \mathcal{T}^{(r)} \to \times_{p\in P} P(T_p\Sigma)$.

To a marked Riemann surface, we define

$$\mathcal{T}_{\mathbf{\Sigma}} = \pi_p^{-1}(V).$$

These are the spaces we will be interested in, and these are contractible.

Now, we turn to formal neighbourhoods. Let C be a Riemann surface and $q \in C$. A *formal n'th order neighbourhood* is a choice of isomorphism

$$\mathcal{O}_{C,q}/m_q^{n+1} \cong \mathbb{C}[\![\xi]\!]/(\xi^{n+1}),$$

and a formal neighbourhood of infinite order is a choice of isomorphism with the limit

$$\hat{\mathcal{O}}_{C,q} = \lim_{n\to\infty} \mathcal{O}_{C,q}/m_q^n \overset{\eta}{\cong} \mathbb{C}[\![\xi]\!].$$

The first order neighbourhood is

$$\mathcal{O}_{C,q}/m_q^2 = \mathbb{C} \oplus T_q^*C \cong \mathbb{C} \oplus \mathbb{C}.$$

For a choice of 1 is the latter, let $\alpha \in T_q^*C$ be the corresponding element under the isoimorphsim. We see that specifying a first order neighborhood is the same as choosing $v \in T_qC$, $\alpha(v) = 1$. In particular, we can not get rid with these choices.

Consider now $\mathcal{X} = (C, q_1, \dots, q_n, \eta_1, \dots, \eta_n)$, where η_j are formal coordinates (we need an ordering here, since we take ordered tensor products).

Definition 4.1.2. A family of pointed Riemann surfaces with formal coordinates are the following: C, B complex manifolds, $\dim C = \dim B + 1$, a holomorphic submersion $\pi : C \to B$, sections $s_j : B \to C$, $j = 1, \dots, n$ (that will trace out the points in the family), formal neighbourhoods $\eta_j : \hat{\mathcal{O}}_{s_j} = \lim_n \mathcal{O}_C/I_j^n \cong \mathcal{O}_B[\![\xi]\!]$ (where I_j^n is the n'th power of the ideal of the particular section).

The general scheme now is that we want to look at families, where we have identified each fiber with a particular surface, and gluing all families together in a canonical way to obtain a vector bundle over Teichmüller space. In order to do that, we need to talk about special good families.

Suppose now $Y = \Sigma \times B$ (thought of as a smooth manifold) and suppose we have a family $\mathcal{F} = (\pi : C \to B, \vec{s}, \vec{\eta})$ as above. We would like to have a map $Y \overset{\Phi_{\mathcal{F}}}{\to} C$ over B, taking points of Y to sections, such that the diagram with the coverings commute. This here will immediately induce (by the universal property Teichmüller space has) a map $\Psi_F : B \to \mathcal{T}_{(\Sigma,P)}$. The family $\mathcal{F}$ will be called a family on (Σ, P). A *good family* is one where B is biholomorphic to an open polydisk (open contractible complex manifold really), such that the map $\Psi_{\mathcal{F}} : B \to \mathcal{T}_{(\Sigma,P)}$ is a biholomorphism onto its open image.

Proposition 4.1.3. *We can cover $\mathcal{T}_{(\Sigma,P)}$ by images of good families.*

Suppose we have good families $\mathcal{F}_1, \mathcal{F}_2$ with the same images in Teichmüller space. Then we have the diagram

$$\begin{array}{ccccccc}
\Sigma \times B_1 & \xrightarrow{\Phi_{\mathcal{F}_1}} & C_1 & \xrightarrow{\Phi} & C_2 & \xleftarrow{\Phi_{\mathcal{F}_2}} & \Sigma \times B_2 \\
& \searrow & \downarrow & & \downarrow & \swarrow & \\
& & B_1 & \xrightarrow{\Psi^{-1}_{\mathcal{F}_2} \circ \Phi_{\mathcal{F}_1}} & B_2 & &
\end{array}$$

and the map

$$\Phi^{-1}_{\mathcal{F}_2} \Phi \Phi_{\mathcal{F}_1} : (B_1 \times \Sigma, P) \to (B_2 \times \Sigma, P)$$

is isotopic to $\Psi^{-1}_{\mathcal{F}_2} \circ \Psi_{\mathcal{F}_1} \times \mathrm{Id}$.

The sheaf of vacua for any simple Lie algebra: Consider a family $\mathcal{F} = (\pi : C \to B, \vec{s}, \vec{\eta})$. Recall that we considered

$$\hat{\mathfrak{g}}_N(B) = \mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{j=1}^{N} \mathcal{O}_B((\xi_j)) \oplus \mathcal{O}_B \cdot c.$$

Let $S = \bigsqcup_{j=1}^N s_j(B)$. Then we put

$$\hat{\mathfrak{g}}(\mathcal{F}) = \mathfrak{g} \otimes_{\mathbb{C}} \pi_*(\mathcal{O}_C(*S)),$$

which is the space of meromorphic functions from C to $\mathfrak{g}$ with poles along S.

Let now $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ be the labels in $P_l(\mathfrak{g})$. Then we let

$$\begin{aligned}
\mathcal{H}_{\vec{\lambda}}(B) &= \mathcal{O}_B \otimes \mathcal{H}_{\lambda}, \\
\mathcal{H}^{\dagger}_{\vec{\lambda}}(B) &= \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{H}_{\vec{\lambda}}(B), \mathcal{O}_B).
\end{aligned}$$

Then $\mathfrak{g}(B)$ acts on $\mathcal{H}_{\vec{\lambda}}(B)$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}(B)$. Laurent expansion allows us to view $\hat{\mathfrak{g}}(\mathcal{F})$ as a subspace of $\hat{\mathfrak{g}}_N(B)$.

Definition 4.1.4. We define the *sheaf of vacua* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{V}_{\vec{\lambda}}(B), \mathcal{O}_B)$, where $\mathcal{V}_{\vec{\lambda}}(B) = \mathcal{H}_{\vec{\lambda}}(B)/\hat{\mathfrak{g}}(\mathcal{F})\mathcal{H}_{\vec{\lambda}}(B)$ is the *sheaf of covacua.*

These, we have seen, are locally free sheaf. What we will do now is cover Teichmüller space by good families, over each of these we take the bundle coming from the above and specify the precise transition functions to give a bundle over Teichmüller space.

Change of coordinates: Take $D = \mathrm{Aut}\mathbb{C}((\xi))$. As we have seen, $D \cong \{\sum_{n=0}^{\infty} a_n \xi^{n+1} \mid a_0 \neq 0\}$ mapping $h \to h(\xi)$. What we will need now

$$D^0_+ = \{h \in D \mid a\xi + \cdots, a > 0\},$$

such that $\exp : d^0_+ := \{l(\xi)\frac{d}{d\xi} \mid l(\xi) = a\xi + \cdots, a \in \mathbb{R}\} \to D^0_+$ is an isomorphism. We have $\exp(2\pi i \xi \frac{d}{d\xi}) = \mathrm{Id}$. For $h \in D^0_+$ we define an operator

$$G[h] = \exp(-T[l]) : \mathcal{H}^{(\dagger)}_{\lambda} \to \mathcal{H}^{(\dagger)}_{\lambda},$$

where $T[l]$ is as defined previous in these lectures. Consider now

$$\mathcal{F}_1 = (\pi : C \to B, s_1, \eta_j),$$
$$\mathcal{F}_2 = (\pi : C \to B, s_j, h_j(\eta_j)),$$

for $h_j \in D^0_+$. Then

$$G[h_1] \otimes \cdots \otimes G[h_n] : \mathcal{H}^{(\dagger)}_{\vec{\lambda}} \to \mathcal{H}^{(\dagger)}_{\vec{\lambda}}$$

induces *isomorphisms* $\mathcal{V}^{(\dagger)}_{\vec{\lambda}_1}(\mathcal{F}_1) \to \mathcal{V}^{(\dagger)}_{\vec{\lambda}_2}(\mathcal{F}_2)$.

Remark that if we have $\Phi : \mathcal{F}_1 \to \mathcal{F}_2$, covering the identity on B, then there is a canonically induced isomorphism

$$\mathcal{V}^\dagger(\Phi) : \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathcal{F}) \to \mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathcal{F}),$$

with $\Phi^*(\vec{\lambda}_2) = \vec{\lambda}_1$, induced by the identity on $\mathcal{H}^{(\dagger)}_{\vec{\lambda}}$. Consider now good families $\mathcal{F}_1, \mathcal{F}_2$, with $\Psi_{\mathcal{F}_1}(B_1) = \Psi_{\mathcal{F}_2}(B_2)$, so we have biholomorphisms $\Phi_{12} : C_1 \to C_2$ covering $B_1 \overset{\Psi^{-1}_{\mathcal{F}_2}\Psi_{\mathcal{F}_1}}{\to} B_2$. We consider $(\Psi^{-1}_{\mathcal{F}_2}\Psi_{\mathcal{F}_1})^*\mathcal{V}^\dagger(\mathcal{F}_2)$. There is a permutation s which handles the order of the sections in $\mathcal{F}_1, \mathcal{F}_2$, and there is some unique coordinate change h and maps

$$\begin{array}{ccccccc}
\mathcal{H}^\dagger_{\vec{\lambda}_1}(B_1) & \xrightarrow{s} & \mathcal{H}_{\lambda_2}(B_1) & \xrightarrow{G(\vec{h})} & \mathcal{H}_{\lambda_2}(B_1) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathcal{F}_1) & \longrightarrow & \mathcal{V}^\dagger_{\vec{\lambda}_2}(s\mathcal{F}_1) & \longrightarrow & \mathcal{V}^\dagger_{\vec{\lambda}_2}(\tilde{\mathcal{F}}_1) & \xrightarrow{\mathcal{V}^\dagger(\Phi_{12})} & (\Psi^{-1}_{\mathcal{F}_2}\Psi_{\mathcal{F}_1})^*\mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathcal{F}_2)
\end{array}$$

The composite isomorphism, we denote

$$G_{12} : \mathcal{V}^\dagger_{\vec{\lambda}_1}(\mathcal{F}_1) \to (\Psi^{-1}_{\mathcal{F}_2}\Psi_{\mathcal{F}_1})\mathcal{V}^\dagger_{\vec{\lambda}_2}(\mathcal{F}_2).$$

Thus all in all, this defines the transition functions of the desired bundle over Teichmüller space.

5 April 25th, 2012

5.1 Andersen III

Recall that we were looking at smooth surfaces (Σ, P) and considering Teichmüller space $\mathcal{T}_{(\Sigma,P)}$ with the feature of remembering first order jets of the tangent spaces at the marked points. We ended up with the following:

Theorem 5.1.1. *There is a holomorphic bundle $\mathcal{V}^\dagger_{\vec{\lambda}}(\Sigma, P) \to \mathcal{T}_{(\Sigma,P)}$, where $\lambda : P \to \Lambda = P_l(\mathfrak{g})$, such that for all families $\mathcal{F}$, the map $\Psi^*_{\mathcal{F}}(\mathcal{V}^*\ \dagger_\lambda\ (\Sigma, P)) = \mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$.*

The connection in this bundle: This needs a little bit of preparation. What we will do is first do it in good families and show that it is invariant under the transformations defining the bundle. Let $\mathcal{F}$ be a good family of pointed Riemann surfaces with formal coordinates $\mathcal{L}(\mathcal{F}) = \bigoplus_{j=1}^N \mathcal{O}_B[\xi_j^{-1}]\frac{d}{d\xi_j}$. As we have seen from Kodaira–Spencer theory, there is a short exact sequence of sheaves

$$0 \to \pi_*\Theta_{(C/B)}(*S)\mathcal{L}(\mathcal{F}) \xrightarrow{\theta} \Theta_B \to 0.$$

What we would like is to take a tangent vector on the base and somehow define how it infinitesimally acts on the space on conformal blocks. We will define an action of $\vec{l}$ in $\mathcal{L}(\mathcal{F})$, with $\vec{l}$ mapping to a given tangent vector on the base.

Recall that for $\mathcal{O}_B(U)$, $U \subseteq B$, $|\Phi\rangle \in \mathcal{H}_\lambda$, we defined the operators

$$D(\vec{l})(F \otimes |\Phi\rangle) = \theta(\vec{l})(F) \otimes |\Phi\rangle - F\left(\sum_{j=1}^{N} \rho_j(T[l_j])\right)|\Phi\rangle.$$

This operator has almost all of the properties of being a connection, except it is defined on $\mathcal{L}(\mathcal{F})$ rather than Θ_B.

Definition 5.1.2. Suppose we have $\omega \in H^0(C \times_B C, \omega_{C\times_B C}(2\Delta))$ a bidifferential, such that ω in local coordinates near the diagonal can be written

$$\omega = \frac{1}{(x-y)^2} dx\, dy + \text{holomorphic terms.}$$

Then we can define a holomorphic quadratic differential

$$S_\omega^{(\xi_j)} \xi_j = 6 \lim_{\eta\to\xi_j, \xi\to\xi_j} \omega(\eta,\xi) - \frac{d\eta\, d\xi}{(\eta-\xi)^2}$$

We then define $b_\omega : \mathcal{L}(\mathcal{F}) \to \mathcal{O}_B$ by

$$b_\omega(\vec{l}) = \frac{1}{6}\sum_{j=1}^{N} \operatorname{Res}_{\xi_j=0}(l_j(\xi_j) S_\omega(\xi_j\, d\xi_j))$$

We are then able to define

$$\nabla_X^{(\omega)}(F \otimes |\Phi\rangle) = D(\vec{l})(F \otimes |\Phi\rangle) - \frac{c}{2} b_\omega(\vec{l}) F \otimes |\Phi\rangle.$$

Here, $\theta(\vec{l}) = X$, and c is the central charge.

Theorem 5.1.3. *$\nabla^{(\omega)}$ is a well-defined holomorphic connection in $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathcal{F}) \to B$. Moreover, if we have a family on (Σ, P) (recall that this means that we have a fixed parametrization $\Phi_{\mathcal{F}} : B \times \Sigma \to C$), and we let $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ a symplectic basis of $H^1(\Sigma, \mathbb{Z})$, this defines a bidifferential ω on $\mathcal{F}$, and we write $\nabla^{(\alpha,\beta)} = \nabla^{(\omega)}$. If $\mathcal{F}_1, \mathcal{F}_2$ are good families on Σ giving $\nabla_1^{(\alpha,\beta)}, \nabla_2^{(\alpha,\beta)}$, with $\Psi_{\mathcal{F}}(B_1)\Psi_{\mathcal{F}}(B_2)$, then*

$$G_{12}^*(\nabla_2^{(\alpha,\beta)}) = \nabla_1^{(\alpha,\beta)}.$$

On the dependence of (α, β): If $\Lambda \in \mathrm{Sp}(2g, \mathbb{Z})$, $\Lambda = \begin{matrix} A & B \\ C & D \end{matrix}$, then

$$\nabla^{\Lambda(\alpha,\beta)} - \nabla^{(\alpha,\beta)} = -\frac{c}{2}\pi^*(d \log \det(C\tau + D)),$$

where π and τ are ... In particular if $C = 0$, the dependence vanishes, and we see that Lagrangian subspaces play a role here.

Recall that the connection has curvature

$$R^{(\alpha,\beta)}(X,Y) = \frac{c}{2}\left(-b_\omega[\vec{l},\vec{m}]_{\mathcal{F}} + X(b_\omega(\vec{m})) - Y(b_\omega(\vec{l})) - \frac{1}{6}\sum_{j=1}^{N}\operatorname{Res}_{\xi_j=0}\left(\frac{d^3 l_j}{d\xi_j^3} m_j\, d\xi_j\right)\right),$$

where $\theta(X) = \vec{l}$, $\theta(Y) = \vec{m}$, and the commutator is

$$[\vec{l},\vec{m}]_{\mathcal{L}(\mathcal{F})} = [\vec{l},\vec{m}] + X(\vec{m}) - Y(\vec{l}).$$

If the curvature were 0, we could just use $\nabla^{(\omega)}$ as our connection and consider projectively flat sections with respect to that, but it is not. As we have seen, we can fix this by considering the abelian case; all that has been said in the last two lectures, can be done in the abelian theory, replacing every c by -2. More precisely, we have a line bundle $\mathcal{V}_{ab}(\Sigma, P) \to \mathcal{T}_{(\Sigma,P)}$ and a choice of (α, β) gives a connection $\nabla^{(\alpha,\beta)}_{ab}$. Now, everything fits together perfectly, and all we need to do is consider

$$\tilde{\mathcal{V}}^{\dagger}_{\lambda}(\Sigma, P) = \mathcal{V}^{\dagger}_{\lambda}(\Sigma, P) \otimes (\mathcal{V}^{\dagger}_{ab}(\Sigma, P))^{-c/2}.$$

Remember here that $c \in \mathbb{Q}$, so the last term needs some explanation. If it is possible though, this defines a vector bundle over $\mathcal{T}_{(\Sigma,P)}$ with a flat connection.

The preferred section: Let again $\mathcal{F} : (C \to B, S_j, \xi_j)$ be a family. Then we have seen that there is a holomorphic section $s^{(\alpha,\beta)}_{\mathcal{F}} : B \to \mathcal{V}_{ab}(\mathcal{F})$, which evaluated at $t \in B$ gives

$$s^{(\alpha,\beta)}_{t} = (\langle \omega(\mathcal{X}_t, \{\alpha(t), \beta(t)\})|)^{\otimes 2},$$

where if for fixed $L = \operatorname{span}\{\beta_1, \dots, \beta_g\}$, then the above transforms by some symplectic matrix (that went a little fast). Thus we really consider s as a section $B \to \mathcal{V}_{ab}(\mathcal{F})^{\otimes 2}$, and we really want to consider $\mathcal{V}_{ab}(\mathcal{F})^{-c/2} = (\mathcal{V}_{ab}(\mathcal{F})^{\otimes 2})^{-c/4}$.

For two families $\mathcal{F}_1, \mathcal{F}_2$, $G^{ab}_{12}(s^{(\alpha,\beta)}_{\mathcal{F}_1}) = s^{(\alpha,\beta)}_{\mathcal{F}_2}$, so we have our preferred section $s^{(\alpha,\beta)}_{\Sigma} : \mathcal{T}_{\Sigma} \to \mathcal{V}_{ab}(\Sigma, P)^{\otimes 2}$. One sees that the projective corrections depend not on the points. We define $\widetilde{\mathcal{V}_{ab}(\Sigma)}^{\otimes 2}$ to be the fiberwise universal cover of $\mathcal{V}_{ab}(\Sigma)^{\otimes 2}$ based at S_{Σ} (drawing). We have $\mathbb{Z}$ acting on $\widetilde{\mathcal{V}_{ab}(\Sigma)}^{\otimes 2}$ by ρ and now consider the action of $\mathbb{Z}$ by $\alpha\rho$, $\alpha/2 \in \mathbb{C}^*$. Define now simply

$$\mathcal{V}_{ab}(\Sigma, L)^{\alpha} = \widetilde{\mathcal{V}_{ab}(\Sigma)}^{\otimes 2} / \alpha\rho \otimes \mathbb{C},$$

with the usual action of $\mathbb{C}^*$ on $\mathbb{C}$. Now we can do as proposed before, namely define the **flat bundle**

$$\tilde{\mathcal{V}}^{\dagger}_{\lambda}(\Sigma, P) = \mathcal{V}^{\dagger}_{\lambda}(\Sigma, P) \otimes \mathcal{V}_{ab}(\Sigma, L)^{c/2}.$$

The modular functor: Really, what we have done so far is completely functorial: The association $(\mathbf{\Sigma}, \lambda) \mapsto \tilde{\mathcal{V}}^{\dagger}_{\lambda}(\Sigma, P)$ is a functor. The modular functor is simply just this functor, composed with the functor that takes covariantly constant section (where really we restrict to $\mathcal{T}_{\mathbf{\Sigma}}$ which is a contratible space, so the last step makes sense):

Definition 5.1.4. The vector space part of the modular functor is given by $V^{op}_{\lambda}(\mathbf{\Sigma})$ is the covariant constant sections of $\tilde{\mathcal{V}}^{\dagger}_{\lambda}(\Sigma, P)$ over $\mathcal{T}_{\mathbf{\Sigma}}$. Recall though that part of a modular functor is gluing, so that needs to be specified.

Recall that if $\mathcal{F}$ is a good family on $(\Sigma, P, V, L) = \mathbf{\Sigma}$, we can just let $V^{g}_{\lambda}(\mathcal{F})$ to be the space of covariantly constant sections over B of $\mathcal{V}^{\dagger}_{\lambda}(\mathcal{F}) \otimes \mathcal{V}_{ab}(\mathcal{F})^{-c/2}$ and then $V^{g}_{\lambda}(\mathcal{F}) \cong V^{g}_{\lambda}(\mathbf{\Sigma})$. The main idea is the following: Consider some family $\mathcal{F} = (C \to B, s_1, \dots, s_N, s_+, s_-, \xi_1, \dots, \xi_N, \xi_+, \xi_-)$, i.e. the family before gluing. Consider the glued family $B \times D = B_c$ (drawing of neighbourhoods $U_{\pm}$ around $p_{\pm}$ with coordinates $x_{\pm}$ and $\tau \in D$). The canonical family we glue on to this using these coordinates is $C^1 = \{(z, w, \tau) \in D^{\times 3} \mid zw = \tau\}$. Take

$$C^2_c = \{(y, \tau) \in \mathcal{C} \times D \mid y \in U_{\pm} \Rightarrow |x_{\pm}(y)| > |\tau|\},$$

$C_c = C^1_c \cup_{\varphi} C^2_c$, where the gluing map $\varphi : ((U_- \setminus p_-) \times D \cup (U_+ \setminus p_+) \times D) \cap C^2_c \to C^1_c$ is given by

$$\varphi(y, \tau) = \begin{cases} (x_-(y), \tau/x_-(y), \tau, \pi(y)) & y \in Y_+ \setminus p_+ \\ (\tau/x_+(y), x_+(y), \tau, \pi(y)) & y \in U_- \setminus p_-. \end{cases}$$

(Editor's note: This works and defines the sewing.)

6 April 26th, 2012

6.1 Andersen IV

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