Introduction to gauge theory

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Disclaimer

These are notes from a course given by Kim Frøyshov in Aarhus during the fall of 2011. They have been written and TeX'ed during the lecture and some parts have not been completely proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what happened with more ease, and to get experience with TeX'ing live. That being said, feel very free to send any comments and or corrections to fuglede@imf.au.dk.

The most recent version of these notes is available at http://home.imf.au.dk/pred.

1st lecture, August 29th 2011

1 Plan and introduction

The plan of the course is the go through the following

- Chern–Weil theory and its significance to gauge theory.
- Dirac operators and Bochner formulas.
- Seiberg–Witten theory.
- Sobolev spaces and elliptic operators.
- Basic theory of moduli spaces

We begin by discussing some examples of gauge theories. Let X be an oriented, compact Riemannian n-manifold, let G be a compact, connected (semi-simple) Lie group. Recall that a compact, connected Lie group is semi-simple if and only if its center is finite. Let $P \to X$ be a principal G-bundle. Given any representation of the Lie group, we can associate to P a vector bundle over X. In particular we can use the adjoint representation. More precisely, we put $\mathfrak{g}_P = P \times_G LG := (P \times LG)/G$, where LG denotes the Lie group of G, and G acts on $P \times LG$ by

$$g \cdot (u, \xi) = (ug^{-1}, \mathrm{Ad}_g(\xi)).$$

Let $E \to X$ be a real or complex vector bundle over X associated (in some way) to P. Let A be a connection in P, and let $\varphi \in \Gamma(E)$ be a section of E. We want to study certain equations for (A, φ). We introduce the moduli space M of solutions (A, φ) to these equations and take the quotient with the symmetry group $\mathcal{G}$ of the theory, which in the simplest case is simply the gruop of all automorphisms of P.

Of particular interest will be equations that are "elliptic modulo the action of $\mathcal{G}$". In those cases, the regular (consisting of points with surjective linearization) and irreducible part (consisting of points with trivial stabilizer) M will a finite-dimensional smooth manifold with dimension (called the expected dimension of the moduli space) the index of a certain elliptic operator over X. The first goal of the course is to understand this picture in more detail.

Examples of "elliptic" equations are the following:

1. Let $n = \dim X = 2$. One equation one could study is $F_A = 0$, where F_A is the curvature of A. In this case, the vector bundle is trivial, and the irreducible part of the moduli space has dimension

$$\dim M^* = \dim G\dot{(}2g - 2),$$

where g is the genus of the surface X.

2. Consider again $n = 2$. Another example is the *vortex equations*. Here, let $L \to X$ be an Hermitian line bundle and consider pairs (A, φ), with A a unitary connection in L, and $\varphi \in \Gamma(L)$. I.e. here P is the frame bundle of L, which is a $G = \mathrm{U}(1)$-bundle. The vortex equations are

$$\partial_A \varphi = 0,$$
$$F_A = \frac{i}{2} * (1 - |\varphi|^2).$$

In this case, M is either empty or it is $M = \mathrm{Sym}^r X$ for some r.

3. Consider now the case $n = \dim X = 3$. Here we have the *Bogomolny equations* $*F_A = d_A \varphi$, where $\varphi \in \Gamma(\mathfrak{g}_P)$. The expected dimension of the moduli space is 0 in this case.

4. Also, for $n = 3$, we have the 3-dimensional Seiberg–Witten equations.

5. Now, let $n = 4$. Similar to the first equation (which is only interesting in dimension 2, since it is overdetermined in higher dimensions), in 4 dimensions, we have the anti-self-dual equations $F_A^+ = 0$, where $^+$ denotes the self-dual part of a 2-form on a 4-manifold.

6. The 4-dimension Seiberg–Witten equations. In this case, let X be equipped with a spin^c-structure. Then X also comes with two spin bundles S^+, S^-, i.e. certain rank 2 vector bundles, and a line bundle $L = \det S^+$. Now, let A be a connection in L, $\varphi \in \Gamma(S^+)$. We have the so-called *Dirac operator*, which is an operator $D_A : \Gamma(S^\pm) \to \Gamma(S^\pm)$, and the *Seiberg–Witten equations* are $F_A = q(\varphi), D_A \varphi = 0$, where $q(\varphi)$ is a certain expression, quadratic in φ. Miraculously, it turns out that M is always compact.

2 Instanton theory

2.1 Connections and curvature

In this part, let X be any smooth manifold, and let $E \to X$ be a $\mathbb{F}$-vector bundle where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. We consider the space

$$\Omega^r(X; E) = \Gamma(\wedge^r T^* X \otimes E)$$

of r-forms on X with values in E. For any point $p \in X$, an element $\varphi \in \Omega^r(X; E)$ defines a $\mathbb{R}$-multilinear alternating map

$$\varphi_p : T_p X \times \cdots \times T_p X \to E_p.$$

Our notation for connections will be ∇, ∇', etc. In physics, they are often denoted A, B. To describe the corresponding covariant derivatives, one usually uses ∇_A, ∇_B, etc. We will switch between these notations.

Let A be a connection in E. The covariant derivative maps

$$\nabla_A : \Omega^0(X; E) \to \Omega^1(X; E).$$

There is a natural extension to a map

$$d_A : \Omega^r(X; E) \to \Omega^{r+1}(X; E), \quad r \geq 0,$$

satisfying

$$d_A(\alpha \wedge \varphi) = d\alpha \wedge \varphi = (-1)^r \alpha \wedge d_A \varphi$$

for any $\alpha \in \Omega^r(X)$, $\varphi \in \Omega^s(X, E)$. This property uniquely characterizes d_A. The curvature of A is denoted $F_A \in \Omega^2(X; \mathrm{End}(E))$, defined by the equation $d_A(d_A \varphi) = F_A \wedge \varphi$ for any $\varphi \in \Omega^r(X; E)$.

Proposition 2.1.1. *Let E, E' be $\mathbb{F}$-vector bundles with connections ∇, ∇' respectively. Then there is a unique connection ∇'' in $\mathrm{Hom}(E, E')$ such that*

$$\nabla'(us) = \nabla'' u \cdot s + u \nabla s \tag{1}$$

for all $s \in \Gamma(E), u \in \Gamma(\mathrm{Hom}(E, E'))$.

To prove this, one simply notes that the (1) determines ∇''.

Proposition 2.1.2. *If $E, E', E'' \to X$ are $\mathbb{F}$-vector bundles equipped with connections, then for all homomorphisms $E \xrightarrow{u} E' \xrightarrow{v} E''$, one has*

$$\nabla(vu) = \nabla v \cdot u + v \nabla u$$

Proof. Let $s \in \Gamma(E)$. Then by the previous proposition,

$$\begin{aligned} \nabla(vu) \cdot s + vu\nabla s &= \nabla(vus) \\ = \nabla v \cdot us + v\nabla(us) &= \nabla v \cdot us + \nabla u \cdot s + u \nabla s, \end{aligned}$$

which gives us what we wanted. □

Proposition 2.1.3. *Let $E, E', E'' \to X$ be $\mathbb{F}$-vector bundles with connections A, A', A''. Let $E \otimes E' \to E''$ be a homomorphism, denoted $s \otimes t \mapsto s \cdot t$, such that*

$$\nabla_{A''}(s \cdot t) = \nabla_A(s) \cdot t + s \nabla_{A'}(t)$$

for all $s \in \Gamma(E), t \in \Gamma(E')$. Then for all $\varphi \in \Omega^k(X; E), \psi \in \Omega^l(X; E')$, one has

$$d_{A''}(\varphi \wedge \varphi) = d_A(\varphi) \wedge \psi + (-1)^j \varphi \wedge d_{A'}(\psi)$$

where in $\varphi \wedge \psi$ etc., we use the map $E \otimes E' \to E''$.

Proof. The operator d_A is local in the sense that $d_A \varphi|_U$ depends only on $\varphi|_U$ for $U \subseteq X$ an open subset. Therefore, we can assume that $X = \mathbb{R}^n$. Write $\varphi = dx_I \otimes s$, where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\psi = dx_J \otimes t$, where $dx_j = dx_{j_1} \wedge \cdots \wedge d_{j_l}$. Then

$$\varphi \wedge \psi = (dx_I \wedge dx_J) \otimes (s \cdot t).$$

Applying $d_{A''}$, we obtain

$$\begin{aligned} d_{A''}(\varphi \wedge \psi) &= (-1)^{k+l} dx_I \wedge dx_J \otimes \nabla_{A''}(s \cdot t) \\ &= (-1)^{k+l} dx_I \wedge dx_J \otimes (\nabla_A s \cdot t + s \cdot \nabla_{A'} t) \\ &= (-1)^k (dx_I \wedge \nabla_A s) \wedge (dx_j \otimes t) + (-1)^{k+l} (dx_I \otimes s) \wedge (dx_j \otimes \nabla_{A'} t) \\ &= d_A(dx_I \otimes s) \wedge (dx_J \otimes t) + (-1)^k (dx_I \otimes s) \wedge d_A(dx_J \otimes t) \\ &= d_A(\varphi) \wedge \psi + (-1)^k \varphi \wedge d_A(\psi). \end{aligned}$$

Here, we used that $\nabla_A s = \sum_i \alpha_i \otimes s_i$ for 1-forms α_i and sections s_i, and we see that

$$\begin{aligned} dx_I \wedge dx_J \otimes (\alpha_i \otimes s_i) \cdot t &= dx_I \wedge dx_J \wedge \alpha_i \otimes s_i \cdot t \\ &= (-1)^l dx_I \wedge (\alpha_i \otimes s_i) \wedge dx_J \otimes t \end{aligned}$$

□

Theorem 2.1.4 (The Bianchi identity)**.** *If A is a connection in $E \to X$, then $d_A F_A = 0$.*

Note that there is a corresponding result for connections in principal bundles. In that case, the proof is actually more transparent than the following.

Proof. For all $s \in \Gamma(E)$, the previous proposition tells us that

$$\begin{aligned} F_A \wedge d_A s = d_A d_a(d_A s) &= d_A(d_A d_A s) = d_A(F_A s) \\ &= d_A F_A \cdot s + F_A \wedge d_A s, \end{aligned}$$

and it follows that $d_A F_A \cdot s = 0$ for all s. □

2.2 Chern–Weil theory

Example 2.2.1. If A is a connection in a complex line bundle $L \to X$. The corresponding endomorphism bundle $\mathrm{End}(L) = X \times \mathbb{C}$ is trivial, since L is a line bundle, and in this bundle A induces the product connection, because $\nabla_A \mathrm{Id} = 0$, and $F_A \in \Omega^2(X;\mathbb{C})$, and by the Bianchi identity, F_A is closed, and in turns out that the corresponding cohomology class

$$[F_A] \in H^2_{dR}(X, \mathbb{C})$$

does not depend on the connection A, and it is a multiple of the so-called Chern class, as we will discuss next time.

2nd lecture, August 31st 2011

Before we continue our discussion of Chern–Weil theory, we recall the concept of a pullback bundle and pullback connections.

Proposition 2.2.2. *Let ∇ be a connection in $E \to M$ and let $f : M' \to M$ be a smooth function. We can then pull back E to a bundle f^*E*

$$\begin{array}{ccc} F^*E & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ M' & \xrightarrow{f} & M \end{array}$$

where $f^*E = \{(x, e) \in M' \times E \mid f(x) = \pi(e)\}$. Then there is a unique connection $\nabla' = f^*\nabla$ in $E' = f^*E$ characterized by the property that for all $s \in \Gamma(E)$, $p \in M'$, $v \in T_pM'$, one has

$$\nabla'_v(f^*s) = \nabla_{f_*v}(s).$$

A few remarks are in order: Note that sections in the pullback bundle correspond to maps $t : M' \to E$ with $\pi \circ t = f$. So the pullback section f^*s corresponds simply to $s \circ f$. In the claim of the proposition we identify $E'_p = E_{f(p)}$. Note also that this is a generalization of the concept of covariant differentiation along curves from Riemannian geometry. We leave out the proof and instead prove the following.

Proposition 2.2.3. *The curvature of $f^*\nabla$ satisfies $F(f^*\nabla) = f^*F(\nabla)$.*

Proof. Let $\{s_j\}$ be a (local) basis of sections of $E|_U$, $U \subseteq M$ open. Then

$$d_\nabla s_j = \nabla s_j = \sum_i \omega_{ij} \otimes s_i,$$

for $\omega_{ij} \in \Omega^1(U)$. In terms of these 1-forms, the curvature of ∇ is

$$\begin{aligned} F(\nabla) \cdot s_j &= d_\nabla d_\nabla s_j \\ &= \sum_i \left(d\omega_{ij} \otimes s_i - \omega_{ij} \wedge \sum_k \omega_{ki} \otimes s_k \right) \\ &= \sum_i \left(d\omega_{ij} \otimes s_i + \sum_{ik} \omega_{ki} \wedge \omega_{ij} \otimes s_k \right) \\ &= \sum_i \left(d\omega_{ij} \otimes s_i + \sum_{ik} \omega_{ik} \wedge \omega_{kj} \otimes s_i \right) \\ &= \sum_i \left(d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} \right) \otimes s_i \\ &= \sum_i \Omega_{ij} \otimes s_i. \end{aligned}$$

Write $s'_j := f^*s_j \in \Gamma(E'|_{f^{-1}(U)})$ and $\nabla' = \sum_i \omega'_{ij} \otimes s'_i$. Then by the characterization of $f^*\nabla$, we have $\omega'_{ij} = f^*\omega_{ij}$, and so

$$F(\nabla') \cdot s'_j = \sum_i \Omega'_{ij} \otimes s_i,$$

where $\Omega'_{ij} = f^*\Omega_{ij}$ by the same calculation as above. Let $e_{ij} : E|_U \to E|_U$ be the map defined by $s_j \mapsto s_i$, $s_k \mapsto 0$. Then

$$F(\nabla) = \sum_{ij} \Omega_{ij} \otimes e_{ij}$$
$$F(\nabla') = \sum_{ij} \Omega'_{ij} \otimes e_{ij} = f^*F(\nabla).$$

□

2.2.1 Chern classes

Let $E \to M$ be a complex vector bundle of rank r with a connection ∇. Let $\{s_j\}$ be a local basis of sections and write as before $F(\nabla) \cdot s_j = \sum_i \Omega_{ij} \otimes s_i$. Changing the local basis $\{s_j\}$ means replacing Ω by $A\Omega A^{-1}$ for some $A : U \to \mathrm{GL}_r(U)$. We can use this to construct from Ω forms that are independent of the choice of local basis. Namely, consider

$$\det(I + t\Omega) = \sum_{k=0}^{r} \sigma_k(\Omega)t^k \in \Omega^{\text{even}}(U, \mathbb{C})[t],$$

where $\sigma_k(\Omega) \in \Omega^{2k}(U)$ is independent of the $\{s_j\}$. This implies that $\sigma_k(\Omega) \in \Omega^{2k}(M; \mathbb{C})$ is well-defined.

Lemma 2.2.4. *The forms $\sigma_k(\Omega)$ are closed.*

Proof. Observe that one can find a local basis of covariantly constant sections in a neighbourhood of any point in the manifold. Namely, given $x \in U$, we can find $\{s_i\}$ with $(\nabla s_i)_x = 0$. Then $(\nabla e_{ij})_x = 0$ as well because $e_{ij}s_j = s_i$ and by the Bianchi identity, we obtain

$$0 = d_\nabla F_\nabla = \sum_{ij} (d\Omega_{ij} \otimes e_{ij} + \Omega_{ij} \wedge \nabla e_{ij}),$$

which at x equals $\sum_{ij} d\Omega_{ij} \otimes E_{ij}$. Thus $d\Omega_{ij} = 0$ at x. Since $\sigma_k(\Omega)$ is a polynomial in Ω_{ij} we see that $d\sigma_k(\Omega) = 0$ at x. □

Lemma 2.2.5. *The cohomology class $[\sigma_k(\Sigma)] \in H^{2k}(M; \mathbb{C})$ is independent of ∇.*

Proof. Let ∇, ∇' be connections in E. Define a connection $\tilde{\nabla}$ in $\mathbb{R} \times E \to \mathbb{R} \times M = \{(t, x)\}$ by $\tilde{\nabla} = (1 - t)\pi_2^*\nabla + t\pi_2^*\nabla'$. Here we use that in general, for functions for functions f_i with $\sum f_i = 1$, we obtain a connection $\sum_i f_i\nabla_i$. Define $j_t : M \to \mathbb{R} \times M$ by $j_t(x) = (t, x)$, and in general $j^*(\sum_i f_i\nabla_i) = \sum_i j^*f_i j^*\nabla_i$. Pulling back $\tilde{\nabla}$ by j_t, and using that $\pi_2 \circ j_t = \mathrm{id}$, we thus obtain

$$(j_t)^*\tilde{\nabla} = (1 - t)\nabla + t\nabla'.$$

In particular $j_0^*\tilde{\nabla} = \nabla$ and $j_1^*\tilde{\nabla} = \nabla'$. We obtain that

$$[\sigma_k(\Omega)] = [j_0^*\sigma_k(\tilde{\Omega})] = [j_1^*\sigma_k(\tilde{\Omega})] = [\sigma_k(\Omega')],$$

where the second equality follows from the Poincaré Lemma which says that $\pi_2^* : H^*(M) \to H^*(M \times \mathbb{R})$ is an isomorphism, and the composition $H^*(M) \overset{\pi_2^*}{\to} H^(\mathbb{R} \times M) \overset{(j_t)^*}{\to} H^*(M)$ is the identity, and so $(j_t)^*$ is independent of t. □

Theorem 2.2.6. *The form* $(-2\pi i)^{-k}[\sigma_k(\Sigma)] \in H^{2k}(X;\mathbb{C})$ *is the image of the complex reduction of the Chern class* $c_k(E) \in H^{2k}(X;\mathbb{Z})$*.*

A proof of this theorem can be found in Milnor and Stasheff.

For today, we define $c_k(E) = (-2\pi i)^{-k}\sigma_k(\Omega)$.

Proposition 2.2.7 (Properties of Chern classes)**.** Naturality *with respect to pullbacks: If* $E \to M$ *is a complex vector bundle and* $f : M' \to M$ *is smooth, then* $f^*c_k(E) = c_k(f^*E)$*.*

Product formula*: If* $E, F \to M$ *are complex vector bundles, then*

$$c_k(E \oplus F) = \sum_{j=0}^{k} c_j(E) \cdot c_{k-j}(F).$$

Dual bundles*: For the dual bundle* E^* *to a vector bundle* E*, we have* $c_k(E^*) = (-1)^k c_k(E)$*.*

Proof. We already proved naturality. For the product formula, choose connections ∇, ∇' in E, F respectively. Define a connection $\tilde{\nabla}$ in $E \oplus F$ by

$$\tilde{\nabla}_v(s, s') = (\nabla_v s, \nabla'_v s').$$

Choose local bases of sections for E and F to obtain matrices Ω, Ω', and $\tilde{\Omega}$ as before. Then

$$\tilde{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega' \end{pmatrix},$$

and we obtain

$$\det(I + t\tilde{\Omega}) = \det(I + t\Omega) \cdot \det(I + t\Omega'),$$

and it follows that

$$\sum_k t^k \sigma_k(\tilde{\Omega}) = [\sum_i t^i \sigma_i(\Omega)][\sum_j t^j \sigma_j(\Omega')].$$

Multiplying this out, we see that

$$\sigma_k(\tilde{\Omega}) = \sum_{i+j} \sigma_i(\Omega)\sigma_j(\Omega'),$$

which immediately implies the product formula.

For the behaviour of dual bundles, remark that if ∇ is a connection in E, then the induced connection ∇^* in E^* is characterized by the formula

$$d(\varphi s) = \nabla^*\varphi \cdot s + \varphi \cdot \nabla s \tag{2}$$

for $s \in \Gamma(E)$, $\varphi \in \Gamma(E^*)$. Choose a local basis of sections $\{s_j\}$ of E with a dual local basis $\{\varphi_j\}$ of E^* such that $\varphi_j \cdot s_i = \delta_{ij}$. As before, consider $\nabla s_j = \sum_i \omega_{ij} \otimes s_i$ and $\nabla^*\varphi_j = \sum_i \omega^*_{ij} \otimes \varphi_i$. Differentiating $\varphi_j \cdot s_i$ and plugging in (2), we get

$$\begin{aligned} 0 = d(\varphi_i s_j) &= (\sum_k \omega^*_{ki} \otimes \varphi_k) \cdot s_j + \varphi_i \sum_k \omega_{kj} \otimes s_k \\ &= \omega^*_{ji} + \omega_{ij}. \end{aligned}$$

We obtain

$$\begin{aligned} \Omega^*_{ij} &= d\omega^*_{ij} + \sum_k \omega^*_{ik} \wedge \omega^*_{kj} \\ &= -d\omega_{ji} + \sum_k \omega_{ki} \wedge \omega_{jk} \\ &= -d\omega_{ji} - \sum_k \omega_{jk} \wedge \omega_{ki} = -\Omega_{ji}. \end{aligned}$$

Thus $\det(I + t\Omega^*) = \det(I + t\Omega^t) = \det(I - t\Omega)$. This implies that $\sigma_k(\Omega^*) = (-1)^k \sigma_k(\Omega)$. $\square$

3rd lecture, September 5th 2011

2.2.2 Pontryagin classes

Today we discuss Pontryagin classes. Let $E \to M$ be a real vector bundle with a connection ∇. Then we can play the same game as we previously did with complex line bundle. So let $s_1, \ldots, s_n$ be a local basis of sections of E and describe the curvature of ∇ in terms of a matrix of 2-forms,

$$F(\nabla) \cdot s_j = \sum_i \Omega_{ij} \otimes s_i.$$

Applying σ_k to this matrix, we obtain

$$\sigma_k(\Omega) \in \Omega^{2k}(M; \mathbb{R}).$$

Lemma 2.2.8. *If k is odd, $[\sigma_k(\Omega)] = 0$ in $H^{2k}(M; \mathbb{R})$.*

Proof. A choice of Euclidean metric on E yields an isomorphism $E \to E^*$. As in the proof of the last proposition, this gives $[\sigma_k(\Omega)] = (-1)^k[\sigma_k(\Omega)]$. □

Definition 2.2.9. For $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$ Let $p_k(E) = [(2\pi)^{-2k}\sigma_{2k}(\Omega)] \in H^{4k}(M; \mathbb{R})$, be the *k'th Pontryagin class with real coefficients.*

Proposition 2.2.10. *We have $p_k(E) = (-1)^k c_{2k}(E \otimes_{\mathbb{R}} \mathbb{C})$.*

Proof. If we pick a connection in E, the induced connection in the complexification has the same curvature matrix, and the only difference is in the normalization, but $(-i)^{2k} = (-1)^k$. □

Definition 2.2.11. If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{F}$, then the *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ given by

$$B(x, y) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y),$$

where here $\operatorname{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is the map $z \mapsto [x, z]$

Example 2.2.12. Let $\mathfrak{g} = \mathfrak{so}(n)$. A simple computation shows that

$$B(X, Y) = (n-2)\operatorname{tr}(XY) = -(n-2)\operatorname{tr}(XY^t),$$

so the Killing form in this case is negative definite. This is always the case for semi-simple Lie algebras.

We will take $(X, Y) \mapsto -\operatorname{tr}(XY)$ as an inner product on $\mathfrak{so}(n)$. This is invariant under the adjoint action of the group $\mathrm{SO}(n)$.

Proposition 2.2.13. *If A is an orthogonal connection in the Euclidean vector bundle $E \to X$. Then $\operatorname{tr}(F_A \wedge F_A)$ is closed and represents*

$$-8\pi^2 p_1(E) \in H^4(X; \mathbb{R})$$

In the expression $\operatorname{tr}(F_A \wedge F_A)$, we take the wedge product of the form part and the trace of the composition of endomorphisms.

Note that we thus have a description of $p_1(E)$ which is not defined in terms of a local basis. Note also that if A is a connection in any principal G-bundle $P \to X$, then $F_A \wedge_B F_A$ is closed, and $[F_A \wedge_B F_A] \in H^4(X; \mathbb{R})$ is independent of A. Here $\wedge_B$ denotes the combination of the wedge product with the Killing form.

Proof of Proposition. We will show the statement on the level of differential forms; namely, that

$$\mathrm{tr}(F_A \wedge F_A) = -2\sigma_2(\Omega),$$

where Ω as before is the matrix of 2-forms describing the curvature in terms of a local basis of sections. So let $\{s_j\}$ be local *orthonormal* basis of sections of E. As before we have the elementary endomorphisms e_{ij} mapping $s_j \mapsto s_i$, $s_k \mapsto 0$, and write $f_A = \sum_{ij} \Omega_{ij} \otimes e_{ij}$. Since A is orthogonal, $\Omega_{ji} = -\Omega_{ij}$. We find that

$$\begin{aligned}\mathrm{tr}(F_A \wedge F_A) &= \mathrm{tr}\left[\sum_{ik}\left(\sum_j \Omega_{ij}\Omega_{jk}\right)\otimes e_{ik}\right] \\ &= \sum_{ij}\Omega_{ij}\Omega_{ji} = -\sum_{ij}(\Omega_{ij})^2 \\ &= -2\sum_{i<j}(\Omega_{ij})^2 = -2\sigma_2(\Omega).\end{aligned}$$

Here, the last equality follows from the definition, $\det(I + t\Omega) = \sum_k t^k \sigma_k(\Omega)$. $\square$

2.3 Instantons

We specify now to the 4-dimensional case to see the relevance of the expression $\mathrm{tr}(F_A \wedge F_A)$ to instantons. For that we need the Hodge star operator.

Let V be an oriented n-dimensional Euclidean vector space (in practice the tangent space of a 4-manifold). Then for $k \in \mathbb{N}_0$, the *Hodge star operator* is the map

$$* : \wedge^k V \to \wedge^{n-k} V$$

characterized by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \omega$$

for any $\alpha, \beta \in \wedge^k V$, where $\omega \in \wedge^n V$ is the volume element. More explicitly, if $e_1, \ldots, e_n$ is a positive orthonormal basis of V, then

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_n$$

In the case $n = 4$, the map $* : \wedge^2 V \to \wedge^2 V$ satisfies $*^2 = 1$, so $\wedge^2 V$ splits into the $+1$- and -1-eigenspaces of $*$, $\wedge^2 V = \wedge^+ \oplus \wedge^-$. Explicitly, a basis for $\wedge^+$ is given by

$$e_1 \wedge e_2 + e_3 \wedge e_4,\ e_1 \wedge e_3 - e_2 \wedge e_4,\ e_1 \wedge e_4 + e_2 \wedge e_3,$$

Note that for $\alpha \in \wedge^+$, $\beta \in \wedge^-$, we have

$$\alpha \wedge \alpha = |\alpha|^2\omega,\ \beta \wedge \beta = -|\beta|^2\omega,\ \alpha \wedge \beta = 0.$$

The last equality implies that $\wedge^+ \perp \wedge^-$.

We now turn to instantons. Let X be an oriented Riemannian 4-manifold. Using the above, we obtain decompositions

$$\wedge^2 T^*X = \wedge^+ \oplus \wedge^-,\ \Omega^2(X) = \Omega^+ \oplus \Omega^-.$$

Let $E \to X$ be an Euclidean vector bundle, and let A be an orthogonal connection in E. Then we have a decomposition

$$\Omega^2(X; \mathfrak{so}(E)) = \Omega^+(X; \mathfrak{so}(E)) \oplus \Omega^-(X; \mathfrak{so}(E)),$$

and so the curvature $F_A \in \Omega^2(X; \mathfrak{so}(E))$ decomposes into its selfdual and anti-selfdual parts, $F_A = F_A^+ + F_A^-$.

Definition 2.3.1. A connection A is called an *instanton* or an *anti-selfdual connection* if $F_A^+ = 0$.

Of course, the same definition applies to other gauge groups, and often one instead uses $G = \mathrm{U}(2)$.

It turns out that the instantons minimize a certain functional, called the *energy*, on the space of connections:

Proposition 2.3.2. *If X is closed and A is any orthogonal connection in $E \to X$, then*

$$-8\pi^2 \int_X p_1(E) \leq \int_X |F_A|^2$$

with equality if and only if A is an instanton.

Proof. We find that

$$\begin{aligned} -8\pi^2 \int_X p_1(E) &= \int_X \mathrm{tr}(F_A \wedge F_A) = \int_X |F_A^-|^2 - |F_A^+|^2 \\ &\leq \int_X |F_A^-|^2 + |F_A^+|^2 = \int_X |F_A|^2. \end{aligned}$$

□

Note that in particular, the L^2-norm is the same of all instantons and that we have a bound bound on this norm. This is the starting point of the whole discussion of compactness of moduli spaces.

Our next goal is to describe instantons over $\mathbb{R} \times Y^3$. To do this, we need to say a bit more about the curvature.

Proposition 2.3.3. *If A is a connection in the (real or complex) vector bundle $E \to X$, and suppose $a \in \Omega^1(X; \mathrm{End}(E))$. Then $A + a$ is again a connection and has curvature*

$$F(A + a) = F(A) + d_A a + a \wedge a.$$

Note that in the term $a \wedge a$ we could both mean taking composition of endomorphisms or taking their Lie bracket. We do the former, even though the latter is the more general. Had we instead used the Lie bracket, we should have multiplied the term $a \wedge a$ by $\frac{1}{2}$.

Proof of Proposition. Write $A' = A + a$ and let $s \in \Gamma(E)$. Then by definition,

$$\begin{aligned} F(A')s &= (d_A + a)(d_A + a)s = d_A d_A s + d_A(as) + a \wedge d_A s + a \wedge (as) \\ &= d_A d_A + (d_A a) \wedge s - a \wedge d_A s + a \wedge d_A s + a \wedge (as) \\ &= (F_A + d_a a + a \wedge a) \cdot s. \end{aligned}$$

□

Definition 2.3.4. Let $E_1, E_2 \to X$ be two vector bundles over X with an isomorphism $u : E_1 \to E_2$. If A is a connection in E_1, we define a connection $u(A)$ in E_2 by

$$\nabla_{u(A)}(s) := u \cdot \nabla_A(u^{-1}s).$$

for $s \in \Gamma(E_2)$.

Proposition 2.3.5. *The connection $u(A)$ satisfies*

$$u(A) = A - (\nabla_A u) \cdot u^{-1}.$$

Proof. Observe that $0 = \nabla_A(uu^{-1}) = \nabla_A u \cdot u^{-1} + u \cdot \nabla_A(u^{-1})$. It follows that

$$\begin{aligned}\nabla_{u(A)}(s) &= u \cdot \left(\nabla_A(u^{-1}) \cdot s + u^{-1}\nabla_A s\right) \\ &= \left(-\nabla_A(u) \cdot u^{-1} + \nabla_A\right) s.\end{aligned}$$

□

Let Y be any manifold and $E \to Y$ a vector bundle over Y. Let A be a connection in the pullback bundle $\mathbb{R} \times E \to \mathbb{R} \times Y = \{(t, y)\}$ obtained by pulling back a connection in Y. Then $\nabla^A_{\partial_t}$ is linear over functions, so

$$\nabla^A_{\partial_t} = \partial_t + \varphi$$

for some $\varphi \in \Gamma(\mathrm{End}(\mathbb{R} \times E))$. Here, $\partial_t = \frac{\partial}{\partial_t}$.

Definition 2.3.6. If $\varphi = 0$, then we say that A is in *temporal gauge.*

Proposition 2.3.7. *For any connection A in $\mathbb{R} \times E \to \mathbb{R} \times Y$, there is a bundle automorphism $u : \mathbb{R} \times E \to \mathbb{R} \times E$ such that $u(A)$ is in temporal gauge.*

Proof. Let $\nabla^A_{\partial_t} = \partial_t + \varphi$. We find that

$$\begin{aligned}\nabla^{u(A)}_{\partial_t} &= \nabla^A_{\partial_t} - \nabla^A_{\partial_t} u \cdot u^{-1} \\ &= (\partial_t + \varphi) - (\partial_t + \varphi)u \cdot u^{-1} \\ &= \partial_t + \varphi - (\partial_t u + \varphi u - u\varphi)u^{-1} \\ &= \partial_t - (\partial_t u - u\varphi) \cdot u^{-1}.\end{aligned}$$

noting in the second equality that φ acts on u through the commutator. Thus we want to find u such that $\partial_t u = u\varphi$. Fixing a point in the manifold Y, this is just a first order linear differential equation which has a unique solution to a given initial value, say $u(0, \cdot) = I$. □

Note that one can do this proof more conceptually using principal bundles and the holonomy of connections.

Proposition 2.3.8. *Let A be a connection in $\mathbb{R} \times E \to \mathbb{R} \times Y$ in temporal gauge. Then its curvature is given by*

$$F(A) = dt \wedge \frac{\partial A_t}{\partial t} + F(A_t),$$

where here $A_t = (j_t)^* A$, where $j_t : Y \times \mathbb{R} \times Y$ maps $y \mapsto (t, y)$, and

$$\frac{\partial A_t}{\partial t} = \lim_{s \to 0} \frac{1}{s}(A_{t+s} - A_t),$$

which makes sense since $A_{t+s} - A_t \in \Omega^1(Y, \mathrm{End}(E))$ can be differentiated in the naive way.

4th lecture, September 7th 2011

Recall from last lecture the following proposition.

Proposition 2.3.9. *Let A be a connection in $\mathbb{R} \times E \to \mathbb{R} \times Y$ in temporal gauge. Then*

$$F(A) = dt \wedge \frac{\partial A_t}{\partial t} + F(A_t)$$

where $A_t := (j_t)^* A$, $j_t : Y \to \mathbb{R} \times Y$, $j_t(y) = (t, y)$.

Proof. Choose local coordinates $\{y^j\}$ on a small open set $U \subseteq Y$ with a trivialization of $E|_U$. On $\mathbb{R} \times U$, $A = d + \sum_j dy^j \otimes a_j =: d + a$, where d is the product connection. Since both A and d is in temporal gauge, $a_j \in \Gamma(\mathrm{End}(\mathbb{R} \times E|_U))$. Let d_Y be the product connection in $E|_U$. Then

$$a(t) = (j_t)^*(A - d) = A_t - d_Y,$$

and since d is flat, this equation tells us that

$$\begin{aligned} F(A) &= da + a \wedge a \wedge a \\ &= -\sum_j dy^j \wedge \left(dt \wedge \frac{\partial a_j}{\partial t} + d_Y a_j(t) \right) + a \wedge a \\ &= dt \wedge \frac{\partial A_t}{\partial t} + d_y a(t) + a \wedge a = dt \wedge \frac{\partial A_t}{\partial t} + F(A_t). \end{aligned}$$

□

We are now in a position to describe instantons on cylinders.

Proposition 2.3.10. *Let Y be an oriented Riemannian 3-manifold, and let A be any connection in the pullback bundle $\mathbb{R} \times E \to \mathbb{R} \times Y$ with $\nabla^A_{\partial_t} = \partial_t + \varphi$, for some $\varphi \in \Gamma(\mathrm{End}(\mathbb{R} \times E))$. Then $F_A^+ = 0$ if and only if*

$$\frac{\partial A_t}{\partial t} = - *_Y F(A_t) + \nabla_{A_t}(\varphi(t)),$$

*where $*_Y$ is the Hodge *-operator on Y.*

Before proving this, we note two corollaries:

Corollary 2.3.11. *If A is in temporal gauge (i.e. $\varphi = 0$), then $F_A^+ = 0$ if and only if the flow equation $\frac{\partial A_t}{\partial t} = - *_Y F(A_t)$ is satisfied.*

Corollary 2.3.12. *If A is translationary invariant, i.e. $A_t \equiv B$ for suitable B and all t, and similarly $\varphi(t) = \psi$ for some ψ for all t, then $F_A^+ = 0$ if and only if $*_Y F(B) = \nabla_B(\psi)$. This is the Bogomolny equation.*

Proof of Proposition 2.3.10. Define the connection $A' = A - dt \otimes \varphi$ which is in temporal gauge and on the t slice is $(A')_t = A_t$. Then by Proposition 2.3.9, we have

$$F(A) = F(A') + d_{A'}(dt \otimes \varphi) = dt \frac{\partial A_t}{\partial t} + F(A_t) - dt \wedge \nabla_{A_t}(\varphi(t)). \tag{3}$$

In general, if $\omega \in \Omega^2(\mathbb{R} \times Y)$, then $\omega = dt \wedge \alpha + \beta$, where $\partial_t \lrcorner \alpha = 0$, $\partial_t \lrcorner \beta = 0$. Then

$$*\omega = *_Y \alpha + dt \wedge *_Y \beta.$$

This means that $\omega^+ = 0$ if and only if $*\omega = -\omega$ which is the same as saying that $\alpha = - *_Y \beta$ and $\beta = - *_Y \alpha$. On a 3-manifold, $*^2 = 1$, these two equations are equivalent. Applying this to (3) we find that $F_A^+ = 0$ if and only if

$$\frac{\partial A_t}{\partial t} - \nabla_{A_t}(\varphi(t)) = - *_Y F(A_t).$$

□

3 Spin structures and spin^c structures

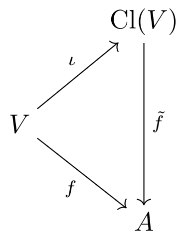
Instead of developing the instanton theory further, we postpone the harder parts and instead introduce the Seiberg–Witten equations which are in some sense easier to deal with. For example because the moduli spaces of solutions turn out to always be compact.

3.1 Clifford algebras and spinor groups

The reference for this section is [LM89]. Let V be a finite dimensional Euclidean vector space, i.e. a real inner product space. A *Clifford algebra* for V is a real associative algebra $\mathrm{Cl}(V)$ with a unit 1 together with a linear map $\iota : V \to \mathrm{Cl}(V)$ such that the following hold:

1. For all $v \in V$, we have $\iota(v)^2 = -|v|^2 \cdot 1$.

2. The Clifford algebra is universal with respect to this property: That is, if A is any other real associative algebra with a unit, and $f : V \to A$ a linear map such that for all $v \in V$, $f(v)^2 = -|v|^2 \cdot 1$, then f factors uniquely through the Clifford algebra, i.e. there is a unique algebra homomorphism $\tilde{f} : \mathrm{Cl}(V) \to A$ such that the following diagram commutes:



These properties tell us that $(\mathrm{Cl}(V), \iota)$ is unique up to isomorphism if it exists. We construct it as follows: Let T be the tensor algebra of V

$$T := \bigoplus_{r \geq 0} (\otimes^r V) \,,$$

where $\otimes^r V = V \otimes \cdots \otimes V$ (r times). Let J be the two-sided ideal in T generated by all elements $v \otimes v + |v|^2 \cdot 1$, for $v \in V$. We put

$$\mathrm{Cl}(V) := T/J,$$

and define ι to be the composition $V = \otimes^1 V \subset T \to \mathrm{Cl}(V)$.

Since ι is injective, we identify $v \in V$ with $\iota(v)$ in $\mathrm{Cl}(V)$, so one has $v^2 = -|v|^2$. If $v, w \in V$ then

$$0 = (v+w)^2 + |v+w|^2 = vw + wv + 2\langle v, w\rangle,$$

so in particular v and w anti-commute if they are perpendicular. Let $\wedge^* V = \bigoplus_r \lambda^r V$ be the exterior algebra of the vector space V.

Proposition 3.1.1. *There is a canonical vector space isomorphism*

$$h : \wedge^* V \overset{\cong}{\to} \mathrm{Cl}(V).$$

Proof. Define a map $V \times \cdots \times V \to \mathrm{Cl}(V)$ by

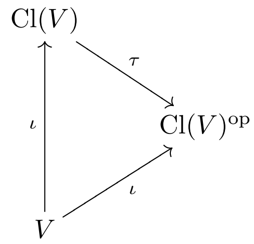
$$(v_1, \ldots, v_r) \mapsto \frac{1}{r!} \sum_\sigma \mathrm{sign}(\sigma) v_{\sigma(1)} \cdots v_{\sigma(r)}.$$

This map is alternating and therefore induces a linear map $\wedge^r V \to \mathrm{Cl}(V)$. The sum of these maps for all r gives the map $h : \wedge^* V \to \mathrm{Cl}(V)$. If $e_1, \ldots, e_n$ is an orthonormal basis for V, then $e_i e_j = -e_j e_i$ for $i \neq j$, so

$$h(e_{i_1} \wedge \cdots \wedge e_{i_r}) = e_{i_1} \cdots e_{i_r}$$

for $1 \leq i_1 < \cdots < i_r \leq n$, which implies that h is surjective. To prove that it is injective, one has to go through some algebraic technicalities, and we refer to [LM89]. □

There are two natural endomorphisms of Clifford algebras. Consider the map $V \xrightarrow{-1} V$, $v \mapsto -v$ and note that $(-v)^2 = -|-v|^2$. This gives rise to an algebra homomorphism $\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ satisfying $\alpha^2(v) = v$. Since elements of V generate $\mathrm{Cl}(V)$, $\alpha^2 = \mathrm{id}$ on $\mathrm{Cl}(V)$. This gives rise to an eigenspace decomposition $\mathrm{Cl}(V) = \mathrm{Cl}_0 \oplus \mathrm{Cl}_1$, with $\alpha|_{\mathrm{Cl}_0} = \mathrm{id}$, $\alpha|_{\mathrm{Cl}_1} = -\mathrm{id}$. Now Cl_0 is a subalgebra of $\mathrm{Cl}(V)$. On the other hand, the opposite algebra $\mathrm{Cl}(V)^{\mathrm{op}}$, which agrees with $\mathrm{Cl}(V)$ as a vector space and has the product $x \cdot_{\mathrm{op}} y = y \cdot x$. We still have an inclusion $\iota : V \to \mathrm{Cl}(V)^{\mathrm{op}}$ which factors through $\mathrm{Cl}(V)$ to give a map $\tau : \mathrm{Cl}(V) \to \mathrm{Cl}(V)^{\mathrm{op}}$,



Here $\tau|_V = \mathrm{id}$, $\tau(v_1 \dots v_r) = v_r \tau(v_1 \cdots v_{r-1}) = v_r \cdots v_1$, and $\alpha\tau = \tau\alpha$.

Our next goal is to construct the Spin groups. Recall that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}/r$ for $n \geq 3$. Let $\mathrm{Spin}(n)$ be the universal cover of $\mathrm{SO}(n)$,

$$1 \to \{\pm 1\} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

We will show that $\mathrm{Spin}(n)$ in fact sits in the Clifford algebra which allows us to construct certain representations of $\mathrm{Spin}(n)$ that do not factor through $\mathrm{SO}(n)$. The idea is that the Clifford algebras are in fact isomorphic to matrix groups which we know how to represent, and from that we obtain representations of $\mathrm{Spin}(n)$.

Let $N : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ be the norm map $x \mapsto x \cdot \alpha\tau(x)$. Pick an orthonormal basis $\{e_i\}_i$ of V. If an element $x \in \mathrm{Cl}(V)$ is given by

$$x = \sum_I x_I e_I,$$

where, for $I = (i_1, \dots, i_r)$, we let $e_I = e_{i_1} \cdots e_{i_r}$, then

$$N(x) = \sum_I (x_I)^2 + \sum_{I \neq \emptyset} a_I e_I.$$

Let $\mathrm{Cl}(V)^*$ be the group of invertible elements of $\mathrm{Cl}(V)$, and define

$$G := \{x \in \mathrm{Cl}(V) \mid N(x) = 1\}.$$

Lemma 3.1.2. *The group G is a compact Lie group, and a subgroup of* $\mathrm{Cl}(V)^*$.

Proof. If $x, y \in \mathrm{Cl}(V)$ with product $xy = 1$, then $yx = 1$ since left multiplication by x is a surjective linear endomorphism $l_x : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ which is injective since V is finite-dimensional, and since $l_x \circ l_y = \mathrm{id}$, we obtain $l_y \circ l_x = \mathrm{id}$. Hence $G \subseteq \mathrm{Cl}(V)^*$.

Define $\sigma = \alpha\tau$. If $N(x) = 1 = N(y)$ for elements x and y, then

$$N(xy) = xy\sigma(xy) = xy\sigma(y)\sigma(x) = x\sigma(x) = 1,$$
$$N(x^{-1}) = x^{-1}\sigma(x^{-1}) = x^{-1}\sigma(x)^{-1} = (\sigma(x)x)^{-1} = 1,$$

which proves that $G \subseteq \mathrm{Cl}(V)^*$ is a subgroup. Furthermore, G is closed in $\mathrm{Cl}(V)$ and hence in $\mathrm{Cl}(V)^*$, and G is a submanifold of $\mathrm{Cl}(V)^*$ and hence a Lie group. Also $G \subseteq \mathrm{Cl}(V)$ is bounded and hence compact. □

Define the *adjoint representation* $\mathrm{Ad} : \mathrm{Cl}(V)^* \to \mathrm{Aut}(\mathrm{Cl}(V))$ by

$$\mathrm{Ad}_x(y) = xyx^{-1}.$$

For an element $w \in V \setminus \{0\}$, $\mathrm{Ad}_w(V) = V$ since $\mathrm{Ad}_w|_V$ is the reflection in $\mathbb{R}w$, which is orientation-preserving if and only if $n = \dim V$ is odd. We correct this by defining $\widetilde{\mathrm{Ad}} : \mathrm{Cl}(V)^* \to \mathrm{Aut}(\mathrm{Cl}(V))$ by $\widetilde{\mathrm{Ad}}_x(y) = \alpha(x)yx^{-1}$. Then $\widetilde{\mathrm{Ad}}_w|_V = -\mathrm{Ad}_w|_V$ is reflection in $w^\perp$ which is always orientation-reversing.

Definition 3.1.3. Define the compact groups $\mathrm{Pin}(V)$ and $\mathrm{Spin}(V)$ by

$$\begin{aligned}\mathrm{Pin}(V) &:= \{x \in G \mid \widetilde{\mathrm{Ad}}_x(V) = V\},\\ \mathrm{Spin}(V) &:= \mathrm{Pin}(V) \cap \mathrm{Cl}_0(V).\end{aligned}$$

5th lecture, September 12th 2011

We continue our discussion of Clifford algebras and spin groups. Recall that for a finite dimensional Euclidean vector space V, we let

$$\mathrm{Pin}(V) := \{x \in \mathrm{Cl}(V) \mid N(x) = 1, \widetilde{\mathrm{Ad}}_x(V) = V\}$$

where $\widetilde{\mathrm{Ad}}_x(y) = \alpha(x)yx^{-1}$, and we let Spin be the even part of Pin,

$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}_0(V).$$

These are compact Lie groups.

Theorem 3.1.4. *We have a short exact sequence, explaining the importance of* $\mathrm{Pin}(V)$*:*

$$1 \to \{\pm 1\} \to \mathrm{Pin}(V) \stackrel{\widetilde{\mathrm{Ad}}}{\to} \mathrm{O}(V) \to 1.$$

Here, $\mathrm{O}(V)$ *denotes the orthogonal transformations of* V*.*

Proof. First off, we note that $\widetilde{\mathrm{Ad}}$ is in fact a homomorphism, since α is an algebra homomorphism, so

$$\widetilde{\mathrm{Ad}}_{xy}(z) = \alpha(x)\alpha(y)zy^{-1}x^{-1} = \widetilde{\mathrm{Ad}}_x \circ \widetilde{\mathrm{Ad}}_y(z).$$

Furthermore, $\widetilde{\mathrm{Ad}}$ has image in $\mathrm{O}(V)$ since for $x \in \mathrm{Pin}(V)$ and $v \in V$, we have

$$\begin{aligned}|\widetilde{\mathrm{Ad}}_x(v)|^2 &= N(\alpha(x)vx^{-1}) = \alpha(x)vx^{-1} \cdot \tau\alpha(\alpha(x)vx^{-1})\\ &= \alpha(x)vx^{-1} \cdot \tau\alpha(x^{-1})(-v)\tau(x) = \alpha(x)vN(x^{-1})(-v)\tau(x)\\ &= |v|^2\alpha(x)\tau(x) = |v|^2\alpha(x \cdot \alpha\tau(x)) = |v|^2\alpha(N(x)) = |v|^2.\end{aligned}$$

The next step is to show that $\widetilde{\mathrm{Ad}}$ is surjective, which follows from the fact every element of $\mathrm{O}(V)$ is the product of (at most $n := \dim V$) reflections in hyperplanes. Any such reflection is in the image of $\widetilde{\mathrm{Ad}}$ since $\widetilde{\mathrm{Ad}}_v$ is the reflection in $v^\perp$.

It is clear that $\{\pm 1\}$ is in $\ker \widetilde{\mathrm{Ad}}$, and in fact nothing else is: Let $x \in \mathrm{Pin}(V)$ with $\widetilde{\mathrm{Ad}}_x = I$. This is the same as saying that for every $v \in V$, we have $vxv^{-1} = \alpha(x)$. Choose an orthonormal basis $\{e_i\}$ of V, and set as before $e_I := e_{i_1} \cdots e_{i_r}$ for $I = (i_1, \dots, i_r)$, for $i_1 < i_2 < \cdots < i_r$. Recall that the e_I form a basis of the Clifford algebra as a vector space, so $x = \sum_I x_I e_I$. If $j \in I$, then

$$e_j e_I e_j^{-1} = (-1)^{|I|-1} e_I = -\alpha(e_I),$$

and hence the only non-zero x_I comes from $I = \emptyset$, so $x \in \mathbb{R} \cdot 1$, and since $N(x) = 1$, we find $x = \pm 1$. □

This theorem allows us to give a much more explicit description of the groups Pin and Spin.

Proposition 3.1.5. *We have*

$$\mathrm{Pin}(V) = \{v_1 \cdots v_r \mid r \geq 0, v_j \in V, |v_j| = 1\},$$
$$\mathrm{Spin}(V) = \{v_1 \cdots v_{2r} \mid r \geq 0, v_j \in V, |v_j| = 1\}.$$

Proof. For any element $x \in \mathrm{Pin}(V)$, then as before, $\widetilde{\mathrm{Ad}}_x$ is a product of reflections in hyperplanes, so

$$\widetilde{\mathrm{Ad}}_x = \widetilde{\mathrm{Ad}}_{v_1} \circ \cdots \circ \widetilde{\mathrm{Ad}}_{v_r}$$

for some $v_j \in V$, $|v_j| = 1$. Since $\ker(\widetilde{\mathrm{Ad}}) = \{\pm 1\}$, we have $x = (\pm v_1) v_2 \cdots v_r$. □

Corollary 3.1.6. *We have a short exact sequence*

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1.$$

Proof. By Proposition 3.1.5, $\mathrm{Spin}(V) = \left(\widetilde{\mathrm{Ad}}\right)^{-1} \mathrm{SO}(V)$. □

Proposition 3.1.7. *We have* $\mathrm{Spin}(V) \cong \mathrm{U}(1)$ *for* $n = 2$*, and* $\mathrm{Spin}(V)$ *is simply-connected for* $n \geq 3$.

Proof. Since the kernel of the Lie group homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ is discrete, $\widetilde{\mathrm{Ad}}|_{\mathrm{Spin}(V)}$ is a covering projection. Since $\mathrm{SO}(2) \cong \mathrm{U}(1)$, and since $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}/2$ for $n \geq 3$, it suffices to prove that $\mathrm{Spin}(V)$ is always connected or, equivalently since $\mathrm{SO}(V)$ is connected, that we can find a path from 1 to -1 in $\mathrm{Spin}(V)$. More precisely, given $x, y \in \mathrm{Spin}(V)$, we can find a path from x to either y or $-y$ as a lift of a path from $\widetilde{\mathrm{Ad}}_x$ to $\widetilde{\mathrm{Ad}}_y$ in $\mathrm{SO}(V)$, and by multiplication by y a path from 1 to -1 in $\mathrm{Spin}(V)$ gives a path from y to $-y$.

To find the path from -1 to 1, choose $v, w \in V$ with $|v| = |w| = 1$ and $v \perp w$. Define $\gamma : [-\frac{\pi}{4}, \frac{\pi}{4}] \to \mathrm{Spin}(V)$ by

$$\begin{aligned}\gamma(t) &= (\cos t \cdot v + \sin t \cdot w)(\sin t \cdot v + \cos t \cdot w) \\ &= -2\cos t \cdot \sin t + (\cos^2 t - \sin^2 t) \cdot vw = -\sin(2t) + \cos(2t) \cdot vw,\end{aligned}$$

so $\gamma(\frac{\pi}{4}) = -1, \gamma(-\frac{\pi}{4}) = 1$. □

Note that $\mathrm{O}(V)$ has two connected components, and hence $\mathrm{Pin}(V)$ has two connected components $\mathrm{Pin}(V)^{\pm}$ where $\mathrm{Pin}(V)^+ = \mathrm{Spin}(V)$.

3.2 Representations of spinor groups

Definition 3.2.1. Given an orientation of V, the *volume element* in $\mathrm{Cl}(V)$ is defined by $\omega := e_1 \cdots e_n$, where $e_1, \ldots, e_n \in V$ is any positive orthonormal basis.

The easiest way to see that the above definition is independent of the choice of basis is that the same statement is true for the exterior algebra: Namely, consider the map $\wedge^n V \to \mathrm{Cl}(V)$ mapping $e_1 \wedge \cdots \wedge e_n \mapsto e_1 \cdots e_n$. Now $\mathrm{GL}(V)$ acts on $\wedge^n V$, and for $A \in \mathrm{GL}(V)$, $A(e_1 \wedge \cdots \wedge e_n) = \det(A) e_1 \wedge \cdots \wedge e_n$.

The square of the volume element is

$$\begin{aligned}\omega^2 &= e_1 \cdots e_n e_1 \cdots e_n = (-1)^{n-1} e_1^2 e_2 \cdots e_n e_2 \cdots e_n \\ &= (-1)^{1+\cdots+n-1} e_1^2 e_2^2 \cdots e_n^2 = (-1)^{1+\cdots+n} = (-1)^{n(n+1)/2}.\end{aligned}$$

If $\omega^2 = 1$, then the Clifford algebra splits as $\mathrm{Cl}(V) = \mathrm{Cl}^+ \oplus \mathrm{Cl}^-$, where $\mathrm{Cl}^{\pm} = (1 \pm \omega)\,\mathrm{Cl}(V)$ are the (± 1)-eigenspace of $\mathrm{Cl}(V) \to \mathrm{Cl}(V)$, $x \mapsto \omega x$.

Note that $\omega \in Z(\mathrm{Cl}(V))$ if and only if $e_i \omega = \omega e_i$ for all i, which happens if and only if n is odd.

Hence, for $n \equiv 3 \bmod 4$, $\omega^2 = 1$, and ω is central, so $\mathrm{Cl}^{\pm} \subseteq \mathrm{Cl}(V)$ is a two-sided ideal.

Definition 3.2.2. Let $\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ be the *complexified Clifford algebra*. The *complex volume element* in $\mathbb{C}\mathrm{l}(V)$ is defined by

$$\omega_{\mathbb{C}} := i^{\lfloor \frac{n+1}{2} \rfloor} \omega.$$

One finds that $(\omega_{\mathbb{C}})^2 = 1$ for all n, so we always have decomposition $\mathbb{C}\mathrm{l}(V) = \mathbb{C}\mathrm{l}^+ \oplus \mathbb{C}\mathrm{l}^-$, where $\mathbb{C}\mathrm{l}^{\pm} = (1 \pm \omega_{\mathbb{C}})\, \mathbb{C}\mathrm{l}(V)$. If n is odd, then $\mathbb{C}\mathrm{l}^{\pm} \subseteq \mathbb{C}\mathrm{l}(V)$ is an ideal.

In fact, the Clifford algebras can be classified, and all of them are matrix algebras or a sum of two such.

Definition 3.2.3. Write $\mathrm{Cl}(n) := \mathrm{Cl}(\mathbb{R}^n)$, where $\mathbb{R}^n$ has the Euclidean metric, and write $\mathbb{C}\mathrm{l}(n) := \mathbb{C}\mathrm{l}(\mathbb{R}^n)$.

The classification for low n is shown in Table 1.

n	1	2	3	4	5	6	7	8
$\mathrm{Cl}(n)$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}\mathrm{l}(n)$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

Table 1: The Clifford algebras $\mathrm{Cl}(n)$ and $\mathbb{C}\mathrm{l}(n)$ for low values of n.

In the table, $\mathbb{F}(k) = M_{k \times k}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. One notices that the complex Clifford algebras have a 2-periodicity, and similarly the real Clifford algebras express a periodocity with period 8. A proof of this and the above can be found in [LM89]. Note that we see explicitly the splittings into ideals in the these cases.

Proposition 3.2.4. *We have* $\mathrm{Cl}(n) \cong \mathrm{Cl}_0(n+1)$.

Proof. Consider the map $f : \mathbb{R}^n \to \mathrm{Cl}_0(n+1)$ mapping $e_j \mapsto e_j e_{n+1}$. Then for $1 \leq i, j \leq n$, $e_i e_{n+1} e_j e_{n+1} = e_i e_j$, which implies that $f(v)^2 = -|v|^2$, so f induces an algebra homomorphism $\hat{f} : \mathrm{Cl}(n) \to \mathrm{Cl}_0(n+1)$. This homomorphism is in fact surjective, since the image contains all expressions $e_i e_j$ by the relation above, and the even part of $\mathrm{Cl}(n+1)$ is generated by such products as well as 1. Finally, the map is an isomorphism since $\dim \mathrm{Cl}(n) = \frac{1}{2} \dim \mathrm{Cl}(n+1) = \dim \mathrm{Cl}_0(n+1)$. □

Since we now know that Clifford algebras have the structures of matrix groups, we also know their representations.

Definition 3.2.5. A module M over a ring R is called *irreducible* if M has no submodules other than 0 and M.

Proposition 3.2.6. *Let S be a finite dimensional complex vector space and $\mathbb{C}\mathrm{l}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ a representation (i.e. a complex linear algebra homomorphism, and we can view S as a $\mathbb{C}\mathrm{l}(n)$-module). Then S is a direct sum of irreducible submodules $S = \bigoplus_{i=1}^r S_i$.*

Proof. For finite groups, this result is completely standard, and we will use that $\mathbb{C}\mathrm{l}(n)$ contains a certain nice finite subgroup. Set

$$\Gamma = \{\pm e_{i_1} \cdots e_{i_k} \in \mathbb{C}\mathrm{l}(n) \mid 1 \leq i_1 < \cdots < i_k \leq n,\, k \geq 0\},$$

where $e_1, \ldots, e_n \in \mathbb{R}^n$ is the standard basis. Then Γ is a subgroup of $\mathbb{C}\mathrm{l}(n)^*$ of order $|\Gamma| = 2^{n+1}$. We want to choose a certain inner product on S, which is invariant under Γ as follows: Let $\langle\, , \rangle'$ be any inner product on S and define an inner product $\langle\, , \rangle$ on S by

$$\langle x, y \rangle := \sum_{g \in \Gamma} \langle gx, gy \rangle'.$$

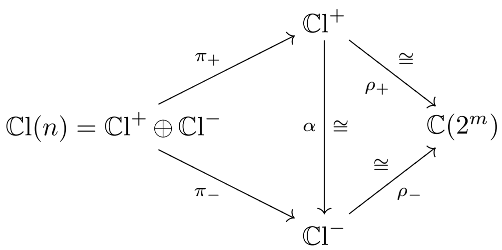
Then $\langle\,,\rangle$ is Γ-invariant in the sense that $\langle gx, gy\rangle = \langle x, y\rangle$ for all $g \in \Gamma$, $x, y \in S$. If $T \subseteq S$ is a $\mathbb{C}\mathrm{l}(n)$-submodule, then the orthogonal complement $T^\perp \subseteq S$ with respect to $\langle\,,\rangle$ satisfies $\Gamma T^\perp \subseteq T^\perp$ and is therefore a $\mathbb{C}\mathrm{l}(n)$-submodule, $\mathbb{C}\mathrm{l}(n)\cdot T^\perp \subseteq T^\perp$. Now $S = T \oplus T^\perp$. If one of these is not irreducible, we continue slitting as long as possible; the process has to stop, since S is finite-dimensional. □

For a proof of the following, see [Lan02].

Proposition 3.2.7. *The irreducible representations of the complex Clifford algebras are the following:*

- *(i) If $n = 2m + 1$, $m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, then $\mathbb{C}\mathrm{l}(n) \cong \mathbb{C}(2^m) \oplus \mathbb{C}(2^m)$ has exactly two equivalence classes of irreducible representations, given by the two projections onto $\mathbb{C}(2^m)$.*
- *(ii) If $n = 2m$, $m \in \mathbb{N}$, then $\mathbb{C}\mathrm{l}(n) \cong \mathbb{C}(2^m)$ has exactly one equivalence class of irreducible finite dimensional representations.*

Remark 3.2.8. For $n = 2m + 1$, $\alpha(\omega) = -\omega$, and so α maps $\mathbb{C}\mathrm{l}^+ \to \mathbb{C}\mathrm{l}^-$. Hence, the two inequivalent representations of $\mathbb{C}\mathrm{l}(n)$ restrict to equivalent representations of $\mathbb{C}\mathrm{l}_0(n)$. To see this, let $\rho_- : \mathbb{C}\mathrm{l}^- \to \mathbb{C}(2^m)$ be an isomorphism, and let $\rho_+ := \rho_- \circ \alpha$. Then $\rho_+\pi_+ = \rho_l\pi_-$ on $\mathbb{C}\mathrm{l}_0(V)$, where the $\pi_\pm$ are projections as in the following diagram:



Definition 3.2.9. The complex *spin representation* of $\mathrm{Spin}(n)$ is the homomorphism $\Delta_n : \mathrm{Spin}(n) \to \mathrm{GL}_{\mathbb{C}}(S)$, where S is a finite dimensional vector space, given by restricting an irreducible complex representation of $\mathbb{C}\mathrm{l}(n) \xrightarrow{\rho} \mathrm{End}_{\mathbb{C}}(S)$ to $\mathrm{Spin}(n) \subseteq \mathbb{C}\mathrm{l}_0(n) \subseteq \mathbb{C}\mathrm{l}(n)$.

Remark 3.2.10. If n is odd, then the spin representation Δ_n is independent of the choice ρ of irreducible representation of $\mathbb{C}\mathrm{l}(n)$ up to equivalence, and Δ_n is irreducible because $\mathrm{Spin}(n)$ generates $\mathbb{C}\mathrm{l}_0(n)$, and $\rho|_{\mathbb{C}\mathrm{l}_0(n)}$ is irreducible for dimensional reasons.

6th lecture, September 14th 2011

We continue where we left off last time.

Proposition 3.2.11. *If n is odd, then the spin representation Δ_n is independent of the choice ρ of irreducible representation of $\mathbb{C}\mathrm{l}(n)$ up to equivalence, and Δ_n is irreducible.*

If n is even, then $S = S^+ \oplus S^-$, where $S^\pm = (1 \pm \omega_{\mathbb{C}})S$ are inequivalent irreducible representations of $\mathrm{Spin}(n)$.

Proof. We already discussed the first part of the Proposition in Remark 3.2.10. For the second one, note that $\mathbb{C}\mathrm{l}(n) \cong \mathbb{C}(2^{n/2})$ is a simple ring and therefore (see [Lan02]) has no (two-sided) ideal other than 0 and $\mathbb{C}\mathrm{l}(n)$. Since

$$(1 + \omega_{\mathbb{C}})(1 - \omega_{\mathbb{C}}) = 1 - (\omega_{\mathbb{C}})^2 = 0,$$

we have $\mathbb{C}\mathrm{l}(n)^+ \cdot S^-$ (here, we redefine $\mathbb{C}\mathrm{l}(n)^\pm := \mathbb{C}\mathrm{l}(n)(1 \pm \omega_{\mathbb{C}})$). If now $S^+ = 0$, then $\mathbb{C}\mathrm{l}(n)^+ \subseteq \ker(\rho)$, but $\ker(\rho)$ is a two-sided ideal which means that $\rho = 0$. Hence, $S^+ \neq 0$, and similarly one shows that $S^- \neq 0$.

The two irreducible representations of $\mathbb{Cl}_0(n) \cong \mathbb{Cl}(n-1)$ have dimension $2^{\frac{n}{2}-1}$ whereas $\dim S = 2^{\frac{n}{2}}$. Therefore S^+ and S^- must both be irreducible representations of $\mathbb{Cl}_0(n)$. Because $\mathrm{Spin}(n)$ contains a linear basis of $\mathbb{Cl}_0(n)$, S^+ and S^- are also irreducible representations of $\mathrm{Spin}(n)$, and they are in fact inequivalent since $\omega \in \mathrm{Spin}(n)$ acts as $\pm i^{-\lfloor \frac{n+1}{2} \rfloor}$ on $S^\pm$. □

Definition 3.2.12. Let S be a Euclidean vector space. A representation $\mathrm{Cl}(n) \to \mathrm{End}_{\mathbb{R}}(S)$ is called *orthogonal*, if

$$\langle v \cdot x, v \cdot y \rangle = \langle x, y \rangle$$

for all $v \in \mathbb{R}^n$ with $|v| = 1$ and all $x, y \in S$.

Lemma 3.2.13. *Let S be a finite dimensional real vector space and $\varphi : \mathrm{Cl}(n) \to \mathrm{End}_{\mathbb{R}}(S)$ a representation. Then there exists an inner product $\langle\, , \rangle$ on S with respect to which φ is orthogonal.*

Proof. Let $\Gamma = \{e_I\} \subseteq \mathrm{Cl}(n)^*$ as in the proof of Proposition 3.2.6 and let $\langle\, , \rangle$ be a Γ-invariant inner product on S. If $v \in \mathbb{R}^n$, $|v| = 1$, then $\varphi(v)^2 = \varphi(v^2) = -I$. This means that $\varphi(v)$ is orthogonal if and only if $\varphi(v)$ is skew-symmetric. Let $v = \sum_{i=1}^n v^i e_i$. Then $\varphi(v) = \sum_{i=1}^n v^i \varphi(e_i)$ is also skew-symmetric, and hence orthogonal. □

3.3 Dirac operators

Recall first if $P \to X$ is a principal G-bundle, and G acts on the finite dimensional vector space V, then we can define an *associated vector bundle* $E := P \times_G V := (P \times V)/G$, where in the last expression, G acts on $P \times V$ by $g \cdot (p, v) = (pg^{-1}, gv)$. Note that if $x \in X$, any $p \in P_x$ defines an isomorphism $V \to E_x$ by $v \mapsto [p, v]$. Secondly, any connection in P defines a connection in E.

Definition 3.3.1. Let X be a Riemannian n-manifold, and let $P_{\mathrm{O}}(X) \to X$ be the principal $\mathrm{O}(n)$-bundle of orthonormal frames on X. Then, the fibre $P_{\mathrm{O}}(X)_x$ at $x \in X$ is the set of all linear isometries $\mathbb{R}^n \xrightarrow{\cong} T_x X$.

The *Clifford bundle* $\mathrm{Cl}(X)$ is defined by $\mathrm{Cl}(X) := P_{\mathrm{O}}(X) \times_{\mathrm{O}(n)} \mathrm{Cl}(n)$, where the action of $\mathrm{O}(n)$ on $\mathrm{Cl}(n)$ is induced by its action on $\mathbb{R}^n$. Let $\mathrm{Cl}(X)$ as a vector bundle have the connection induced by the Riemannian connection in $P_{\mathrm{O}}(X)$.

Notice that the Clifford bundle is well-defined whether or not X is oriented, but if X is oriented, we can identify $\mathrm{Cl}(X) = P_{\mathrm{SO}}(X) \times_{\mathrm{SO}(n)} \mathrm{Cl}(n)$, where $P_{\mathrm{SO}}(X)$ is the principal $\mathrm{SO}(n)$-bundle of positive orthonormal frames.

Remark 3.3.2. For any point $x \in X$ there is a canonical isomorphism $\mathrm{Cl}(X)_x \xrightarrow{\cong} \mathrm{Cl}(T_x X)$. This isomorphism is given by mapping $[u, \varphi] \mapsto u_*(\varphi)$ where $u \in P_{\mathrm{O}}(X)_x$, $\varphi \in \mathrm{Cl}(n)$, and $u_* : \mathrm{Cl}(n) \xrightarrow{\cong} \mathrm{Cl}(T_x X)$.

Definition 3.3.3. A *Dirac bundle* over a Riemannian manifold X is a Euclidean vector bundle $E \to X$ with an orthogonal connection ∇, together with a homomorphism of bundles of real algebras $\mathrm{Cl}(X) \to \mathrm{End}_{\mathbb{R}}(E)$ such that the following hold:

(i) $\langle e \cdot v, e \cdot w \rangle = \langle v, w \rangle$ for all $x \in X$, $v, w \in E_x$ and $e \in T_x X$ with $|e| = 1$.

(ii) $\nabla(\varphi \cdot s) = \nabla\varphi \cdot s + \varphi \cdot \nabla s$ for all $\varphi \in \Gamma(\mathrm{Cl}(X))$, $s \in \Gamma(E)$.

Note that for every $x \in X$, we have a representation $\mathrm{Cl}(T_x X) = \mathrm{Cl}(X)_x \to \mathrm{End}_{\mathbb{R}}(E_x)$ and condition (i) says that all of these representations are orthogonal.

Definition 3.3.4. The *Dirac operator* associated to a Dirac bundle as above is the map $D : \Gamma(E) \to \Gamma(E)$ defined by

$$D(s) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}(s),$$

where $e_1, \ldots, e_n$ is a local orthonormal frame of X

Example 3.3.5. Let $E := \mathrm{Cl}(X)$. We equip E with a Eucledian metric, inherited from the canonical identification $\mathrm{Cl}(X) \cong \wedge^* TX$, and the connection already defined on $\mathrm{Cl}(X)$. The module structure is simply given by the Clifford multiplication

$$\mathrm{Cl}(X)_x \times E_x \to E_x$$

mapping $(\varphi, \psi) \mapsto \varphi \cdot \psi$. Condition (i) holds since as before orthogonality corresponds to skew-symmetry.

Later we will see that the corresponding Dirac operator can be described in terms of the exterior derivative and its adjoint (the so-called Euler characteristic operator).

Definition 3.3.6. A *spin structure* on an oriented Riemannian n-manifold X is a principal $\mathrm{Spin}(n)$-bundle $P_{\mathrm{Spin}} \to X$ together with a smooth fibre-preserving map $f : P_{\mathrm{Spin}} \to P_{\mathrm{SO}}(X)$ which is $\mathrm{Spin}(n)$-equivariant in the sense that $f(ug) = f(u)\pi(g)$ for all $u \in (P_{\mathrm{Spin}})_x$, $g \in \mathrm{Spin}(n)$, where $\pi = \mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$.

Note that X admits a spin structure if and only if the second Stiefel–Whitney class $w_2(X) = w_2(TX) \in H^2(X, \mathbb{Z}/2)$ vanishes.

7th lecture, September 19th 2011

3.3.1 Detour on principal bundles

Before we go ahead and define Dirac operators associated to spin structures, we make some basic remarks on principal bundles (see [KN96]).

Definition 3.3.7. Let $P \to X$ be a principal G-bundle, and $Q \to X$ a principal H-bundle. A fibre-preserving smooth map $f : P \to Q$ is called an *i-homomorphism* (i for identity) of principal bundles with respect to a homomorphism $\varphi : G \to H$ of Lie groups, if $f(pg) = f(p)\varphi(g)$ for all $p \in P$, $g \in G$.

An i-homomorphism f induces a map $f_* : \mathcal{A}(P) \to \mathcal{A}(Q)$, where $\mathcal{A}(P), \mathcal{A}(Q)$ are the spaces of all connections in P and Q. If moreover $L\varphi : LG \xrightarrow{\varphi} LH$ is an isomorphism (where LG and LH denote the corresponding Lie algebras), or, equivalently, if φ is a local diffeomorphism, then f_* is a bijection.

If $P \to X$ and $Q \to X$ are as above, then pulling back the $(G \times H)$-bundle $P \times Q \to X \times X$ by the diagonal map $\Delta : X \to X \times X$ mapping $x \mapsto (x, x)$ yields a $(G \times H)$-bundle denoted $P + Q := \Delta^*(P \times Q) \to X$. Let $\pi_1 : P + Q \to P$ and $\pi_2 : P + Q \to Q$ be the natural projections (which are i-homomorphisms with respect to the projections $G \times H \to G$ and $G \times H \to H$ respectively).

Putting these together, we obtain a bijection

$$(\pi_1)_* \times (\pi_2)_* : \mathcal{A}(P + Q) \to \mathcal{A}(P) \times \mathcal{A}(Q).$$

Let $f : P \to Q$ be an i-homomorphism with respect to a homomorphism $\varphi : G \to H$, and let $\rho : H \to \mathrm{Aut}(V)$ be a representation. Then there is an vector bundle associated to ρ, but we could also have obtained this from P. Namely, there is an isomorphism of associated vector bundles

$$P \times_{\rho \circ \varphi} V \xrightarrow{\cong} Q \times_\rho V$$

given by $[p, v] \mapsto [f(p), v]$, where $Q \times_\rho V$ is an alternative notation for $Q \times_H V$. To see this, note that it is clearly a homomorphism of vector bundles, which is an isomorphism on each fibre.

Finally, if $f : P \to Q$ is an i-homomorphism as above, then there is an isomorphism of H-bundles,

$$P \times_G H \xrightarrow{\cong} Q, \quad [p, h] \mapsto f(p) \cdot h,$$

where G acts from the left on H by $(g, h) \mapsto \varphi(g) \cdot h$.

If φ is surjective and $K := \ker(\varphi)$, then there is an $H := G/K$-equivariant diffeomorphism

$$P/K \xrightarrow{\cong} P \times_G H,$$

and hence P/K is a principal G/K-bundle.

If φ is surjective and $K := \ker(\varphi)$, then $P \times_G H$ is isomorphic to the quotient P/K, which is an $H = G/K$-bundle.

3.3.2 Back to Dirac operators

Let $P_{\mathrm{Spin}} := P_{\mathrm{Spin}}(X) \to P_{\mathrm{SO}}(X)$ be a spin structure with an orthogonal representation $\rho : \mathrm{Cl}(n) \to \mathrm{End}_{\mathbb{R}}(S)$ where $n = \dim X$. Then $\rho|_{\mathrm{Spin}(n)}$ is an orthogonal representation (and note that this is not a tautological statement). The associated Euclidean vector bundle

$$\mathbb{S} := P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} S$$

is called a *real spinor bundle*. We will now make this bundle into a Dirac bundle.

First we define the connection in $\mathbb{S}$. Let P_{Spin} have the connection which maps to the Levi-Civita connection in $P_{\mathrm{SO}}(X)$. This defines an orthogonal connection in $\mathbb{S}$. Next we make $\mathbb{S}$ a Clifford module. Let $\mathrm{Spin}(n)$ act on $\mathrm{Cl}(n)$ by conjugation. In fact, this action factors through the standard representation of $\mathrm{SO}(n)$ defined earlier, i.e. we have maps

$$\begin{array}{ccc} \mathrm{Spin}(n) & \xrightarrow{\ \mathrm{conj}\ } & \mathrm{Aut}(\mathrm{Cl}(n)) \\ & {\scriptstyle \mathrm{Ad}}\searrow \quad \nearrow & \\ & \mathrm{SO}(n) & \end{array}$$

As remarked earlier, we can identify

$$P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} \mathrm{Cl}(n) \xrightarrow{\cong} P_{\mathrm{SO}} \times_{\mathrm{SO}(n)} \mathrm{Cl}(n) = \mathrm{Cl}(X).$$

Observe that the homomorphism $\rho : \mathrm{Cl}(n) \to \mathrm{End}_{\mathbb{R}}(S)$ is $\mathrm{Spin}(n)$-equivariant with respect to the conjugation action on both spaces, and therefore it induces a homomorphism of vector bundles $\mathrm{Cl}(X) \to \mathrm{End}_{\mathbb{R}}(\mathbb{S})$: In general, if G acts on V, any fixed point $v \in V$ gives a section of $P \times_G V$, since the equivalence class $[p, v]$ does not depend on p.

Thus, $\mathbb{S}$ is a Dirac bundle (the conditions are easily verified). The corresponding Dirac operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is called the *Atiyah–Singer operator*.

3.4 $\mathrm{Spin}^c(n)$ and spin^c-structures

Definition 3.4.1. We define a Lie group $\mathrm{Spin}^c(n) := (\mathrm{Spin}(n) \times \mathrm{U}(1))/\pm(1, 1)$.

We can think of this group as sitting in the complex Clifford algebra as follows: Consider the group homomorphism

$$\mu : \mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathbb{Cl}(n)^*$$

defined by $\mu(x, z) = x \otimes z$. If $x \otimes z = 1 \otimes 1$, then z must be real so $xz = 1$ in $\mathrm{Cl}(n)$, hence $x = z = \pm 1$. Therefore $\ker \mu = \{\pm(1, 1)\}$, and so μ induces an embedding

$$\mathrm{Spin}^c(n) \hookrightarrow \mathbb{Cl}(n)^*.$$

Associated to $\mathrm{Spin}^c(n)$, there are the following three short exact sequences:

$$1 \to \mathrm{U}(1) \xrightarrow{\alpha'} \mathrm{Spin}^c(n) \xrightarrow{\alpha} \mathrm{SO}(n) \to 1.$$

Here α is determined by the projection $\mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathrm{Spin}(n)$; more precisely, we have the following picture:

$$\begin{array}{ccc} \mathrm{Spin}(n) \times \mathrm{U}(1) & \longrightarrow & \mathrm{Spin}(n) \\ \downarrow & & \downarrow \\ \mathrm{Spin}^c(n) & \xrightarrow{\alpha} & \mathrm{SO}(n) \end{array}$$

The maps are given by $\alpha'(z) = [(1, z)]$, $\alpha([x, z]) = \mathrm{Ad}_x$. Similarly, we have

$$1 \to \mathrm{Spin}(n) \overset{\beta'}{\to} \mathrm{Spin}^c(n) \overset{\beta}{\to} \mathrm{U}(1) \to 1,$$

where $\beta'(x) = [(1, x)]$ and $\beta([x, z]) = z^2$. Lastly,

$$1 \to \{\pm 1\} \to \mathrm{Spin}^c(n) \overset{\alpha \times \beta}{\to} \mathrm{SO}(N) \times \mathrm{U}(1) \to 1.$$

This sequence is obtained by considering the 4:1-map $\mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathrm{SO}(n) \times \mathrm{U}(1)$, which factors through the 2:1-map $\mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$. In particular, $\alpha \times \beta$ is a double covering.

Definition 3.4.2. A spin^c*-structure* on an oriented Riemannian manifold X is a principal $\mathrm{Spin}^c(n)$-bundle $P_{\mathrm{Spin}^c} \to X$ together with an i-homomorphism

$$f : P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}}(X)$$

with respect to $\alpha : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$.

Remark 3.4.3. We make the following remarks:

(i) To any spin^c-structure, there is an assocated U(1)-bundle $P_{\mathrm{Spin}^c} \times_\beta \mathrm{U}(1) \to X$.

(ii) An oriented Riemannian manifold X carries a spin^c-structure if and only if $w_2(X) \in H^2(X; \mathbb{Z}/2)$ has a lift to $H^2(X; \mathbb{Z})$, which is weaker than the obstruction to having a spin structure. For a proof, see the appendix of [LM89].

(iii) By (ii), one can show that every compact oriented 4-manifold carries a spin^c-structure.

(iv) The 5-manifold SU(3)/SO(3) has no spin^c-structure.

3.4.1 Dirac bundles from spin^c-structures

As in the spin case, one can use spin^c-structures to construct Dirac bundles. Let $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}}(X)$ be a spin^c-structure, and let $\rho : \mathbb{C}\mathrm{l}(n) \to \mathrm{End}_{\mathbb{C}}(S)$ be a represenation. In the case of $\mathrm{Spin}(n)$, we required the representation to be orthogonal; here, we require it to be *unitary*: Choose an Hermitian inner product $\langle \, , \, \rangle$ on S, such that $\langle vx, vy \rangle = \langle x, y \rangle$ for all $v \in \mathbb{R}^n \subseteq \mathbb{C}\mathrm{l}(n)$ with $|v| = 1$ and all $x, y \in S$ (that this is possible can be shown as in the real case). Then $\rho|_{\mathrm{Spin}^c(n)}$ is a unitary representation, and the associated Hermitian vector bundle

$$\mathbb{S} := P_{\mathrm{Spin}^c(n)} \times_{\mathrm{Spin}^c(n)} S$$

is called the *complex spinor bundle*. Just as in the real case, the representation ρ gives rise to a homomorphism $\mathbb{C}\mathrm{l}(X) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S})$, and it remains to specify the connection in $\mathbb{S}$. Let $P_{\mathrm{U}} \to X$ be the U(1)-bundle associated to P_{Spin^c}. Then we have an i-homomorphism $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}} + P_U$ with respect to $\alpha \times \beta : \mathrm{Spin}^c(n) \to \mathrm{SO}(n) \times \mathrm{U}(1)$, since we have a homomorphism $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}} \times P_U|_{\Delta(X)} \cong P_{\mathrm{SO}} + P_{\mathrm{U}}$.

Since $L(\alpha \times \beta)$ is an isomorphism, the i-homomorphism above induces a bijection $\mathcal{A}(P_{\mathrm{Spin}^c}) \to \mathcal{A}(P_{\mathrm{SO}}) \times \mathcal{A}(P_{\mathrm{U}})$. Let P_{Spin^c} have the connection which maps to the Riemannian connection in P_{SO} and to any connection A in P_{U}. This induces a unitary connection in $\mathbb{S}$ giving $\mathbb{S}$ the structure of a Dirac bundle. The corresponding Dirac operator $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is complex linear.

8th lecture, September 21st 2011

Example 3.4.4. Let $P_{\text{Spin}} \to P_{\text{SO}}(X)$ be a spin structure and $P_U \to X$ a U(1)-bundle, and let $n = \dim X$. Define the $\text{Spin}^c(n) = (\text{Spin}(n) \times \text{U}(1))/\pm(1,1)$-bundle

$$P_{\text{Spin}^c} := (P_{\text{Spin}} + P_{\text{U}})/\pm(1,1).$$

This is a spin^c-structure whose associated U(1)-bundle is $P_{\text{U}}/\pm 1$, which we can also think of as the U(1)-bundle associated to P_{U} and the homomorphism $\text{U}(1) \to \text{U}(1)$, $z \mapsto z^2$. Let $\rho : \mathbb{C}\text{l}(n) \to \text{End}_{\mathbb{C}}(S)$ be a unitary representation. Set

$$\mathbb{S} := P_{\text{Spin}} \times_{\text{Spin}(n)} S,$$

where $\text{Spin}(n)$ acts on S by $\rho|_{\text{Spin}(n)}$. Of course, we also get a spinor bundle from the spin^c-structure, and the two bundles are related as follows: Let $L \to X$ be the Hermitian line bundle associated to P_{U} and the standard representation of U(1) on $\mathbb{C}$. Then

$$P_{\text{Spin}^c} \times_{\text{Spin}^c(n)} S \overset{\cong}{\leftarrow} (P_{\text{Spin}} + P_U) \times_{\text{Spin}(n)\times\text{U}(1)} (S \otimes_{\mathbb{C}} \mathbb{C}) = \mathbb{S} \otimes L,$$

by one of the remarks made in the beginning of Section 3.3.1. Here we let $\text{Spin}(n)$ act on S and U(1) on $\mathbb{C}$ in the first tensor product.

Any connection in $P_{\text{U}}/\pm 1$ induces a connection in P_{U} (recall that connections in P_{U} are in 1-1 correspondance with those in $P_{\text{U}}/\pm 1$, since we have a map $P_{\text{U}} \to P_{\text{U}}/\pm 1$ induced by the 2:1 map $\text{U}(1) \to \text{U}(1)$, $z \mapsto z^2$), and hence it induces an Hermitian connection in L. The spin^c Dirac operator

$$D_A : \Gamma(\mathbb{S} \otimes L) \to \Gamma(\mathbb{S} \otimes L)$$

is given by

$$D_A(s) = \sum_{j=1}^{n} e_j \nabla^A_{e_j}(s),$$

where ∇^A is the tensor product connection in $\mathbb{S} \otimes L$.

The Dirac operator is in particular a differential operator, and we now turn to a general discussion of these.

3.5 Differential operators

Definition 3.5.1. In $\mathbb{R}^n$, for every multiindex $\alpha = (\alpha_1, \dots, \alpha_k)$, $\alpha_j \geq 0$, set

$$\begin{aligned} \partial^\alpha &:= \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha_n}, \\ |\alpha| &:= \alpha_1 + \cdots + \alpha_n, \\ \partial_\alpha &:= i^{-|\alpha|}\partial^\alpha. \end{aligned}$$

The point in the last definition is that ∂_α fits better with the Fourier transform and is in fact self-adjoint. For example, on $\mathbb{R}$,

$$\langle if', g\rangle = \int_{\mathbb{R}} if'(t)\overline{g(t)}\,dt = \int_{\mathbb{R}} f(t)\overline{ig'(t)}\,dt,$$

if $\text{supp}(f) \cap \text{supp}(g)$ is compact. In this sense, $i\partial$ is self-adjoint.

Definition 3.5.2. Let X be a smooth n-manifold and let $E, F \to X$ be two $\mathbb{F}$-vector bundles, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. A map $P : \Gamma(E) \to \Gamma(F)$ is called a *differential operator* of order $\leq r$, if P is a locally finite sum of operators of the form

$$s \mapsto a\nabla_{V_1} \cdots \nabla_{V_k}(s), \quad s \in \Gamma(E),$$

where $0 \leq k \leq r$, $a \in \Gamma(\mathrm{Hom}(E, F))$, ∇ is a connection in E, and each V_j is a vector field on X.

The choice of connection in this definition does not matter, since for another connection ∇', we have $\nabla'_V = \nabla_V + b(V)$ where $b(V) \in \Gamma(\mathrm{Hom}(E, F))$. In the case $\mathbb{F} = \mathbb{R}$, there is an induced differential operator $\Gamma(E \otimes_{\mathbb{R}} \mathbb{C}) \to \Gamma(F \otimes_{\mathbb{R}} \mathbb{C})$.

In terms of local coordinates $x^1, \dots, x^n$ on an open set $U \subseteq X$ and trivializations of $E|_U$, $F|_U$, the differential operator P can be expressed as

$$Ps = \sum_{|\alpha| \leq r} a_\alpha \partial^\alpha s$$

over U.

Definition 3.5.3. Let $\mathrm{Diff}_{\leq r}(E, F)$ denote the set of differential operators $\Gamma(E) \to \Gamma(F)$ of order at most r, and let $\mathrm{Diff}_r(E, F)$ denote those of order exactly r, $\mathrm{Diff}_r = \mathrm{Diff}_{\leq r} \setminus \mathrm{Diff}_{\leq r-1}$.

3.5.1 Symbols of differential operators

Assume from now on that $E, F \to X$ are complex vector bundles.

Definition 3.5.4. The *r-symbol* of $P \in \mathrm{Diff}_{\leq r}(E, F)$ is the map

$$\sigma_{P,r} = \sigma_P : T^*X \to \mathrm{Hom}_{\mathbb{C}}(E, F)$$

defined as follows: Let $x \in X$, $\xi \in T^*_x X$, $v \in E_x$. Choose a smooth function $f \in C^\infty(X)$ and $s \in \Gamma(E)$ with $f(x) = 0$, $df_x = \xi$, $s(x) = v$. Define

$$\sigma_P(x, \xi)v := \frac{i^r}{r!} P(f^r \cdot s)(x).$$

Note that the symbol is a vector bundle homomorphism for $r = 1$, and in general $\sigma_{P,r}(x, t\xi) = t^r \sigma_{P,r}(x, \xi)$ for $x \in X, \xi \in T^*_x X, t \in \mathbb{R}$.

To show that $\sigma_P(x, \xi)$ does not depend on the choices of f and s, let $X = \mathbb{R}^n$, $E = X \times \mathbb{C}^p$, $F = X \times \mathbb{C}^q$, and write $P = \sum_{|\alpha| \leq r} a_\alpha \partial_\alpha$ for smooth maps $a_\alpha : \mathbb{R}^n \to M_{q \cdot p}(\mathbb{C})$. Then

$$\frac{i^r}{r!} P(f^r s)(x) = \sum_{|\alpha| = r} \xi^\alpha a_\alpha(x) v,$$

where $\xi^\alpha = (\xi_1)^{\alpha_1} \cdots (\xi_n)^{\alpha_n}$ for $\xi = (\xi_1, \dots, \xi_n)$. Since neither f nor s appears on the right hand side, $\sigma_P(x, \xi) = \sum_{|\alpha|} \xi^\alpha a_\alpha \in M_{p \cdot q}(\mathbb{C})$ is independent of the choices made.

Proposition 3.5.5. *If $E, E', E'' \to X$ are complex vector bundles, and $\Gamma(E) \xrightarrow{P} \Gamma(E') \xrightarrow{Q} \Gamma(E'')$ is a composition of differential operators P and Q of orders $\leq r$ and $\leq r'$ respectively, then $QP \in \mathrm{Diff}_{\leq r+r'}(E, E'')$, and*

$$\sigma_{QP, r+r'}(x, \xi) = \sigma_{Q,r'}(x, \xi) \circ \sigma_{P,r}(x, \xi)$$

for $x \in X$, $\xi \in T^*_x X$.

Proof. The proof is a local calculation. Let $X = \mathbb{R}^n$, let E, E', E'' be product bundles, and write

$$P = \sum_{|\alpha| \le r} a_\alpha \partial_\alpha, \quad Q = \sum_{|\beta| \le r'} b_\beta \partial_\beta.$$

The composition becomes

$$\begin{aligned} QPs &= \sum_{|\beta| \le r'} b_\beta \partial_\beta \sum_{|\alpha| \le r} a_\alpha \partial_\alpha s \\ &= \sum_{|\gamma| = r+r'} \left(\sum_{\alpha+\beta=\gamma} b_\beta a_\alpha \right) \partial_\gamma s + \sum_{|\gamma| < r+r'} c_\gamma \partial_\gamma s \end{aligned}$$

for suitable matrix valued functions c_γ. Hence

$$\begin{aligned} \sigma_{QP, r+r'}(x, \xi) &= \sum_{|\gamma| = r+r'} \left(\sum_{\alpha+\beta=\gamma} b_\beta(x) a_\alpha(x) \right) \xi^\gamma \\ &= \left(\sum_{|\beta| = r'} b_\beta(x) \xi^\beta \right) \left(\sum_{|\alpha| = r} a_\alpha(x) \xi^\alpha \right) \\ &= \sigma_{Q,r'}(x, \xi) \circ Q_{P,r}(x, \xi). \end{aligned}$$

□

Definition 3.5.6. A differential operator $P \in \mathrm{Diff}_r(E, F)$ is called *elliptic* if

$$\sigma_{P,r}(x, \xi) : E_x \to F_x$$

is invertible for every $x \in X$, $\xi \in T_x^* X \setminus \{0\}$.

Example 3.5.7. The following are examples of differential operators:

1. The *Cauchy–Riemann operator* on $\mathbb{C} = \{z = x + iy\}$ is the operator $\overline{\partial} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$. Its 1-symbol is

   $$\sigma_{\overline{\partial}}(z, \xi) = i(\xi_1 + i\xi_2) = i\xi_1 - \xi_2,$$

   which is non-zero when ξ is, so $\overline{\partial}$ is elliptic.

2. The *Laplace operator* on Euclidean $\mathbb{R}^n$ is the operator

   $$\Delta = -\sum_{j=1}^n \left(\frac{\partial}{\partial x^j} \right)^2 = \sum_{j=1}^n \left(\frac{1}{i} \partial_j \right)^2$$

   acting on complex functions on $\mathbb{R}^n$. Its symbol is

   $$\sigma_\Delta(x, \xi) = |\xi|^2,$$

   which is positive for $\xi \neq 0$, so Δ is elliptic.

3. Let ∇ be a connection in the complex vector bundle $E \to X$. Then

   $$\nabla : \Gamma(E) \to \Gamma(T^* X \otimes_{\mathbb{R}} E)$$

   if a differential operator of order 1. If $f \in C^\infty(X)$, $f(x) = 0$, $df_x = \xi$, $s \in \Gamma(E)$, $s(x) = v$, we find

   $$\sigma_\nabla(x, \xi) v = i \nabla(fs)(x) = i(df_x \otimes v + f(x)(\nabla s)_x) = \xi \otimes iv.$$

   In this case, if $\xi \neq 0$, then $\sigma_\nabla(x, \xi)$ is injective but not surjective unless $\mathrm{rk}\, E = 1$.

9th lecture, September 26th 2011

3.6 Dirac operators revisited

Proposition 3.6.1. *Let X be a Riemannian n-manifold and $\mathrm{Cl}(X) \to \mathrm{End}_{\mathbb{C}}(\mathbb{S})$ a complex Dirac bundle. Then the Dirac operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ mapping $D\varphi = \sum_{j=1}^n e_j \nabla_{e_j}(\varphi)$ is a differential operator of order 1. The symbols of the Dirac operator D and the* Dirac Laplacian *D^2 are given by*

$$\begin{aligned}\sigma_{D,1}(x,\xi) &= i\xi^*,\\ \sigma_{D^2,2}(x,\xi) &= |\xi|^2\end{aligned}$$

where ξ^* is the tangent vector at x dual to ξ with respect to the Riemannian metric, and $i\xi^*$ denotes Clifford multiplication with $i\xi^*$. In particular the Dirac operator D and its square D^2 are elliptic operators.

Proof. Let $\varphi \in \Gamma(\mathbb{S})$, $f \in C^\infty(X)$, with $f(x) = 0$, $df_x = \xi$. Then

$$\begin{aligned}\sigma_{D,1}(x,\xi)\cdot\varphi(x) = iD(f\varphi)(x) &= i\sum_{j=1}^n e_j \nabla_{e_j}(f\varphi)_x\\ &= i\sum_{j=1}^n e_j \left(df_x(e_j)\cdot\varphi(x) + f(x)\nabla_{e_j}(\varphi)_x\right)\\ &= i\sum_{j=1}^n e_j\cdot\xi(e_j)\cdot\varphi(x) = i\xi^*\cdot\varphi(x).\end{aligned}$$

For $v \in \mathbb{S}_x$,

$$\begin{aligned}\sigma_{D^2,2}(x,\xi)\cdot v = \sigma_{D,1}(x,\xi)^2\cdot v &= i\xi^*\cdot(i\xi^*\cdot v)\\ &= -(\xi^*)^2\cdot v = |\xi^*|^2\cdot v = |\xi|^2\cdot v.\end{aligned}$$

□

3.6.1 The adjoint of an operator

For a Riemannian n-manifold X and $f \in C_c(X)$, we write $\int_X f := \int_X f\,d\mu$, where μ is the *volume measure* defined by the Riemannian metric. If X is oriented, then $\int_X f\,d\mu = \int_X f\,\omega$, where $\omega \in \Omega^n(X)$ is the volume form on X.

Proposition 3.6.2. *Let E, F be Hermitian vector bundles over a Riemannian manifold X, and let $P \in \mathrm{Diff}_r(E,F)$. Then there is a uniquely defined operator, denoted $P^* \in \mathrm{Diff}_r(F,E)$ characterized by*

$$\int_X \langle Ps, t\rangle = \int_X \langle s, P^*t\rangle$$

for all $s \in \Gamma_c(E)$, $t \in \Gamma_c(F)$. Moreover

$$\sigma_{P^*,r}(x,\xi) = \sigma_{P,r}(x,\xi)^* : F_x \to E_x.$$

Proof. By a continuity argument, uniqueness follows from the characterizing equation. Existence is essentially a local problem: Replacing s by $\sum_\lambda \beta_\lambda s$ for a suitable partition of unity $\{\beta_\lambda\}_\lambda$ – defined over a coordinate patch on the manifold – reduces the problem to the case, where $X = \mathbb{R}^n$ as smooth manifolds, $E = X \times \mathbb{C}^p$, $F = X \times \mathbb{C}^q$ as Hermitian vector bundles, $P = \sum_{|\alpha|\le r} a_\alpha \partial_\alpha$, and $\mu = hm$ for $h \in C^\infty(X)$, $h > 0$, and m the Lebesgue measure on $\mathbb{R}^n$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \langle Ps, t\rangle h\, dm &= \sum_{|\alpha|\le r} \int_{\mathbb{R}^n} \langle a_\alpha \partial_\alpha s, ht\rangle\, dm \\ &= \sum_{|\alpha|\le r} \int_{\mathbb{R}^n} \langle s, \partial_\alpha(a_\alpha^* ht)\rangle\, dm \\ &= \int_{\mathbb{R}^n} \langle s, \sum_{|\alpha|=r} a_\alpha^* \partial_\alpha t + \sum_{|\alpha|<r} b_\alpha \partial_\alpha t\rangle h\, dm\end{aligned}$$

for suitable functions $b_\alpha : \mathbb{R}^n \to M_{q,p}(\mathbb{C})$ independent of s and t (but involving both a_α, h, and their derivatives). In the second equality, we have used integration by parts together with the fact that s is compactly supported and ∂_α is self-adjoint. □

Definition 3.6.3. The operator P^* is called the *(formal) adjoint* of P, and P is called *self-adjoint* if $P = P^*$.

Proposition 3.6.4. *Let X be a Riemannian n-manifold and $E \to X$ an Hermitian vector bundle with connection A (which will always be assumed to be Hermitian). Then the (formal) adjoint of*

$$d_A : \Omega^p(X; E) \to \Omega^{p+1}(X; E)$$

is the operator

$$d_A^* = (-1)^{1+pn} * d_A * : \Omega^{p+1}(X; E) \to \Omega^p(X; E).$$

Note that we did not assume X to be oriented, so a priori we have no Hodge star, but the expression $*d_A*$ is defined with respect to any local orientation, it is independent of the orientation, and so makes sense globally.

Proof of Proposition. The problem is local, so we may assume that X is oriented. Let $\varphi \in \Omega^p_c(X; E)$, $\psi \in \Omega^{p+1}(X; E)$. Combining the wedge product on forms with the Hermitian metric on the vector bundle, Proposition 2.1.3 (with $E'' = \mathbb{C}$, $A'' = d$) tells us that

$$d\langle \varphi \wedge *\psi\rangle = \langle d_A\varphi \wedge *\psi\rangle + (-1)^p \langle \varphi \wedge d_A * \psi\rangle.$$

By Stokes' theorem and since $*^2 = (-1)^{p(n-p)}$ on $(n-p)$-forms, we find that

$$\begin{aligned}\int_X \langle d_A\varphi, \psi\rangle = \int_X \langle d_A\varphi \wedge *\psi\rangle &= (-1)^{p+1} \int_X \langle \varphi \wedge d_A * \psi\rangle \\ &= (-1)^{p+1+p(n-p)} \int_X \langle \varphi \wedge * * d_A * \psi\rangle = (-1)^{1+pn} \int_X \langle \varphi, *d_A * \psi\rangle.\end{aligned}$$

□

Our next goal is to express $d_A + d_A^*$ as a Dirac operator. To do this, we need to find an expression for d_A in terms of covariant derivatives.

Proposition 3.6.5. *Let X be a smooth n-manifold and ∇ a connection in TX. Define d' : $\Omega^p(X) \to \Omega^{p+1}(X)$ by*

$$d'\varphi = \sum_{j=1}^n e_j^* \wedge \nabla_{e_j}(\varphi)$$

for any local basis of vector fields $\{e_j\}$ on X with dual local basis of 1-forms $\{e_j^*\}$. Then the following are equivalent:

(i) The operator d' agrees with the exterior derivative.

(ii) The connection ∇ is torsion-free.

Proof. By the characterizing properties of the exterior derivative (rewriting $d^2 = 0$ slightly), it suffices to prove the following:

(a) For any $f \in C^\infty(X)$, we have $d'f = df$.

(b) For $\varphi \in \Omega^p(X)$, $\psi \in \Omega^q(X)$, we have $d'(\varphi \wedge \psi) = d'\varphi \wedge \psi + (-1)^p \varphi \wedge d'\psi$.

(c) If $X = \mathbb{R}^n$, then $d'(dx^j) = 0$ for all j, if and only if ∇ is torsion-free.

First, (a) is easy, since

$$d'f = \sum_j e_j^* df(e_j) = df.$$

We also have

$$\begin{aligned} d'(\varphi \wedge \psi) &= \sum_j e_j^* \wedge (\nabla_{e_j}\varphi \wedge \psi + \varphi \wedge \nabla_{e_j}\psi) \\ &= d'\varphi \wedge \psi + (-1)^p \varphi \wedge d'\psi, \end{aligned}$$

which proves (b). To see (c), write $\partial_j = \frac{\partial}{\partial x^j}$ and write

$$\nabla \partial_k = \sum_i \omega_k^i \otimes \partial_i$$

for $\omega_k^i \in \Omega^1(\mathbb{R}^n)$. Then $\Gamma_{jk}^i = \omega_k^i(\partial_j)$ are the Christoffel-symbols of ∇. As seen earlier, $\nabla dx^k = \sum_i \omega_i^k \otimes dx^i$, so

$$\nabla_{\partial_j}(dx^k) = \sum_i \Gamma_{ji}^k dx^i.$$

Applying d', we obtain

$$\begin{aligned} d'(dx^k) &= \sum_j dx^j \wedge \nabla_{\partial_j}(dx^k) = \sum_j dx^j \wedge \sum_i \Gamma_{ji}^k dx^i \\ &= \sum_{ij} \Gamma_{ji}^k dx^j \wedge dx^i = \sum_{i<j} (\Gamma_{ji}^k dx^j \wedge dx^i + \Gamma_{ij}^k dx^i \wedge dx^j) \\ &= \sum_{i<j} (\Gamma_{ij}^k - \Gamma_{ij}^k) dx^i \wedge dx^j. \end{aligned}$$

Now (c) follows since ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j, k. $\square$

Convention 3.6.6. Let V be a finite dimensional vector space over $\mathbb{F}$. Then the duality pairing,

$$\begin{gathered} \wedge^p V \times \wedge^p(V^*) \to \mathbb{F}, \\ (v_1 \wedge \cdots \wedge v_p, \alpha_1 \wedge \cdots \wedge \alpha_p) \mapsto \det(\alpha_i(v_j)), \end{gathered}$$

defines the isomorphism $\wedge^p(V^*) \xrightarrow{\cong} (\wedge^p V)^*$. Some authors use the expression $\frac{1}{p!}\det(\alpha_i(v_j))$ instead.

Let V be a vector space, $r \geq 1$, $x \in V$. Let $\iota_x : \wedge^r V^* \to \wedge^{r-1} V^*$ be the contraction satisfying $\iota_x = 0$ on $\Omega^0 V^*$ and

$$\iota_x(\alpha \wedge \beta) = \iota_x(\alpha) \wedge \beta + (-1)^p \alpha \wedge \iota_x(\beta)$$

for $\alpha \in \wedge^p V^*$, $\beta \in \wedge^q V^*$. If $\dim V = n < \infty$ and $\omega \in (\wedge^r V)^*$, then under the isomorphism $(\wedge^r V)^* \cong \wedge^r V^*$, one has

$$\iota_x \omega(v_2, \dots v_n) = \omega(x, v_2, \dots, v_n),$$

so in particular $\iota_x \circ \iota_x = 0$. An alternative notation is $x \lrcorner \omega = \iota_x \omega$. If V is a finite dimensional Euclidean vector space, then for $\alpha \in \wedge^r V^*$, $\beta \in \wedge^{r-1} V^*$, we have

$$\langle x \lrcorner \alpha, \beta \rangle = \langle \alpha, x^* \wedge \beta \rangle,$$

when x^* is dual to x with respect to $\langle \cdot, \cdot \rangle$. Thus, we find the adjoint to ι_x as

$$(\iota_x)^* \beta = x^* \wedge \beta.$$

10th lecture, September 28th 2011

Proposition 3.6.7. *Let X be an n-dimensional Riemannian manifold and $E \to X$ an Hermitian vector bundle with connection A. For any p, let $\wedge^p T^* X \otimes_{\mathbb{R}} E$ be equipped with the connection induced by A and the Levi-Civita connection in TX. Then the operator*

$$d_A : \Omega^p(X; E) \to \Omega^{p+1}(X; E)$$

and its adjoint d_A^* are given by

$$\begin{aligned} d_A &= \sum_{j=1}^n e_j^* \wedge \nabla_{e_j}^A, \\ d_A^* &= -\sum_{j=1}^n e_j \lrcorner \nabla_{e_j}^A, \end{aligned}$$

where $\{e_j\}$ is any local orthonormal frame field.

Proof. For $\alpha \in \Omega^p(U)$, $s \in \Gamma(E|_U)$, $U \subseteq X$ open, one has, by the previous proposition, that

$$\begin{aligned} d_A(\alpha \otimes s) &= d\alpha \otimes s + (-1)^p \alpha \wedge \nabla^A(s) \\ &= (\sum_j e_j^* \wedge \nabla_{e_j}(\alpha)) \otimes s + (-1)^p \alpha \wedge \sum_j e_j^* \otimes \nabla_{e_j}^A(s) \\ &= \sum_j e_j^* \wedge \Big(\nabla_{e_j}(\alpha) \otimes s + \alpha \otimes \nabla_{e_j}^A(s) \Big) \\ &= \sum_j e_j^* \wedge \nabla_{e_j}^A(\alpha \otimes s), \end{aligned}$$

which proves the first formula. To prove the second one, it might be tempting to use the first formula directly, but this is difficult in general, since we have to do partial integration (explaining the sign in the formula); it works in Euclidean $\mathbb{R}^n$, where one can take $e_j = \partial_j := \frac{\partial}{\partial x^j}$ and compute the adjoint of $\sum_j dx^j \wedge \nabla_{\partial_j}^A$ directly. In the general case, we use the known formula for d_A^*. The problem is local, so we may assume that X is oriented. Let $x \in X$ and pick a positive orthonormal frame field $\{e_j\}$ in an open neighbourhood $U \subseteq X$ of x which is covariantly constant in x, i.e. such that $(\nabla_{e_j})_x = 0$ for all j. Let $s \in \Gamma(E|_U)$ and

$$\psi = e_1^* \wedge \dots \wedge e_{p+1}^* \otimes s.$$

Since $(\nabla e_j^*)_x = 0$ for all j, at the point x we have

$$\begin{aligned} *d_A * \psi &= *d_A(e_{p+2}^* \wedge \cdots \wedge e_n^* \otimes s) \\ &= * \sum_{j=1}^{n} e_j^* \wedge \nabla_{e_j}^A (e_{p+2}^* \wedge \cdots \wedge e_n^* \otimes s) \\ &= * \sum_{j=1}^{p+1} e_j^* \wedge e_{p+2}^* \wedge \cdots \wedge e_n^* \otimes \nabla_{e_j}^A s. \end{aligned}$$

Now $*$ applied to $e_j^* \wedge e_{p+2}^* \wedge \cdots \wedge e_n^*$ can be determined since

$$\begin{aligned} & e_j^* \wedge e_{p+2}^* \wedge \cdots \wedge e_n^* \wedge e_1^* \wedge \cdots \wedge \hat{e_j^*} \wedge \cdots \wedge e_{p+1}^* \\ &\quad = (-1)^{p(n-p-1)} e_j^* \wedge e_1^* \wedge \cdots \wedge \hat{e_j^*} \wedge \cdots \wedge e_n^* \\ &\quad = (-1)^{pn+j-1} e_1^* \wedge \cdots \wedge e_n^*. \end{aligned}$$

Here, $\hat{e_j^*}$ means that e_j^* is removed from the expression. Note also that if $v \in V$ for some vector space V and $\alpha_j \in V^*$, then

$$v \lrcorner (\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{j=1}^{k} (-1)^{j-1} \alpha_j(v) \cdot \alpha_1 \wedge \cdots \wedge \hat{\alpha_j} \wedge \cdots \wedge \alpha_k.$$

This means that

$$\begin{aligned} *d_A * \psi &= \sum_{j=1}^{p+1} (-1)^{pn+j-1} e_1^* \wedge \cdots \wedge \hat{e_j^*} \wedge \cdots \wedge e_{p+1}^* \otimes \nabla_{e_j}^A s \\ &= \sum_{j=1}^{n} (-1)^{pn} e_j \lrcorner (e_1^* \wedge \cdots \wedge e_{p+1}^*) \otimes \nabla_{e_j}^A s \\ &= \sum_{j=1}^{n} (-1)^{pn} e_j \lrcorner \nabla_{e_j}^A (\psi). \end{aligned}$$

We introduced a frame field which was covariantly constant at x, but the last expression is independent of the frame. Thus finally, for any orthornormal frame field $\{e_j\}$,

$$d_A^* \psi = (-1)^{1+pn} * d_A * \psi = -\sum_{j=1}^{n} e_j \lrcorner \nabla_{e_j}^A (\psi).$$

□

3.6.2 Self-adjointness of the Dirac operator

Lemma 3.6.8. *Let V be a Euclidean vector space and let $f : \wedge^* V \xrightarrow{\cong} \mathrm{Cl}(V)$ be the canonical vector space isomorphism. Then for all $v \in V$ and $\varphi \in \wedge^* V$, one has*

$$f(v \wedge \varphi - v^* \lrcorner \varphi) = v \cdot f(\varphi),$$

where $v \in V^*$ is the dual of v with respect to $\langle \cdot, \cdot \rangle$, and $v \cdot f(\varphi)$ is Clifford multiplication.

Proof. Choose an orthonormal basis $e_1, \ldots, e_n$ for V with $v \in \mathbb{R}e_1$. Recall that for $1 \le i_1 < \cdots < i_p \le n$ we have

$$f(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{i_1} \cdots e_{i_p}.$$

If $i_1 = 1$,

$$e_1 e_{i_1} \cdots e_{i_p} = -e_{i_2} \cdots e_{i_p}.$$

Hence for $\varphi = e_{i_1} \wedge \cdots \wedge e_{i_p}$, we have

$$f(e_1 \wedge \varphi - e_1^* \lrcorner \varphi) = e_1 \cdot f(\varphi).$$

Similarly, the formula holds if $i_1 > 1$. □

From now on, to simplify notation write $\wedge^p(X) = \wedge^p T^* X$ and $\wedge^*(X) = \wedge^* T^* X$.

Theorem 3.6.9. *Let X be a Riemannian n-manifold and $E \to X$ an Hermitian vector bundle with connection A. Then under the caoninical vector bundle isomorphism $f : \wedge^* X \otimes E \xrightarrow{\cong} \mathrm{Cl}(X) \otimes E$, the operator $d_A + d_A^*$ on $\wedge^*(X) \otimes E$ corresponds to the Dirac operator D_A on $\mathrm{Cl}(X) \otimes E$.*

Proof. Note first that the vector bundle isomorphism f preserves the connection in the two bundles: The connection in $\wedge^*(X)$ comes from the Levi-Civita connection in $P_{\mathrm{O}}(X) \times_{\mathrm{O}(n)} \wedge^* \mathbb{R}^n$, and the connection in $\mathrm{Cl}(X)$ from $P_{\mathrm{O}}(X) \times_{\mathrm{O}(n)} \mathrm{Cl}(\mathbb{R}^n)$. Since the connection is preserved under the isomorphism $P_{\mathrm{O}}(X) \times_{\mathrm{O}(n)} \wedge^* \mathbb{R}^n \cong P_{\mathrm{O}}(X) \times_{\mathrm{O}(n)} \mathrm{Cl}(\mathbb{R}^n)$, f preserves the connection.

Combining this with the two previous results, for $\varphi \in \Gamma(\wedge^*(X) \otimes E) = \Omega^*(X; E)$, we have

$$\begin{aligned} f((d_A + d_A^*)\varphi) &= f(\sum_j (e_j^* \wedge \nabla^A_{e_j} \varphi - e_j \lrcorner \nabla^A_{e_j} \varphi)) \\ &= \sum_j e_j f(\nabla^A_{e_j} \varphi) = \sum_j e_j \nabla^A_{e_j} (f\varphi) = D_A(\varphi). \end{aligned}$$

□

Proposition 3.6.10. *Let X be a Riemannian n-manifold, and let $\mathbb{S} \to X$ be a complex Dirac bundle (the Proposition could also be formulated for real Dirac bundles). Then the Dirac operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is (formally) self-adjoint.*

Proof. Let $\varphi, \psi \in \Gamma(\mathbb{S})$. Then pointwise on X, since ∇ is Hermitian and since in general $V\langle s, t\rangle = \langle \nabla_V s, t\rangle + \langle s, \nabla_V t\rangle$ for Hermitian connections, we find that

$$\begin{aligned} \langle D\varphi, \psi\rangle &= \sum_j \langle e_j \nabla_{e_j}(\varphi), \psi\rangle = -\sum_j \langle \nabla_{e_j}(\varphi), e_j \psi\rangle \\ &= \sum_j \langle \varphi, \nabla_{e_j}(e_j \psi)\rangle - \sum_j e_j \langle \varphi, e_j \psi\rangle \\ &= \sum_j \langle \varphi, \nabla_{e_j}(e_j) \cdot \psi + e_j \nabla_{e_j}(\psi)\rangle - \sum_j e_j \cdot \alpha(e_j), \end{aligned}$$

where $\alpha \in \Omega^1(X, \mathbb{C})$ is defined by

$$\alpha = \sum_j \langle \varphi, e_j \psi\rangle e_j^*,$$

which is independent of the choice of frame field. Assume now that $\{e_j\}$ is covariantly constant at a given $x \in X$, $(\nabla e_j)_x = 0$ for all j. Then at x, we have

$$\begin{aligned} \sum_j e_j \cdot \alpha(e_j) &= \sum_j (\nabla_{e_j}(\alpha) \cdot e_j + \alpha \cdot \nabla_{e_j}(e_j)) \\ &= \sum_j e_j \lrcorner \nabla_{e_j}(\alpha) = -d^* \alpha. \end{aligned}$$

Therefore, at x we have

$$\langle D\varphi, \psi\rangle = \sum_j \langle \varphi + e_j \nabla_{e_j}(\psi)\rangle + d^*\alpha = \langle \varphi, D\psi\rangle + d^*\alpha,$$

and since α is independent of frame field $\{e_j\}$, this equation holds over all of X. If now at least one of φ and ψ has compact support, then α has compact support, and by Stokes' theorem

$$\int_X d^*\alpha = -\int_X *(d*\alpha) = -\int_X d*\alpha = 0.$$

Hence $\int_X \langle D\varphi, \psi\rangle = \int_X \langle \varphi, D\psi\rangle$. □

Proposition 3.6.11. *If $\nabla = \nabla_A$ is a connection in a vector bundle $E \to X$, then*

$$F_A(V, W) \cdot s = \left(\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]}\right) s$$

Proof. In local coordinates, $x_1, \ldots x_n$ on X, by our definition

$$\begin{aligned} F_A \cdot s = d_A^2 \cdot s &= d_A \left(\sum_j dx^j \otimes \nabla_{\partial_j}(s) \right) \\ &= \sum_{i<j} dx^i \wedge dx^j \otimes [\nabla_{\partial_i}, \nabla_{\partial_j}] s. \end{aligned}$$

Hence for $k < l$, we have

$$\begin{aligned} F_A(\partial_k, \partial_l) \cdot s &= \sum_{i<j} (dx^i \wedge dx^j) \cdot (\partial_k \wedge \partial_l) \otimes [\nabla_{\partial_i}, \nabla_{\partial_j}] s \\ &= [\nabla_{\partial_k}, \nabla_{\partial_l}] s. \end{aligned}$$

□

11th lecture, October 3rd 2011

3.7 Distributions and curvature

Definition 3.7.1. A k-dimensional *distribution* on a manifold P is a rank k subbundle of TP. The distribution A is called *involutive* if $[V, W] \in \Gamma(A)$, whenever $V, W \in \Gamma(A)$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields on P.

Theorem 3.7.2 (Frobenius' theorem)**.** *A distribution A is involutive if and only if for every $p \in P$, there is an open neighbourhood $U \subseteq P$ of p and a diffeomorphism $\varphi : U \xrightarrow{\cong} \mathbb{R}^N$ such that $\varphi_*(A_p) = \mathbb{R}^k \times \{0\}$ for all $p \in U$.*

Now let X and P be smooth manifolds and let $P \xrightarrow{\pi} X$ be a submersion. Then $K := \ker(T\pi)$ is a subbundle of TP. A π-*distribution* in P is a subbundle $A \subseteq TP$ such that π_* maps A_p isomorphically onto $T_{\pi(p)}X$ for every $p \in P$. Then π_* induces an isomorphism of vector bundles $A \xrightarrow{\cong} \pi^*TX$, and we have a direct sum decomposition

$$TP = A \oplus K.$$

Sections of A (resp. K) are called *horizontal* (resp. *vertical*) vector fields on P. For every vector field V on X, let V^A be the horizontal vector field on P such that $\pi_*(V^A)_p = V_{\pi(p)}$ for every $p \in P$.

Such a π-distribution has a *curvature*. To define this, let V, W be vector fields on X and set

$$\mathbb{F}(V,W) := [V,W]^A - [V^A, W^A].$$

This is a vertical vector field on P, and one easily verifies that

$$\mathbb{F}(fV,W) = \mathbb{F}(V,fW) = \pi^* f \cdot \mathbb{F}(V,W)$$

for all $f \in C^\infty(X)$. Hence there is for every $p \in P$ exactly one skew-symmetric bilinear map

$$(\mathbb{F}_A)_p = \mathbb{F}_p : T_{\pi(p)}X \times T_{\pi(p)}X \to K_p$$

such that $\mathbb{F}(V,W)_p = \mathbb{F}_p(V_p, W_p)$ for all $V, W \in \Gamma(TX)$ and $p \in P$.

Definition 3.7.3. Define the *curvature* of A to be $\mathbb{F}_A := \{\mathbb{F}(A)_p\} \in \Gamma(\pi^* \wedge^2 (X) \otimes K)$.

Proposition 3.7.4. *We have $\mathbb{F}_A = 0$ if and only if A is involutive.*

Proof. If A is involutive, then $\mathbb{F}_A = 0$ since $[V^A, W^A]$ is then the horizontal lift of $[V,W]$.

Suppose now $\mathbb{F}_A = 0$. To prove that A is involutive we may assume $X = \mathbb{R}^n$. Then every horizontal vector field on P is a linear combination of $\{\partial_j^A\}$. Recall that in general $[fV,W] = f[V,W] - (Wf)V$ for functions f and vector fields V, W. Thus if $a^j, b^j \in C^\infty(P)$, $j,k = 1,\dots,n$, then

$$\begin{aligned} [\sum_j a^j \partial_j^A, \sum_k b^k \partial_k^A] &= \sum_{j,k} a^j b^k [\partial_j^A, \partial_k^A] + H \\ &= \sum_{j,k} a^j b^k [\partial_j, \partial_k]^A + H = H \end{aligned}$$

for some horizontal vector field H. □

3.8 Curvature of connections in principal bundles

Let $\pi : P \to X$ be a principal G-bundle, S a finite dimensional real vector space, and $\rho : G \to \mathrm{Aut}(S)$ a representation. Set $E := P \times_\rho S$. There is a canonical bijection

$$\begin{aligned} \{G\text{-equivariant smooth maps } P \to S\} &\to \Gamma(E) \\ f &\mapsto \tilde{f}. \end{aligned}$$

Here, f is called G-equivariant if $f(pg^{-1}) = g \cdot f(p)$ for all $p \in P$, $g \in G$. Set

$$\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} LG,$$

where $\mathrm{Ad} : G \to \mathrm{Aut}(LG)$ is the adjoint representation. Thus $\mathrm{Ad}(P)$ is a bundle of Lie algebras. Now let A be a *connection* in P, i.e. a G-invariant π-distribution. Then each vertical fibre can be described as $K_p = \{p \cdot \xi \mid \xi \in LG\}$, where

$$p \cdot \xi := \frac{d}{dt}|_0 p \cdot \exp(t\xi) \in K_p \subseteq T_pP.$$

Let $\omega \in \Omega^1(P; LG)$ be the *connection form* of A, i.e. the form satisfying

$$A_p = \ker(\omega_p),\ \omega(p \cdot \xi) = \xi,\ (r_{g^{-1}})^*\omega = \mathrm{Ad}_g \cdot \omega.$$

The *curvature* $F = F_A \in \Omega^2(X; \mathrm{Ad}(P))$ is defined as follows: Let V, W be vector fields on X. Then $\mathbb{F}(V,W)$ is a vertical G-invariant vector field on P. Hence

$$\omega \cdot \mathbb{F} : P \to LG$$

is a G-equivariant map, and the corresponding section of $\mathrm{Ad}(P)$ is by definition $F(V,W)$. The value of $F(V,W)$ at a point $p \in X$ depends only on the values of V, W at p and so we get an element of $\Omega^2(X; \mathrm{Ad}(P))$.

Remark 3.8.1. To relate this definition of curvature to that of [KN96], recall that if Y, Z are two vector fields and α a 1-form on a manifold, then

$$d\alpha(Y, Z) = Y \cdot \alpha(Z) - Z \cdot \alpha(Y) - \alpha([Y, Z]).$$

Hence in our case,

$$\omega \cdot \mathbb{F}(V, W) = -\omega([V^A, W^A]) = d\omega(V^A, W^A).$$

Definition 3.8.2. A connection A is called *trivial* if P admits a horizontal section, or equivalently, if there exists an isomorphism $P \to X \times G$ of G-bundles which maps A to the product connection (here a section $s \in \Gamma(P)$ is called horizontal if $s_x(V)$ is horizontal for every $x \in X, v \in T_pX$).

A connection A is called *locally trivial* if every $x \in X$ has an open neighbourhood $U \subseteq X$ such that $A|_U$ is trivial.

Theorem 3.8.3. *We have $F_A = 0$ if and only if A is locally trivial.*

Proof. That $F_A = 0$ is equivalent to $\mathbb{F}_A = 0$ which we saw happens exactly when A is involutive. By Frobenius' theorem, this is the same as $P \to X$ admitting local horizontal sections. □

We will now relate this definition of curvature to the definition of curvature of connection in vector bundles. The homomorphism $\rho : G \to \mathrm{Aut}(S)$ induces a homomorphism $L\rho : LG \to \mathrm{End}(S)$ of Lie algebras. Since $L\rho$ is G-equivariant, it induces a vector bundle homomorphism

$$\tilde{\rho} : \mathrm{Ad}(P) \to \mathrm{End}(E).$$

Proposition 3.8.4. *Let ∇ be the connection in E induced by A. Then*

$$F(\nabla) = \tilde{\rho} \cdot F(A).$$

Proof. Let V, W be vector fields on X and to simplify notation set

$$Z := [V, W]^A - [V^A, W^A].$$

Let $f : P \to S$ be a G-equivariant map, corresponding to a section $\tilde{f}$ of E. By definition,

$$\nabla_V(\tilde{f}) = (V^A f)^\sim.$$

Since now $\nabla_V \nabla_W \tilde{f} = \nabla_V (W^A f)^\sim = (V^A W^A f)^\sim$, the curvature of ∇ becomes

$$\begin{aligned} F_\nabla(V, W) \cdot \tilde{f} &= (\nabla_V \nabla_W - \nabla_W \nabla_V - \nabla_{[V,W]})\tilde{f} \\ &= [(V^A W^A - W^A V^A - [V, W]^A) f]^\sim = -(Z \cdot f)^\sim. \end{aligned}$$

If $\xi \in LG, p \in P$, then

$$\begin{aligned} df(p \cdot \xi) &= \frac{d}{dt}|_0 f(p \cdot \exp(t\xi)) = \frac{d}{dt}|_0 \rho(\exp(-t\xi)) \cdot f(p) \\ &= -L\rho(\xi) \cdot f(p). \end{aligned}$$

Hence,

$$-(Z \cdot f)(p) = -df(p \cdot \omega(Z_p)) = L\rho(\omega(Z_p)) \cdot f(p).$$

and the corresponding section of E becomes

$$-(Z \cdot f)^\sim = (\tilde{\rho} \cdot F_A(V, W)) \cdot \tilde{f},$$

since the equivariant map corresponding to this section is exactly the composition $P \to LG \times S \to \mathrm{End}(S) \times S \to S$ mapping

$$p \mapsto (\omega(Z_p), f(p)) \mapsto (L\rho(\omega(Z_p)), f(p)) \mapsto L_\rho(\omega(Z_p)) \cdot f(p).$$

□

12th lecture, October 5th 2011

Proposition 3.8.5. *Let $P \to X$ be a G-bundle, $Q \to X$ an H-bundle, and $u : P \to Q$ an i-homomorphism with respect to $\varphi : G \to H$. Let A be a connection in P and set $B := u(A)$. Then*

(i) the pullback of the connection form of B is $u^*\omega_B = L\varphi \cdot \omega_A$, and

(ii) the curvature of B is $F_B = \tilde{\varphi} \cdot F_A$, where $\tilde{\varphi}$ is the map

$$\mathrm{Ad}(P) = P \times_G LG \xrightarrow{\tilde{\varphi}} Q \times_H LH = \mathrm{Ad}(Q)$$
$$[p, \xi] \mapsto [u(p), L\varphi(\xi)].$$

The proof of the first statement can be found in [KN96], and the other one is left as an exercise.

3.9 The Bochner formulae

Recall that a orthogonal or Hermitian vector bundle $\mathbb{S} \to X$ with a connection is called a Dirac bundle if we have a bundle homomorphism $\mathrm{Cl}(X) \to \mathrm{End}(\mathbb{S})$ such that unit tangent vectors preserve the inner product on $\mathbb{S}$, and such that the connection is compatible with Clifford multiplication.

Theorem 3.9.1. *Let X be an n-dimensional Riemannian manifold and $\mathbb{S} \to X$ a (real or complex) Dirac bundle on X, and let ∇ denote the connection on $\mathbb{S}$. Then the Dirac operator $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ satisfies*

$$D^2 = \nabla^*\nabla + \mathcal{R},$$

where $\mathcal{R}$ is the zeroth order operator

$$\mathcal{R} = \sum_{i<j} e_i e_j F(e_i, e_j) = \frac{1}{2} \sum_{i,j} e_i e_j F(e_i, e_j).$$

Here $F = F(\nabla)$, and $\{e_i\}$ is a local orthonormal frame field on X. It is clear that the last expression is independent of the choice of frame field.

Proof. Let $x \in X$ and choose a local orthonormal frame field $\{e_i\}$ around x with $(\nabla e_i)_x = 0$ for all i. Then for $\varphi \in \Gamma(\mathbb{S})$ one has at x that

$$[e_i, e_j] = \nabla_{e_i}(e_j) - \nabla_{e_j}(e_i) = 0,$$

and

$$\begin{aligned} D^2\varphi &= \sum_i e_i \nabla_{e_i} \sum_j e_j \nabla_{e_j}(\varphi) = \sum_{i,j} e_i e_j \nabla_{e_i} \nabla_{e_j}(\varphi) \\ &= -\sum_i (\nabla_{e_i})^2 \varphi + \sum_{i<j} e_i e_j [\nabla_{e_i}, \nabla_{e_j}]\varphi \\ &= -\sum_i (\nabla_{e_i})^2 \varphi + \sum_{i<j} e_i e_j F(e_i, e_j)\varphi \\ &= \nabla^*\nabla\varphi + \mathcal{R}\varphi. \end{aligned}$$

The last equality follows since at x we have (with $\nabla = \nabla^A$),

$$\begin{aligned} \nabla^*\nabla\varphi = d_A^* d_A &= -\sum_i e_i \lrcorner \nabla_{e_i} \sum_j e_j^* \otimes \nabla_{e_j}(\varphi) \\ &= -\sum_{i,j} e_i \lrcorner (e_j^* \otimes \nabla_{e_i} \nabla_{e_j}(\varphi)) \\ &= -\sum_i (\nabla_{e_i})^2 \varphi. \end{aligned}$$

Note that in this last computation we did not use the fact that we have a Dirac bundle. □

Recall that we have an identification of vector spaces $\mathrm{Cl}(X) = \wedge^*(X)$, and in this untwisted case the Dirac operator D on $\mathrm{Cl}(X)$ corresponds to $d + d^*$ on $\wedge^*(X)$. For a proof of the following Theorem, see [LM89].

Theorem 3.9.2. *Let X be a Riemannian manifold. Define the* Hodge–Beltrami operator

$$\Delta = (d + d^*)^2 = dd^* + d^*d : \Omega^*(X) \to \Omega^*(X).$$

Then on 1-forms,

$$\Delta = \nabla^*\nabla + \mathrm{Ric} : \Omega^1(X) \to \Omega^1(X).$$

Theorem 3.9.3 (Lichnerowicz formula)**.** *Let X be an oriented Riemannian n-manifold with a spin-structure $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}(X)$. Let $\mathbb{S} \to X$ be a spinor bundle and $D : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ the Dirac operator. Then*

$$D^2 = \nabla^*\nabla + \frac{s}{4},$$

where $s : X \to \mathbb{R}$ is the scalar curvature.

We need two lemmas as preparation for the proof. We consider the Riemannian curvature tensor $R \in \Omega^2(X; \mathfrak{so}(TX))$. Here $\mathfrak{so}(TX) \cong \mathrm{Ad}(P_{\mathrm{Spin}})$ acts on the Dirac bundle, and we need to understand how.

Lemma 3.9.4. *Under the canonical vector space isomorphism $c : \wedge^*\mathbb{R}^n \to \mathrm{Cl}(n)$, the Lie algebra $\mathfrak{spin}(n)$ is the image of $\wedge^2\mathbb{R}^n$.*

Proof. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$. As we have seen earlier, the path

$$\gamma(t) = \cos t + \sin t \cdot e_i e_j$$

in $\mathrm{Cl}(n)$ is in $\mathrm{Spin}(n)$ when $i \neq j$. Hence

$$e_i e_j = \gamma'(0) \in \mathfrak{spin}(n).$$

This proves the Lemma since

$$\dim \wedge^2(\mathbb{R}^n) = \dim \mathrm{SO}(n) = \dim \mathrm{Spin}(n).$$

□

Lemma 3.9.5. *Let $\rho = \mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the usual double cover. Then the inverse of the Lie algebra isomorphism $L\rho : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ is given by*

$$(L\rho)^{-1}(A) = \frac{1}{2}\sum_{k<l} \langle Ae_k, e_l \rangle e_k e_l = \frac{1}{4}\sum_{k,l} \langle Ae_k, e_l \rangle e_k e_l$$

for $A \in \mathfrak{so}(n)$.

Proof. Recall that for $\varphi \in \mathrm{Spin}(n)$, $v \in \mathbb{R}^n$, we have

$$\rho(\varphi) \cdot v = \varphi v \varphi^{-1}.$$

Hence

$$(L\rho)(\xi) \cdot v = \xi v - v\xi$$

so for $i < j$ we have

$$(L\rho)(e_i e_j) \cdot e_k = \begin{cases} 0 & \text{if } k \neq i, j \\ 2e_j & \text{if } k = i \\ -2e_i & \text{if } k = j \end{cases}$$

and hence,

$$\sum_{k<l} \langle (L\rho)(e_i e_j) \cdot e_k, e_l \rangle e_k e_l = \langle 2e_j, e_j \rangle e_i e_j = 2e_i e_j.$$

□

Proof of Theorem 3.9.3. Assume for concreteness that the Dirac bundle is real (the complex case is similar). Let $\mathrm{Cl}(n) \to \mathrm{End}_{\mathbb{R}}(S)$ be an orthogonal representation and $\mathbb{S} := P_{\mathrm{Spin}} \times_{\mathrm{Spin}(n)} S$. The statement of the theorem is local, so we may assume that P_{Spin} is trivial, i.e. that P_{Spin} admits a (smooth) section. This provides a section of $P_{\mathrm{SO}}(X)$, i.e. a global positive orthonormal frame field $\{e_i\}$ on X, as well as trivializations of the adjoint bundles of P_{Spin} and $P_{\mathrm{SO}}(X)$. Let A be the Levi-Civita connection in $P_{\mathrm{SO}}(X)$ and B the connection in P_{Spin} which maps to A. Then under the isomorphism

$$\Omega^2(X; \mathfrak{spin}(n)) \overset{\cong}{\to} \Omega^2(X; \mathfrak{so}(n)),$$

F_B maps to F_A by Proposition 3.8.5. The Riemannian curvature tensor has components

$$R_{ijkl} = -\langle F_A(e_i, e_j) \cdot e_k, e_l \rangle,$$

and in terms of the Riemannian curvature, by Lemma 3.9.5 the curvature of B becomes

$$\begin{aligned} F_B(e_i, e_j) &= \frac{1}{4} \sum_{k,l} \langle F_A(e_i, e_j) \cdot e_k, e_l \rangle e_k e_l \\ &= -\frac{1}{4} \sum_{k,l} R_{ijkl} \cdot e_k e_l. \end{aligned}$$

Recall from Riemannian geometry the Bianchi identity which says that $R_{ijkl} + R_{jkil} + R_{kijl} = 0$. From Theorem 3.9.1, the curvature term is given by

$$\begin{aligned} \mathcal{R} &= \frac{1}{2} \sum_{i,k} e_i e_j F_B(e_i, e_j) = -\frac{1}{8} \sum_{ijkl} R_{ijkl} e_i e_j e_k e_l \\ &= -\frac{1}{8} \sum_{l} \sum_{i<j<k} (R_{ijkl} + R_{jkil} + R_{kijl} - R_{jikl} - R_{ikjl} - R_{kjil}) e_i e_j e_k e_l \\ &\quad -\frac{1}{8} \sum_{i,j,l,k:=i} R_{ijil} e_i e_j e_i e_l - \frac{1}{8} \sum_{i,j,l,k:=j} R_{ijjl} e_i e_j e_j e_l \\ &= -\frac{1}{8} \sum_{i,j,l} (R_{ijil} e_j e_l + R_{jijl} e_i e_l) \\ &= -\frac{1}{4} \sum_{i,j,l} R_{ijil} e_j e_l = -\frac{1}{4} \sum_{j,l} \mathrm{Ric}(e_j, e_l) e_j e_l \\ &= \frac{1}{4} \sum_{j} \mathrm{Ric}(e_j, e_j) = \frac{s}{4}. \end{aligned}$$

□

Corollary 3.9.6. *Let X be a closed spin manifold (i.e. an oriented Riemannian manifold with a spin structure) and $\mathbb{S} \to X$ a spinor bundle. If X has positive scalar curvature, then there are no harmonic spinors (i.e.* $\ker(D) = 0$*).*

Proof. Let $\varphi \in \Gamma(\mathbb{S})$ with $D\varphi = 0$. Then

$$\begin{aligned} 0 &= \int_X \langle D\varphi, D\varphi \rangle = \int_X \langle D^2\varphi, \varphi \rangle = \int_X \langle (\nabla^*\nabla + \frac{s}{4})\varphi, \varphi \rangle \\ &= \int_X |\nabla\varphi|^2 + \frac{s}{4}|\varphi|^2, \end{aligned}$$

which implies that $\varphi = 0$. □

13th lecture, November 1st 2011

We will prove a formula analogous to the Lichnerowicz formula for $\mathrm{Spin}^c(n)$-bundles. As a warmup we need to say a little bit about connections in these.

The projection

$$\mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathrm{Spin}^c(n)$$

is a double cover and induces an isomorphism of Lie algebras, so we can identify

$$\mathfrak{spin}^c(n) \cong \mathfrak{spin}(n) \oplus i\mathbb{R}.$$

This splitting is preserved by the adjoint action of $\mathrm{Spin}^c(n)$. The action is trivial on $i\mathbb{R}$, since $\mathrm{U}(1)$ is in the center of $\mathrm{Spin}^c(n)$. Now let X be an oriented Riemannian n-manifold and let

$$P_{\mathrm{Spin}^c(n)} \to P_{\mathrm{SO}}(X)$$

be a spin^c-structure. The decomposition of $\mathfrak{spin}^c(n)$ induces a decomposition

$$\mathrm{Ad}(P_{\mathrm{Spin}^c}) = E \oplus i\underline{\mathbb{R}},$$

where $\underline{\mathbb{R}} = X \times \mathbb{R}$. If A is any connection in P_{Spin^c} then we write $F_A + F'_A + F''_A$, where $F'_A \in \Omega^2(X; E)$, $F''_A \in \Omega^2(X; i\mathbb{R})$.

Recall that we have projections $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ and $\mathrm{Spin}^c(n) \to \mathrm{U}(1)$. The composition $\mathrm{Spin}(n) \times \mathrm{U}(1) \to \mathrm{Spin}^c(n) \to \mathrm{U}(1)$ maps $(x, z) \mapsto z^2$. On the level of Lie algebras, the corresponding map $\mathfrak{spin}(n) \oplus i\mathbb{R} = \mathfrak{spin}^c(n) \to i\mathbb{R}$ maps $(\xi, x) \mapsto 2x$. We denote by $\check{A}$ the image of A in the $\mathrm{U}(1)$-bundle $P_{\mathrm{U}} \to X$ associated to the spin^c-structure. Then

$$F_{\check{A}} = 2F''_A.$$

We call A a *spin connection* if it maps to the Levi-Civita connection in $P_{\mathrm{SO}}(X)$.

Theorem 3.9.7 (Bochner formula in the spin^c case)**.** *Let $P_{\mathrm{Spin}^c(n)} \to P_{\mathrm{SO}}(n)$ as above and let $\mathbb{S} \to X$ be a complex spinor bundle associated to the* spin^c*-structure. Then the Dirac operator $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ satisfies*

$$D_A^2 = \nabla_A^2 \nabla_A + F''_A + \frac{s}{4},$$

where s is the scalar curvature of X, and where $F''_A \in \Gamma(\mathbb{C}\mathrm{l}(X))$ acts on $\Gamma(\mathbb{S})$ by Clifford multiplication.

Proof. Under the canonical homomorphism $i \wedge^2 (X) \hookrightarrow \mathbb{C}\mathrm{l}(X)$,

$$F''_A = \sum_{i<j} e_i^* \wedge e_j^* \otimes F''_A(e_i, e_j) \mapsto \sum_{i<j} e_i e_j \otimes F''_A(e_i, e_j).$$

Theorem 3.9.1 and the same calculation as in Theorem 3.9.3 yields the curvature term

$$\mathcal{R} = \sum_{i<j} e_i e_j (F'_A(e_i, e_j) + F''_A(e_i, e_j)) = \frac{s}{4} + F''_A.$$

□

4 Seiberg–Witten theory

4.1 Spin(3) and Spin(4)

Let $\mathbb{H}$ denote the algebra of quaternions, and Sp(1) the group of unit quaternions. Then we can identify $^{(}1)$ with the pure quaternions $\mathbb{H}_0 = \mathrm{span}\{i, j, k\}$, which we identify with $\mathbb{R}^3$. Therefore we have the sequence

$$1 \to \{\pm 1\} \to \mathrm{Sp}(1) \overset{\mathrm{Ad}}{\to} \mathrm{SO}(3) \to 1,$$

and so we have the following diagram:

$$\begin{array}{ccc} \mathrm{Sp}(1) & \xrightarrow{\cong} & \mathrm{Spin}(3) \\ & {\scriptstyle \mathrm{Ad}} \searrow \quad \swarrow & \\ & \mathrm{SO}(3) & \end{array}$$

For $x, y \in \mathrm{Sp}(1)$ we define

$$f(x, y) : \mathbb{H} \to \mathbb{H}, \quad q \mapsto xqy^{-1}.$$

Then $f(x, y) \in \mathrm{SO}(4)$.

Proposition 4.1.1. *The map* $f : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ *is a surjective homomorphism of Lie groups and* $\ker(f) = \{\pm(1, 1)\}$.

Proof. That $f(x, y) = 1$ is equivalent to $xqy^{-1} = q$ for every $q \in \mathbb{H}$, which is the same as saying that $x = qyq^{-1}$ for every $q \in \mathrm{Sp}(1)$. Recall that the splitting $\mathbb{H} = \mathbb{R} \oplus \mathbb{H}_0$ is preserved by conjugation by q, which means that y has to be real and so that $x = y = \pm 1$. Since the kernel is discrete, the induced map on Lie algebras $Lf :^{(} 1) \oplus^{(} 1) \to \mathfrak{so}(4)$ is injective. Since

$$\dim \mathrm{SO}(4) = 6 = 2 \dim \mathrm{Sp}(1),$$

the map Lf is an isomorphism. Recall that the exponential map defines a local diffeomorphism, and since SO(4) is connected, f is surjective by the general fact that topological groups are generated by any neighbourhood of the identity. ☐

We thus obtain an isomorphism $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Spin}(4)$ as a lift of f,

$$\begin{array}{ccc} \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \xrightarrow{\cong} & \mathrm{Spin}(4) \\ & {\scriptstyle f} \searrow \quad \swarrow & \\ & \mathrm{SO}(4) & \end{array}$$

There is also a nice description of the inverse map: The standard action of SO(4) on $\mathbb{R}^4$ induces an orthogonal representation on $\wedge^2\mathbb{R}^4$ which preserves the splitting

$$\wedge^2\mathbb{R}^4 = \wedge^+ \oplus \wedge^-$$

into self-dual and anti-self-dual forms (i.e. ± 1-eigenvectors of the Hodge star operator). The action therefore defines a homomorphism

$$g : \mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3).$$

Proposition 4.1.2. *The composite homomorphism*

$$\mathrm{Sp}(1) \times \mathrm{Sp}(1) \overset{f}{\to} \mathrm{SO}(4) \overset{g}{\to} \mathrm{SO}(3) \times \mathrm{SO}(3)$$

is $\mathrm{Ad} \times \mathrm{Ad}$.

Proof. Since Sp(1) is connected, it is enough to check that $L(g \circ f) = \mathrm{ad} \times \mathrm{ad}$. This can be done explicitly in terms of a basis on $^{(}1)$, which we leave as an exercise. ☐

Corollary 4.1.3. *The map g is surjective, and* $\ker(g) = \{\pm 1\}$.

In summary, we have the following diagram, where the upper map is an isomorphism by general covering space theory:

$$\begin{array}{ccc} \mathrm{Spin}(4) & \xrightarrow{\cong} & \mathrm{Spin}(3) \times \mathrm{Spin}(3) \\ \downarrow & & \downarrow \\ \mathrm{SO}(4) & \xrightarrow[g]{} & \mathrm{SO}(3) \times \mathrm{SO}(3) \end{array}$$

4.2 The Seiberg–Witten equations

Recall that the Clifford algebra Cl(4) of Euclidean $\mathbb{R}^4$ with complexification $\mathbb{C}\mathrm{l}(4) = \mathrm{Cl}(4) \otimes \mathbb{C}$. Let $e_1, \dots, e_4$ be the standard basis for $\mathbb{R}^4$ with real volume element $\omega = e_1e_2e_3e_4$ and complex volume element $\omega_{\mathbb{C}}$, satisfying $\omega_{\mathbb{C}}^2 = 1$, which in dimension 4 is given by $\omega_{\mathbb{C}} = -\omega$. The Clifford algebra splits as $\mathrm{Cl}(4) = \mathrm{Cl}^+ \oplus \mathrm{Cl}^-$, where

$$\mathrm{Cl}^{\pm} = \{\varphi \in \mathrm{Cl}(4) \mid \omega_{\mathbb{C}}\varphi = \pm\varphi\}.$$

Note that $\mathrm{Cl}^{\pm}$ are not ideals, since ω is not central in dimension 4. Let $h : \wedge^*\mathbb{R}^4 \overset{\cong}{\to} \mathrm{Cl}(4)$ be the canonical linear isomorphism. We want to relate the splitting of $\lambda^2\mathbb{R}^4$ into self-dual and anti-self-dual forms with the splitting of Cl(4).

Lemma 4.2.1. *For $\varphi \in \wedge^2\mathbb{R}^4$ we have*

$$h(*\varphi) = \omega_{\mathbb{C}} \cdot h(\varphi).$$

Proof. It suffices to show this for $\varphi = e_1 \wedge e_2$. The left hand side becomes

$$h(*\varphi) = h(e_3 \wedge e_4) = e_3e_4,$$

and the right hand side

$$\omega_{\mathbb{C}} \cdot h(\varphi) = -e_1e_2e_3e_4 \cdot e_1e_2 = e_3e_4.$$

☐

Corollary 4.2.2. *We have $\varphi \in \wedge^{\pm}$ if and only if $h(\varphi) \in \mathrm{Cl}^{\pm}$.*

Let $\rho : \mathbb{C}\mathrm{l}(4) \to \mathrm{End}_{\mathbb{C}}(S)$ be irreducible unitary representation (recall that *unitary* means that unit vectors in $\mathbb{R}^4$ map to unitary endomorphisms). Up to isomorphism, there is a unique such representation, so $\dim_{\mathbb{C}} S = 4$ and

$$S = S^+ \oplus S^-,$$

where $S^{\pm} = \{x \in S \mid \omega_{\mathbb{C}}x = \pm x\}$. Since $\mathbb{C}\mathrm{l}_0(4) \cdot S^{\pm} \subseteq S^{\pm}$, and $\mathbb{C}\mathrm{l}_0(4)$ is a subalgebra of $\mathbb{C}\mathrm{l}(4)$. Thus ρ induces representations

$$\rho_{\pm} : \mathbb{C}\mathrm{l}_0(4) \to \mathrm{End}_{\mathbb{C}}(S^{\pm}).$$

Furthermore, $\mathrm{Cl}^{\pm} \cdot S^{\pm} \subset S^{\pm}$, so

$$\wedge^{\pm} \cdot S^{\mp} \subseteq S^+ \cap S^- = 0.$$

If $0 \neq v \in \mathbb{R}^4$, then $\rho(v)$ restricts to an isomorphism $S^{\pm} \overset{\cong}{\to} S^{\pm}$, so $\dim_{\mathbb{C}} S^{\pm} = \frac{1}{2} \dim_{\mathbb{C}} S = 2$.

For any finite dimensional Hermitian vector space W, define

$$\begin{aligned} \mathrm{Herm}(W) &:= \{\text{Hermitian endomorphisms of } W\}, \\ \mathrm{Herm}_0(W) &:= \{A \in \mathrm{Herm}(W) \mid \mathrm{tr}(A) = 0\}. \end{aligned}$$

Note that an element of $\mathrm{Herm}(S^+)$ is of the form

$$\begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix}, \quad a, b \in \mathbb{R}, c \in \mathbb{C},$$

and so it follows that

$$\begin{aligned} \dim_{\mathbb{R}} \mathrm{Herm}(S^+) &= 4, \\ \dim_{\mathbb{R}} \mathrm{Herm}_0(S^+) &= 3 = \dim \wedge^+. \end{aligned}$$

Proposition 4.2.3. *The representation ρ_+ restricts to an isomorphism*

$$i\wedge^+ \xrightarrow{\cong} \mathrm{Herm}_0(S^+).$$

Proof. If $v, w \in \mathbb{R}^4$, $|v| = |w| = 1$, $v \perp w$, then the product $\rho(vw) = \rho(v)\rho(w)$ is unitary. Since $(vw)^2 = -1$, $\rho(vw)$ is skew-Hermitian, so $\rho(ivw)$ is Hermitian. That proves that $\rho_0(i\wedge^+) \subseteq \mathrm{Herm}(S^+)$.

Since SO(4) acts transitively on the unit sphere in $\wedge^+$, any element of $\wedge^+$ has the form

$$t(a_1a_2 + a_3a_4),$$

where $a_1, \dots, a_4$ is a positive orthonormal basis for $\mathbb{R}^4$ and $t \in \mathbb{R}$. Set $\varphi = \frac{i}{2}(a_1a_2 + a_3a_4)$. Then

$$\varphi^2 = -\frac{1}{4}(-2 + 2\omega) = \frac{1}{2}(1 + \omega_{\mathbb{C}}).$$

Since $\omega_{\mathbb{C}}$ acts identically on S^+, we find $\rho_+(\varphi)^2 = \rho_+(\varphi^2) = 1$, which means that $\rho_+|_{i\wedge^+}$ is injective.

What remains to be proved is that $\mathrm{tr}(\rho_+(\varphi)) = 0$, since then ρ_+ maps $i\wedge^+$ to $\mathrm{Herm}_0(S^+)$, and the two spaces have the same dimension. We see that $\rho_+(\varphi)$ has eigenvalues ± 1, so we have an eigenspace decomposition $S^+ = E_1 \oplus E_{-1}$, and we need to show that these E_1 and E_{-1} have the same dimension, since $\mathrm{tr}(\rho_+(\varphi)) = \dim E_1 - \dim E_{-1} = 0$. Since $\varphi a_2 a_3 = -a_2 a_3 \varphi$, multiplication with a_2a_3 interchanges the eigenspaces, and since $(a_2a_3)^2 = -1$, $\rho(a_2a_3)$ incudes an isomorphism $E_1 \xrightarrow{\cong} E_{-1}$. □

14th lecture, November 3rd 2011

We are now at the point where we can write down the Seiberg–Witten equations. Recall that we considered an irreducible representation $\rho : \mathbb{C}\mathrm{l}(4) \to \mathrm{End}_{\mathbb{C}}(S)$.

Definition 4.2.4. For any $\varphi \in S$, define $\tilde{q} \in \mathrm{Herm}_0(S^+)$ by

$$\tilde{q}(\varphi) \cdot \psi = \langle \psi, \varphi \rangle \varphi - \frac{|\varphi|^2}{2} \psi.$$

Let $q : S \to i\wedge^+$ be the quadratic map, defined by letting $q(\varphi)$ be the element of $i\wedge^+$ which maps to to $\tilde{q}$ under the isomorphism $\rho_+|_{i\wedge^+} i\wedge^+ \xrightarrow{\cong} \mathrm{Herm}_0(S^+)$ from Prop. 4.2.3.

Note that

$$\tilde{q}(\varphi) \cdot \psi = \begin{cases} \frac{|\varphi|^2}{2} \cdot \varphi & \text{if } \psi = \varphi \\ -\frac{|\varphi|^2}{2} \cdot \varphi & \text{if } \psi \perp_{\mathbb{C}} \varphi \end{cases}.$$

Here we write $\perp_{\mathbb{C}}$ to emphasize that the form is Hermitian.

Now let X be any oriented Riemannian 4-manifold with a spin^c-structure $P_{\mathrm{Spin}^c} \to P_{\mathrm{SO}}(X)$. Let $\mathbb{S} \to X$ be a complex spinor bundle associated to a representation $\rho : \mathbb{C}\mathrm{l}(4) \to \mathrm{End}_{\mathbb{C}}(S)$ as above. Let

$$\mathbb{S} = \mathbb{S} \oplus \mathbb{S}^-$$

where $\mathbb{S}^{\pm}_x$ is the (± 1)-eigenspace of the complex volume element in $\mathbb{C}\mathrm{l}(X)_x$ for any $x \in X$.

Definition 4.2.5. A *configuration* on X is a pair (A, Φ) where A is a spin connection in P_{Spin^c} and $\Phi \in \Gamma(\mathbb{S}^+)$. Denote by $\mathcal{C}$ the space of all configurations (A, Φ).

Recall that the curvature of a spin connection A decomposes as $F_A = F_A + F''_A$, where $F''_A \in \Omega^2(X; i\mathbb{R})$. Given a pertubation parameter $\mu \in \Omega^2(X)$, the *Seiberg–Witten equations* for a configuration (A, Φ) are

$$(F''_A + i\mu)^+ = Q(\Phi),$$
$$D_A\Phi = 0,$$

where $Q(\Phi)_x = q(\Phi_x)$ for $x \in X$. In short, we will write these as the SW_μ-equations.

Note that the complex volume element $\omega_{\mathbb{C}}$ is invariant under the action of SO(4), so $\omega_{\mathbb{C}}$ is a section of $\mathbb{C}\mathrm{l}(X) = P_{\mathrm{SO}}(X) \times_{\mathrm{SO}(4)} \mathbb{C}\mathrm{l}(4)$, and $\nabla\omega_{\mathbb{C}} = 0$, where ∇ is the Levi–Civita connection in $\mathbb{C}\mathrm{l}(S)$. This implies that the connection ∇^A in $\mathbb{S}$, induced by the Levi–Civita connection and A, preserves the decomposition $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ (i.e. if $V \in \Gamma(TX), \Phi \in \Gamma(\mathbb{S}^{\pm})$, then $\nabla^A_V\Phi \in \Gamma(\mathbb{S}^{\pm})$).

Hence D_A maps $\Gamma(\mathbb{S}^{\pm})$ into $\Gamma(\mathbb{S}^{\mp})$ (recalling that $D_A\Phi = \sum_{j=1}^4 e_j \nabla^A_{e_j}(\Phi)$ and $e_j\omega_{\mathbb{C}} = -\omega_{\mathbb{C}}e_j$).

4.3 Gauge transformations

Let $\mathcal{G} := \{\text{smooth maps } X \to \mathrm{U}(1)\}$ be the group of gauge transformations. This is the right thing to do, since the spin connection A is essentially a U(1)-connection (see below), and automorphisms of U(1) are exactly elements of $\mathcal{G}$. We will define a group action of $\mathcal{G}$ on $\mathcal{C}$.

Let $u \in \mathcal{G}, (A, \Phi) \in \mathcal{C}$. We define $u\Phi$ by fibrewise multiplication. To define $u(A)$, recall that in the exact sequence

$$1 \to \mathrm{U}(1) \to \mathrm{Spin}^c(4) \to \mathrm{SO}(4) \to 1,$$

we map U(1) to the center of $\mathrm{Spin}^c(4)$. Therefore, u defines an automorphism of P_{Spin^c} (also denoted u), which in the fibre over $x \in X$ is multiplication with $u(x)$. Note that $u(A)$ is again a spin connection.

In this description, alternatively, we could have considered the action of u on the U(1)-connection $\check{A}$, but then we would have to carry around unfortunate factors of 2 in the theory, because the diagram

$$\begin{array}{ccc} P_{\mathrm{Spin}^c} & \xrightarrow[\cong]{u} & P_{\mathrm{Spin}^c} \\ & \searrow \quad \swarrow & \\ & P_{\mathrm{SO}}(X) & \end{array}$$

commutes.

Definition 4.3.1. Define the action $\mathcal{G} \times \mathcal{C} \to \mathcal{C}$ by

$$u(A, \Phi) := (u(A), u\Phi).$$

We need to understand more precisely what $u(A)$ is, and the following proposition tells us the answer.

Proposition 4.3.2. *We have*

$$u(A) = A - u^{-1}du,$$

where here $du \in \Omega^1(X;\mathbb{C})$ *so* $u^{-1}du \in \Omega^1(X; i\mathbb{R})$.

Proof. Recall that $\mathrm{Ad}(P_{\mathrm{Spin}^c}) = E \oplus i\mathbb{R}$, where $E \xrightarrow{\cong} \mathrm{Ad}(P_{\mathrm{SO}}(X))$

Since A and $u(A)$ are both spin connections, we have

$$a := u(A) - A \in \Omega^1(X; i\mathbb{R}).$$

Let $\beta : \mathrm{Spin}^c(4) \to \mathrm{U}(1)$ be the homomorphism defined earlier and let $L := P_{\mathrm{Spin}^c} \times_{\mathrm{U}(1)} \mathbb{C}$ be the associated line bundle, where $\mathrm{U}(1)$ acts on $\mathbb{C}$ in the standard way. For any connection B in P_{Spin^c}, let ∇^B be the induced connection in L. Then

$$\nabla_A + 2a = \nabla_{A+a} = \nabla_{u(A)} = u^2(\nabla_A) = A - u^{-2}d(u^2) = A - 2u^{-1}du,$$

which implies that $a = -u^{-1}du$. Here, the factor of 2 on the left hand side comes from the fact that in general, if $P \to X$ is a G-bundle, and A a connection in P and $a \in \Omega^1(X; \mathrm{Ad}(P))$, then in the vector bundle $E = P \times_G V$, we find $\nabla_{A+a} = \nabla_A + a$, where the action of a on sections of E is determined by the map $LG \to \mathrm{End}(V)$ induced by the representation $G \to \mathrm{Aut}(V)$. □

The next goal is to show that set of solutions to the Seiberg–Witten equations is invariant under the action $\mathcal{G}$.

Proposition 4.3.3. *For any spin connection* A*,* $\Phi \in \Gamma(\mathbb{S})$ *and* $f \in C^\infty(X;\mathbb{C})$*, we have*

$$D_A(f\Phi) = fD_A\Phi + df \cdot \Phi,$$

where $df \in \Gamma(T^*X \otimes \mathbb{C})$*, identifying* $T^*X \cong TX$ *using the metric, and noting that* $TX \otimes \mathbb{C} \subseteq \mathrm{Cl}(X)$*, so* df *acts on spinors.*

Proof. Pick an orthonormal frame $\{e_j\}$, so under the identification $TX \cong T^*X$, e_j is identified with e_j^*. We then have

$$\begin{aligned} D_A(f\Phi) &= \sum_{j=1}^4 e_j \nabla^A_{e_j}(f\Phi) = \sum_j e_j(e_j f \cdot \Phi + f\nabla^A_{e_j}(\Phi)) \\ &= df \cdot \Phi + fD_A\Phi. \end{aligned}$$

□

Remark 4.3.4. The proposition is not special for 4-manifolds and holds in general.

Proposition 4.3.5. *For any spin connection* A *and* $\Phi \in \Gamma(\mathbb{S})$*, one has the naturality property*

$$D_{u(A)}(u\Phi) = u \cdot D_A\Phi.$$

Proof. This follows directly for abstract reasons but we will see the equality from concrete calculation. For any $\Psi \in \Gamma(\mathbb{S})$, $a \in \Omega^1(X; i\mathbb{R})$, we find

$$\begin{aligned} D_{A+a}(\Psi) &= \sum_j e_j \nabla^{A+a}_{e_j}(\Psi) = \sum_j e_j(\nabla^A_{e_j} + a(e_j)) \cdot \Psi \\ &= D_A\Psi + a\Psi. \end{aligned}$$

Taking $a = -u^{-1}du$, we get

$$\begin{aligned} D_{u(A)}(u\Phi) &= D_{A+a}(u\Phi) = D_A(u\Phi) - u^{-1}du \cdot u\Phi \\ &= uD_A\Phi + du \cdot \Phi - du \cdot \Phi = uD_A\Phi. \end{aligned}$$

□

Proposition 4.3.6. *If* (A, Φ) *satisfy the* SW_μ*-equations, then so does* $u(A, \Phi)$.

Proof. We divide the proof into three steps.

(i) Let us first see what happens to the curvature under gauge transformation. In general, for G- and H-bundles $P \to X$, $Q \to X$, an i-homomorphism homomorphism $u : P \to Q$ with respect to $\rho : G \to H$, we have seen that for connections A in P and $B = u(A)$ in Q, the curvature satisfies $F_B = \tilde{u} F_A$, where $\tilde{u} : \mathrm{Ad}(P) \to \mathrm{Ad}(Q)$ is the map $[p, \xi] \mapsto [u(p), L\rho(\xi)]$. In our case, $P = Q$, $G = H$, and $\rho = \mathrm{id}$. Then $[p, \xi] \mapsto [p \cdot z, \xi] = [p, z\xi] = [p, \xi]$, since $z = u(\pi(p))$ is central; that is, $\mathrm{U}(1) \subseteq \mathrm{Spin}^c(4)$ acts trivially on $\mathfrak{spin}^c(4)$, and therefore the automorphism $\tilde{u}$ of $\mathrm{Ad}(P_{\mathrm{Spin}^c})$ induced by u is the identify. Hence $F_{u(A)} = \tilde{u} F_A = F_A$.

(ii) We consider now the quadratic term Q. If Ψ is another section of $\mathbb{S}^+$, then, since $\langle \, , \, \rangle$ is anti-linear in the second factor and since $\overline{u} u = 1$, we have

$$\langle \Psi, u\Phi \rangle u\Phi - \frac{|u\Phi|^2}{2} \Psi = \langle \Psi, \Phi \rangle \Phi - \frac{|\Phi|^2}{2} \Psi,$$

so $Q(u\Phi) = Q(\Phi)$.

(iii) Finally, since $D_A \Phi = 0$, we have $D_{u(A)}(u\Phi) = u D_A \Phi = 0$. □

4.4 The monopole moduli space

Definition 4.4.1. A solution to the SW_μ-equations is called a *monopole*. The *monopole moduli space* is

$$M := \{\text{Solutions } (A, \Phi) \text{ to the } \mathrm{SW}_\mu\text{-equations}\}/\mathcal{G}.$$

We will later define a topology on M with respect to which it is compact. Two important ingredients of the proof of compactness will be ellipticity of the SW_μ-equations and a certain a priori bound on $\|\Phi\|_\infty$ involving the scalar curvature and the perturbation term.

15th lecture, November 8th 2011

5 Compactness of the monopole moduli space

5.1 An a priori bound on norms

Lemma 5.1.1. *Let X be any Riemannian n-manifold and $E \to X$ a Hermitian vector bundle with connection ∇. Let $s \in \Gamma(E)$. If $\{x^j\}$ are normal (or geodesic) coordinates on X around a point $p \in X$. Then*

$$\nabla^* \nabla s = -\sum_j \left(\nabla_{\partial_j}\right)^2 s$$

at p, where $\partial_j = \frac{\partial}{\partial x_j}$.

Note that we proved almost the same thing earlier when we saw the Bochner formula. There, instead of ∂_j we had e_j for $\{e_j\}$ an orthonormal frame field.

Proof. Recall that the Christoffel symbols vanish at p which means that $\nabla \partial_j = 0$ at p for all j. Hence $\nabla(dx^j) = 0$ at p as well. Let $\{e_j\}$ be a local orthonormal frame field around p such that $e_j = \partial_j$ at p. Recall that $d_A = \nabla_A$ on $\Omega^0(X; E)$, and applying the formula for d_A^*, we find that at p,

$$\nabla^* \nabla s = -\sum_j e_j \lrcorner \nabla_{e_j} \sum_k dx^k \otimes \nabla_{\partial_k}(s) = -\sum_{jk} \partial_j \lrcorner \left(dx^k \otimes \nabla_{\partial_j} \nabla_{\partial_k}(s)\right) = -\sum_j (\nabla_{\partial_j})^2 s.$$

□

The special case where $E = X \times \mathbb{C}$ is the trivial bundle and ∇ is the product connection is also interesting. There the lemma says that for any $f \in C^\infty(X, \mathbb{C})$,

$$\Delta f := d^* df = -\sum_j (\partial_j)^2 f$$

at a point p in normal coordinates.

Corollary 5.1.2. *If $f : X \to \mathbb{R}$ has a local maximum at $p \in X$, then*

$$(\Delta f)(p) \geq 0.$$

Proposition 5.1.3. *Let X be an oriented Riemannian 4-manifold with a spin^c-structure and (A, φ) a solution to the SW_μ-equations. If $|\varphi|$ has a local maximum at a point $p \in X$, then*

$$|\varphi|^2 \leq \max\left(0, 2\sqrt{2}|\mu^+| - \frac{s}{2}\right)$$

at p where $s : X \to \mathbb{R}$ is the scalar curvature of X.

Proof. In normal coordinates around p, writing $\nabla_j = \nabla_{\partial_j}$, one has

$$\begin{aligned} 0 \leq \Delta|\varphi|^2 &= -\sum_j (\delta_j)^2 \langle\varphi, \varphi\rangle = -\sum_j \partial_j 2\langle\nabla_j^A(\varphi), \varphi\rangle \\ &= -2\sum_j \left(\langle(\nabla_j^A)^2\varphi, \varphi\rangle + |\nabla_j^A\varphi|^2\right) \leq 2\langle\nabla_A^*\nabla_A\varphi, \varphi\rangle = 2\left\langle\left(D_A^2 - F_A'' - \frac{s}{4}\right)\varphi, \varphi\right\rangle \\ &= 2\left\langle\left(D_A^2 - (F_A'')^+ - \frac{s}{4}\right)\varphi, \varphi\right\rangle = 2\left\langle\left(i\mu^+ - Q(\varphi) - \frac{s}{4}\right)\varphi, \varphi\right\rangle \\ &= -|\varphi|^4 - \frac{s}{2}|\varphi|^2 + 2\langle i\mu^+\varphi, \varphi\rangle \leq -\varphi|^4 - \frac{s}{2}|\varphi|^2 + 2\sqrt{2}|\mu^+||\varphi|^2. \end{aligned}$$

Here, the fourth equality follows from the Bochner formula, the fifth from the fact that $\wedge^-\mathbb{S}^+ = 0$, and the sixth from the SW_μ-equations. The seventh follows

$$\langle Q(\varphi), \varphi\rangle = \left\langle\frac{|\varphi|^2}{2}\varphi, \varphi\right\rangle = \frac{|\varphi|^4}{2}.$$

The formula tells us that either $\varphi = 0$ or $|\varphi|^2 \leq 2\sqrt{2}|\mu^+| - \frac{s}{2}$ at p. □

We will use this a priori bound to prove compactness of the moduli space endowed with a certain topology. In order to describe this we need to go through some analysis.

5.2 Sobolev spaces

Let X be a closed Riemannian n-manifold, and let μ be the volume measure on the Borel algebra. As for any measure space, we can define, for $1 \leq p < \infty$,

$$L^p(X) = \{\text{measurable functions } X \to \mathbb{C} \mid \int_X |f|^p < \infty\}/\sim,$$

where $f_1 \sim f_2$ if $f_1 = f_2$ almost everywhere (i.e. away from a set of measure 0). The L^p-norm is

$$\|f\|_p := \|f\|_{L^p} = \left(\int_x |f|^p\right)^{1/p}.$$

Then $(L^p(X), \|\cdot\|_p)$ is a Banach space (i.e. a complete normed vector space) and $L^2(X)$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_X f\overline{g}.$$

Here, we have not used that X is a manifold. Because it is, there is an alternative description of L^p, since $C^\infty(X) \subseteq L^p(X)$ is dense, and we could define $L^p(X)$ to be the completion of $C^\infty(X)$.

Now let $E \to X$ be a Hermitian vector bundle. Define $L^p(X;E)$ to be the completion of $\Gamma(E)$ with respect to the norm

$$\|s\|_p := \left(\int_X |s|^p\right)^{1/p}.$$

As before, $L^p(X;E)$ is a Banach space, and $L^2(X;E)$ is a Hilbert space with inner product

$$\langle s,t\rangle_{L^2} = \int_X \langle s,t\rangle.$$

Definition 5.2.1. Let $E \to X$ be a Hermitian vector bundle with a connection ∇. For $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$, we define the *Sobolev space* $L^p_k(X;E)$ to be the completion of $\Gamma(E)$ with respect to the norm

$$\|s\|_{k,p} := \|s\|_{L^p_k} := \left(\int_X |\nabla^j s|^p\right)^{1/p}.$$

Here ∇^j is applied to $\Gamma(T^*X \otimes \cdots \otimes T^*X \otimes E)$ by applying the given connection ∇ on E and the Levi-Civita connection on the cotangent bundles.

Since X is compact, the topology on the Sobolov space $L^p_k(X;E)$ does not depend on the Riemannian metric on X and the choice of ∇ in E (that is, the resulting norms will be equivalent).

5.3 The topology of the moduli space

Let X be a Riemannian 4-manifold with a spin^c-structure. For $k \geq 2$, define

$$\mathcal{C}_k := \{(A_0 + a, \varphi) \mid a \in L^2_k(X; i\mathbb{R}), \varphi \in L^2_k(X; \mathbb{S}^+)\},$$

where A_0 is a smooth (*reference*) spin connection in $P_{\mathrm{Spin}^c} \to X$. Then $\mathcal{C}_k$ is a complete metric space with metric

$$d_{L^2_k}((A,\varphi),(B,\psi)) = \left(\|A-B\|^2_{L^2_k} + \|\varphi-\psi\|^2_{L^2_k}\right)^{1/2}$$

We define

$$\mathcal{G}_k = \{u \in L^2_{k+1}(X;\mathbb{C}) \mid \mathrm{Im}(u) \subseteq \mathrm{U}(1)\}.$$

One can show that $\mathcal{G}_k$ is a Hilbert Lie group, i.e. a smooth manifold modelled on Hilbert spaces with group structure (in this case multiplication) such that the group operations are smooth. The action $\mathcal{G} \times \mathcal{C} \to \mathcal{C}$ extends to a smooth action of $\mathcal{G}_k$ on $\mathcal{C}_k$. We define

$$\mathcal{B}_k := \mathcal{C}_k/\mathcal{G}_k$$

with the quotient topology. Let

$$M_k := \{[A,\varphi] \in \mathcal{B}_k \mid (A,\varphi) \text{ satisfies the } \mathrm{SW}_\mu\text{-equations}\}.$$

Here, one should explain what it means for (A,φ) to satisfy the SW_μ-equations which deal with smooth objects, but this is possible. Let $M_k \subseteq \mathcal{B}_k$ have the subspace topology. It now looks like we have an infinite number of moduli spaces, but they turn out to be essentially equivalent. Namely, we will show that

(i) there is a canonical bijection $M \to M_k$ for every k, and that

(ii) if $2 \leq k \leq l$, then the natural map $M_l \to M_k$ is a homeomorphism, and finally that

(iii) M_k is compact.

Let X be a Riemannian n-manifold and let $E \to X$ and $F \to X$ be Hermitian vector bundles. If $P : \Gamma(E) \to \Gamma(F)$ is a differential operator of order less than or equal to r, then P extends to a bounded (i.e. continuous) operator

$$P : L^p_k(X; E) \to L^p_{k-r}(X; F)$$

for $k \geq r$, $1 \leq p < \infty$. If P is elliptic of order r, then $P_{k,p}$ is a *Fredholm operator* (i.e. $\dim \ker P_{k,p} < \infty$ and $\dim \operatorname{coker} P_{k,p} < \infty$) for $p > 1$. We have $\ker(P_{k,p}) = \ker(P)$ and there is a natural isomorphism $\operatorname{coker}(P) \xrightarrow{\cong} \operatorname{coker}(P_{k,p})$, so in particular $\operatorname{ind}(P_{k,p}) = \dim \ker(P_{k,p}) - \dim \operatorname{coker}(P_{k,p})$ does not depend on k and p.

16th lecture, November 10th 2011

5.4 Compact operators and Fredholm operators

5.4.1 Bounded operators

There are three topics of analysis that are particularly relevant for gauge theory:

- The Fredholm theory of elliptic operators, which is required for the linear theory, and
- the Sobolev embedding theorems and
- the Sobolev multiplication theorems, which are required for the non-linear theory.

To fix notation, we review some basics of functional analysis.

Definition 5.4.1. An operator (i.e. a linear map) $T : X \to Y$ between normed spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ is called *bounded* if there is a $C > 0$ such that for all $x \in X$, we have

$$\|Tx\| \leq C\|x\|.$$

Note that T is bounded if and only if it is continuous.

Let $\mathcal{L}(X, Y)$ be the set of all bounded operators $X \to Y$. This is a normed space with the *operator norm*

$$\|T\| := \sup_{\|x\| \leq 1} \|Tx\|.$$

If Y is complete, then $\mathcal{L}(X, Y)$ is also complete. A special case is the *dual space* $X^* := \mathcal{L}(X, \mathbb{K})$ of X. Elements of the dual space will be denoted x^* (which has nothing to do with an element x of X). We write

$$\langle x, x^* \rangle := x^*(x)$$

for $x \in X$. The Hahn–Banach theorem implies that

$$\|x\| = \sup_{\|x^*\| \leq 1} |\langle x, x^* \rangle|.$$

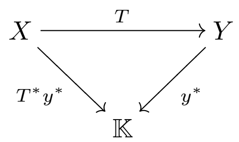
Any bounded operator $T : X \to Y$ induces a bounded operator

$$T^* : Y^* \to X^*$$

called the *adjoint* of T given by

$$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle$$

and fitting into the diagram



The adjoint operator is bounded since its norm is given by

$$\begin{aligned}\|T^*\| &= \sup_{\|y^*\|\leq 1} \|T^*y^*\| = \sup_{\|y^*\|\leq 1} \sup_{\|x\|\leq 1} |\langle x, T^*y^*\rangle| = \sup_{\|y^*\|\leq 1} \sup_{\|x\|\leq 1} |\langle Tx, y^*\rangle| \\ &= \sup_{\|x\|\leq 1} \sup_{\|y^*\|\leq 1} |\langle Tx, y^*\rangle| = \sup_{\|x\|\leq 1} \|Tx\| = \|T\|.\end{aligned}$$

5.4.2 Complemented subspaces

Definition 5.4.2. A closed subspace M of a Banach space X is called *complemented* if there exists a closed subspace $N \subseteq X$ such that $M \cap N = \{0\}$ and $X = M + N$.

One then writes $X = M \oplus N$.

Proposition 5.4.3. *Let $M \subseteq X$ be as above. If at least one of the following conditions holds, then M is complemented:*

(i) If X is a Hilbert space, one can take $X = M \oplus M^\perp$.

(ii) $\dim M < \infty$.

(iii) $\dim X/M < \infty$.

Proof. For a proof of (i), we refer to [Rud91].

To see (ii), choose a linear isomorphism $\alpha : M \xrightarrow{\cong} \mathbb{K}^n$. Since all norms on $\mathbb{K}^n$ are equivalent, α is a homeomorphism. By the Hahn–Banach theorem, α extends to a bounded operator $\beta : X \to \mathbb{K}^n$. Then

$$X = M \oplus \ker(\beta).$$

This follows from the fact that the inclusion map i of the short exact sequence

$$0 \to M \xrightarrow{i} X \to X/M \to 0$$

has a left inverse.

For (iii), let $r : X/M \to X$ be a right inverse of the projection $X \to X/M$. Then $X = M \oplus \mathrm{im}(r)$. □

Note that in [Rud91], one finds examples of uncomplemented subspaces. Let $X = M \oplus N$. Then M and N are Banach spaces in their own right, and the direct sum has the norm $\|(x, y)\| := \|x\| + \|y\|$ for $x \in M, y \in N$. Then the operator $M \oplus N \to X$ mapping $(x, y) \mapsto x + y$ is bounded and bijective, hence an isomorphism (i.e. a linear homeomorphism) by the open mapping theorem.

5.4.3 Compact operators

Definition 5.4.4. A metric space (X, d) is called *totally bounded*, if for every $\varepsilon > 0$, there are finitely many points $x_1, \dots, x_n$ such that

$$X = \bigcup_{i=1}^{n} B_\varepsilon(x_i),$$

where $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$.

For a proof of the following, see [Rud91].

Proposition 5.4.5. *For a metric space X, the following conditions are equivalent:*

(i) X is compact.

(ii) X is sequentially compact.

(iii) X is complete and totally bounded.

Definition 5.4.6. If A is any compact topological space, let $C(A)$ be the space of continuous maps $A \to \mathbb{K}$ with the supremum norm

$$\|f\|_{\sup} := \sup_{a \in A} |f(a)|$$

for $f \in C(A)$. Then $C(A)$ is a Banach space. If $\mathcal{F} \subseteq C(A)$ is any subset, then

(i) $\mathcal{F}$ is called *equicontinuous* if for every $\varepsilon > 0$ and every $a \in A$, there is neighbourhood V of a in A such that for every $b \in V$, $f \in \mathcal{F}$, we have $|f(b) - f(a)| < \varepsilon$,

(ii) $\mathcal{F}$ is called *pointwise bounded* if for all $a \in A$, we have $\sup_{f \in \mathcal{F}} |f(a)| < \infty$.

The proof of the following (also stated in [Rud91]) is an easy exercise.

Proposition 5.4.7 (Arzelà–Ascoli)**.** *Let A be a compact space and $\mathcal{F} \subseteq C(A)$ equicontinuous and pointwise bounded. Then $\mathcal{F}$ is totally bounded (and thus $\overline{\mathcal{F}}$ is compact).*

Definition 5.4.8. An operator $T : X \to Y$ between Banach spaces is called *compact* if $\overline{TA}$ is compact for every bounded subset $A \subseteq X$.

Note that compact operators are always bounded.

Example 5.4.9. If T is bounded and $\dim TX < \infty$, then T is compact.

Example 5.4.10. The inclusion $C^1([0,1]) \to C^0([0,1]) = C([0,1])$ is compact where

$$\|f\|_{C^1} := \|f\|_{\sup} + \|f'\|_{\sup}.$$

This follows from the Arzelà–Ascoli theorem: For $\mathcal{F} \subseteq C^1([0,1])$, and $f \in \mathcal{F}$ we have $f(y) - f(x) = \int_x^y f'$ so $|f(y) - f(x)| \leq \|f'\|_{C^1}$.

Example 5.4.11. If $k : [0,1] \times [0,1] \to \mathbb{C}$ is continuous, then the operator $T : L^1([0,1]) \to C^0([0,1])$ defined by

$$(Tf)(x) := \int_0^1 k(x,y) f(y)\, dy$$

is compact. Again, this follows from the Arzelà–Ascoli theorem.

Observe that the composition of bounded operators with compact operators will always be compact: Let $X \xrightarrow{S} Y \xrightarrow{T} Z$ be bounded operators between bounded operators. If at least one of S and T is compact, then the composition $T \circ S$ is compact, since T maps bounded sets to bounded sets and precompact sets to precompact sets.

Definition 5.4.12. Let $\mathcal{K}(X,Y)$ be the space of compact operators $X \to Y$.

For a proof of the following, we refer again to [Rud91].

Proposition 5.4.13. *The space $\mathcal{K}(X,Y)$ is a closed linear subspace of $\mathcal{L}(X,Y)$.*

The following Proposition is a key step towards Fredholm theory.

Proposition 5.4.14. *If $T : X \to Y$ is a bounded operator between Banach spaces, then T is compact, if and only if the adjoint $T^* : Y^* \to X^*$ is also compact.*

Proof. Suppose first that T is compact. Set

$$D_X := \{x \in X \mid \|x\| \leq 1\},$$
$$D_{Y^*} := \{y^* \in Y^* \mid \|y^*\| \leq 1\}.$$

The set $A := \overline{TD_X}$ is compact in Y. Let $\varphi : Y^* \to C(A)$ be the restriction map $y^* \mapsto y^*|_A$. Then $\varphi(D_{Y^*})$ is pointwise bounded since for all $a \in A$, $y^* \in D_{Y^*}$, we have

$$|\langle a, y^* \rangle| \leq \|a\| \cdot \|y^*\| \leq \|a\|.$$

Also, $\varphi(D_{Y^*})$ is equicontinuous, because for all $a_1, a_2 \in A$, $y^* \in D_{Y^*}$, we have

$$|\langle a_1, y^* \rangle - \langle a_2, y^* \rangle| \leq \|a_1 - a_2\|.$$

By the Arzelà–Ascoli theorem, $\varphi(D_{Y^*})$ is totally bounded. Now, for all $y^* \in Y^*$, by the Hahn–Banach theorem, as before we have

$$\|T^* y^*\| = \sup_{\|x\| \leq 1} |\langle Tx, y^* \rangle| = \sup_{a \in A} |\langle a, y^* \rangle| = \|\varphi(y^*)\|.$$

Hence $\ker(T^*) = \ker(\varphi)$, and there is an isometry $\iota : \operatorname{im}(\varphi) \xrightarrow{\cong} \operatorname{im}^(T^*)$ such that $\iota \circ \varphi = T^*$. Therefore, $T^* D_{Y^*} = \iota(\varphi(D_{Y^*}))$ is totally bounded, so $\overline{T^* D_{Y^*}}$ is compact.

If T^* is compact, then T is compact by a similar argument. $\square$

As a warmup for the Fredholm theory, we consider the following Proposition.

Proposition 5.4.15. *Let $T : X \to Y$ be a bounded operator between Banach spaces. Then the following are equivalent:*

(i) T is injective and $\operatorname{im}(T)$ is closed.

(ii) There exists $C > 0$ such that for all $x \in X$, we have $\|x\| \leq C\|Tx\|$.

Proof. Suppose T is injective and that $\operatorname{im}(T)$ is closed. By the open mapping theorem, T defines an isomorphism $S : \operatorname{im}(T) \xrightarrow{\cong} X$. Hence there exists $C > 0$ such that $\|x\| = \|STx\| \leq C\|Tx\|$.

To see that (ii) implies (i), note first that T is automatically injective. If $\{x_n\}$ is a sequence in X such that $\{Tx_n\}$ is Cauchy, then

$$\|x_n - x_m\| \leq C\|Tx_n - Tx_m\| \to 0$$

as $n, m \to \infty$, i.e. $\{x_n\}$ is Cauchy. Let $x := \lim x_n \in X$. Then $Tx = \lim Tx_n$, so $\operatorname{im}(T)$ is closed. $\square$

17th lecture, November 22th 2011

Throughout today's lecture, X and Y will be Banach spaces.

Proposition 5.4.16. *For any bounded operator $T \in \mathcal{L}(X, Y)$, the following are equivalent:*

(i) T has finite dimensional kernel and closed image.

(ii) There exists a Banach space Z, a compact operator $K : X \to Z$, and a constant $C > 0$, such that for all $x \in X$, we have $\|x\| \leq C(\|Tx\| + \|Kx\|)$.

Proof. To see that (i) implies (ii), set $Z := \ker(T)$. Choose a closed subspace $E \subseteq X$, such that $X = E \oplus Z$ (recall that this is possible by the Hahn–Banach theorem). Let $K : E \oplus Z \to Z$ be the projection onto Z with respect to this decomposition. Since $T|_E$ is injective and has closed image, by Proposition 5.4.15, there exists $C_1 > 0$, such that for all $e \in E$, we have $\|e\| \leq C_1 \|Te\|$. Let $E \oplus Z$ have the norm $\|(l, z)\| = \|e\| + \|z\|$. TFor all $x = (e, z) \in E \oplus Z$, we have

$$\|x\| \leq \|e\| + \|z\| \leq C_1 \|Tx\| + \|Kx\| \leq (C_1 + 1)(\|Tx\| + \|Kx\|).$$

Conversely, for all $x \in \ker(T)$, the inequality of (ii) tells us that $\|x\| \leq C \|Kx\|$. This implies that $K|_{\ker(T)}$ is injective and has closed image, and by the open mapping theorem, $K|_{\ker(T)}$ is an isomorphism onto its image. Then since K is compact, $\ker(T)$ is locally compact and therefore finite dimensional (this holds for general topological vector spaces but is a bit easier to see for Banach spaces, which we leave as an exercise, see [Rud91]).

It remains to prove that $\operatorname{im}(T)$ is closed. Choose as before a closed subspace $E \subseteq X$ such that $X = E \oplus \ker(T)$. We claim that there exists $C_2 > 0$ such that for all $e \in E$, $\|e\| \leq C_2 \|Te\|$. This will imply, by Proposition 5.4.15, that T has closed image.

To prove the claim, assume that there is no such C_2. Then there is a sequence $\{e_n\}$ in E such that $\|e_n\| = 1$ and $\|e_n\| > n \|Te_n\|$. Then $\|Te_n\| < \frac{1}{n}$. Choose a subsequence $\{n_j\}$ such that $\{Ke_{n_j}\}$ converges. To simplify notation, set $x_j := e_{n_j}$. Now

$$\|x_i - x_j\| \leq C(\|Tx_i - Tx_j\| + \|Kx_i - Kx_j\|),$$

and since both $\{Tx_j\}$ and $\{Kx_j\}$ are Cauchy, so is $\{x_j\}$. Let $e := \lim_{j\to\infty} x_j$. Then $\|e\| = \lim \|x_j\| = 1$ and $Te = \lim Tx_j = 0$. Hence $0 \neq e \in E \cap \ker(T) = 0$ which is a contradiction. □

Recall that if $M \subseteq X$ is any closed subspace, then the quotient vector space X/M equipped with the norm

$$\|x + M\| = \inf_{m \in M} \|x + m\|,$$

is a Banach space. Moreover, setting

$$M^\perp := \{x^* \in X^* \mid x^*|_M = 0\},$$

the canonical map $(X/M)^* \to M^\perp$ is a bijective isometry by the Hahn–Banach theorem.

Proposition 5.4.17. *If $T \in \mathcal{L}(X, Y)$ and $\dim \operatorname{coker} < \infty$, then $\operatorname{im}(T)$ is closed.*

Proof. Choose a subspace $F \subseteq Y$ such that $Y = \operatorname{im}(T) \oplus F$ as vector spaces. Let $S : X/\ker(T) \to Y$ be the bounded operator induced by T. Then the map $X/\ker(T) \oplus F \to Y$ mapping $(\alpha, f) \mapsto S\alpha + f$ is an isomorphism by the open mapping theorem. In particular, that means that it maps $X/\ker(T)$ onto a closed subspace of Y, and since $\operatorname{im}(T) = \operatorname{im}(S)$, we see that $\operatorname{im}(T)$ is closed. □

Proposition 5.4.18. *If $T \in \mathcal{L}(X, Y)$ has closed image, then $\operatorname{coker}(T)^* \cong \ker(T^*)$.*

It is also true that if $\operatorname{im}(T)$ is closed, then $\operatorname{im}(T^*)$ is closed and $\operatorname{coker}(T^*) \cong \ker(T)^*$, but we will not prove that.

Proof. By definition, and by the previous proposition, we have

$$\begin{aligned}\operatorname{coker}(T)^* = (Y/\operatorname{im}(T))^* &\cong \operatorname{im}(T)^\perp = \{y^* \in Y^* \mid \forall x \in X : 0 = \langle Tx, y^* \rangle\} \\ &= \{y^* \in Y^* \mid \forall x \in X : 0 = \langle x, T^* y^* \rangle\} = \ker(T^*)\end{aligned}$$

□

5.4.4 Fredholm operators

Definition 5.4.19. An operator $T \in \mathcal{L}(X,Y)$ is called *Fredholm* if $\ker(T)$ and $\operatorname{coker}(T)$ are finite dimensional. The *index* of a Fredholm operator T is

$$\operatorname{ind}(T) := \dim\ker(T) - \dim\operatorname{coker}(T).$$

The space of all Fredholm operators from X to Y will be denoted $\mathcal{F}(X,Y)$. Note that this is not a vector space, unless both X and Y are finite dimensional.

From our previous results, we obtain the following.

Proposition 5.4.20. *Any Fredholm operator has closed image.*

Example 5.4.21. If X, Y are finite dimensional, then any linear map $T : X \to Y$ is Fredholm, and

$$\operatorname{ind}(T) = \dim X - \dim Y.$$

Example 5.4.22. For $1 \le p < \infty$, let

$$\ell^p = L^p(\mathbb{N}, \mu),$$

where μ is the counting measure on $\mathbb{N}$. That is, ℓ^p consists of all maps $f : \mathbb{N} \to \mathbb{C}$, where

$$\|f\|_p := \left(\sum_{n=1}^{\infty} |f(n)|^p\right)^{1/p}$$

is finite. Define $T : \ell^p \to \ell^p$ by

$$(Tf)(n) := \begin{cases} 0, & n = 1 \\ f(n-1), & n > 1 \end{cases}.$$

Then $\ker(T) = 0$, and $\dim\operatorname{coker}(T) = 1$, so $\operatorname{ind}(T) = -1$. Shifting in the other direction, we could obtain an operator of index 1.

We will use the following important result to show that elliptic operators L^2 Sobolev spaces are Fredholm.

Proposition 5.4.23. *If $K \in \mathcal{L}(X,X)$ is compact, then $I - K : X \to X$ is Fredholm.*

Proof. Let $T := I - K$. Then for every $x \in X$, one has

$$\|x\| = \|Tx + Kx\| \le \|Tx\| + \|Kx\|,$$

so by Proposition 5.4.16, T has finite dimensional kernel and closed image. Since T has closed image, Proposition 5.4.18 tells us that $\operatorname{coker}(T)^* \cong \ker(T^*)$, but $T^* = I^* - K^* = I - K^*$, and K^* is compact by Proposition 5.4.14, so the above argument tells us that $\ker(T^*)$ is also finite dimensional. In general by the Hahn–Banach theorem, if $\dim Y = \infty$ for a Banach space Y, then $\dim Y^* = \infty$. Hence in our case, $\dim\operatorname{coker}(T) < \infty$. □

Proposition 5.4.24. *If $T \in \mathcal{L}(X,X)$ and $\|T\| < 1$, then $I - T$ is an isomorphism.*

Proof. For any $R, S \in \mathcal{L}(X,X)$, we have $\|RS\| \le \|R\|\,\|S\|$ (that is, $\mathcal{L}(X,X)$ is a so-called *Banach algebra*), hence $\|T^n\| \le \|T\|^n$ for $n \ge 0$ (where we take $T^0 := I$). This implies that

$$\sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \|T\|^n = \frac{1}{1 - \|T\|} < \infty,$$

so the operator $S := \sum_{n=0}^{\infty} T^n \in \mathcal{L}(X,X)$ is well-defined. Then

$$(I-T)S = (I-T)\lim_{N\to\infty}\sum_{n=0}^{N} T^n = \lim_{N\to\infty}(I-T^{N+1}) = I,$$

and similarly $S(I-T) = I$. $\square$

We prove now that invertibility is an open condition.

Proposition 5.4.25. *The invertible operators form an open subset of $\mathcal{L}(X,Y)$.*

Proof. Let $S,T \in \mathcal{L}(X,Y)$ where S is invertible. Then

$$\left\|S^{-1}T - I\right\| = \left\|S^{-1}(T-S)\right\| \le \left\|S^{-1}\right\| \|T-S\|.$$

Hence, if $\|T-S\| < \left\|S^{-1}\right\|^{-1}$, then $S^{-1}T$ (and hence T) will be invertible. $\square$

Proposition 5.4.26 (Normal form of a Fredholm operator)**.** *For any $T \in \mathcal{F}(X,Y)$, there is a Banach space V and $m,n \in \mathbb{N}_0$, and isomorphisms*

$$X \xrightarrow{\cong} V \oplus \mathbb{K}^m, \quad Y \xrightarrow{\cong} V \oplus \mathbb{K}^n$$

such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \cong\downarrow & & \downarrow\cong \\ V \oplus \mathbb{K}^m & \longrightarrow & V \oplus \mathbb{K}^n \end{array}$$

Here, the bottom map is the map $(a,b) \mapsto (a,0)$.

Proof. Choose closed subspaces $V \subseteq X$, $F \subseteq Y$ such that $X = V \oplus \ker(T)$, $Y = \operatorname{im}(T) \oplus F$. Define $S : V \to \operatorname{im}(T)$ by $x \mapsto Tx$. This is an isomorphism by the open mapping theorem. The composition

$$V \oplus \ker(T) \xrightarrow{T} \operatorname{im}(T) \oplus F \xrightarrow{S^{-1}\oplus I} V \oplus F$$

maps $(a,b) \mapsto (Ta,0) \mapsto (a,0)$. $\square$

18th lecture, November 24th 2011

We want to see that the Fredholm condition is an open condition, and that the index is a locally constant function on the space of Fredholm operators. Through-out, X, Y and Z will denote Banach spaces.

Proposition 5.4.27. *For every $S \in \mathcal{F}(X,Y)$, there is an $\varepsilon > 0$ such that for every $T \in \mathcal{L}(X,Y)$ with $\|T-S\| < \varepsilon$, the following hold:*

(i) T *is Fredholm and* $\operatorname{ind}(T) = \operatorname{ind}(S)$.

(ii) $\dim\ker(T) \le \dim\ker(S)$.

(iii) $\dim\operatorname{coker}(T) \le \dim\operatorname{coker}(S)$.

Proof. We can assume that S is in normal form,

$$S = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : V \oplus \mathbb{K}^m \to V \oplus \mathbb{K}^n,$$

where $I = \mathrm{id} : V \to V$, and V is Banach space. The case $V = \{0\}$ is trivial, so assume that $V \neq \{0\}$. Write T as

$$T = \begin{pmatrix} I + A & B \\ C & D \end{pmatrix} : V \oplus \mathbb{K}^m \to V \oplus \mathbb{K}^N.$$

Let $\iota : V \to V \oplus \mathbb{K}^m$ denote the map $x \mapsto (x, 0)$, and $\pi : V \oplus \mathbb{K}^n \to V$ the map $(x, y) \mapsto x$. Then $A = \pi \circ (T - S)\iota A$, so

$$\|A\| \leq \|\pi\| \, \|T - S\| \, \|\iota\| \leq \varepsilon \, \|\pi\| \, \|\iota\| \, ,$$

so if $\varepsilon < (\|\pi\| \, \|\iota\|)^{-1}$, then $\|A\| < 1$, and $I + A$ is invertible. Set

$$R_1 = \begin{pmatrix} Q^{-1} & 0 \\ -CQ^{-1} & I \end{pmatrix}, \; R_2 = \begin{pmatrix} I & -Q^{-1}B \\ 0 & I \end{pmatrix}.$$

Then

$$R_1 T R_2 = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix}$$

for some $G : \mathbb{K}^m \to \mathbb{K}^n$. Because R_1 and R_2 are invertible, we find that

$$\begin{aligned} \dim \ker(T) &= \dim \ker(G) \leq m = \dim \ker(S), \\ \dim \operatorname{coker}(T) &= \dim \operatorname{coker}(G) \leq n = \dim \operatorname{coker}(S), \text{ and} \\ \operatorname{ind}(T) &= \operatorname{ind}(G) = m - n = \operatorname{ind}(S), \end{aligned}$$

which proves all three statements of the proposition. □

Corollary 5.4.28. *The space $\mathcal{F}(X, Y)$ is open in $\mathcal{L}(X, Y)$, and* ind $: \mathcal{F}(X, Y) \to \mathbb{Z}$ *is locally constant (i.e. continuous).*

Proposition 5.4.29. *For every compact operator $K : X \to X$, one has*

$$\operatorname{ind}(I - K) = 0.$$

Proof. Observe that tK is compact for every $t \in \mathbb{K}$. Therefore we have a continuous family of Fredholm operators $\gamma : \mathbb{K} \to \mathcal{F}(X, Y)$, given by $t \mapsto I - tK$, hence $\operatorname{ind}(I - tK)$ is locally constant, and hence constant. Therefore $\operatorname{ind}(I - K) = \operatorname{ind}(I) = 0$. □

The next proposition says that Fredholm operators are exactly those that are invertible up to compact operators.

Proposition 5.4.30. *A bounded operator $T : X \to Y$ is Fredholm precisely when there are bounded operators $S_1, S_2 : Y \to X$ and compact operators $K_1 : X \to X$, $K_2 : Y \to Y$ such that*

$$S_1 T = I - K_1, \; T S_2 = I - K_2.$$

Furthermore, one can choose $S_1 = S_2$.

Proof. Suppose first that we are given S_1, S_2, K_1, K_2. Then since $\operatorname{im}(T) \supseteq \operatorname{im}(I - K_2)$, we have

$$\begin{aligned} \dim \ker(T) &\leq \dim \ker(I - K_1) < \infty, \\ \dim \operatorname{coker}(T) &\leq \operatorname{coker}(I - K_2) < \infty, \end{aligned}$$

and so T is Fredholm.

Assume that T is Fredholm, in normal form,

$$T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : V \oplus \mathbb{K}^m \to V \oplus \mathbb{K}^n$$

and define

$$S := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : V \oplus \mathbb{K}^n \to V \oplus \mathbb{K}^n.$$

Then $TS = I - K_1$ and $ST = I - K_2$, where

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : V \oplus \mathbb{K}^n \to V \oplus \mathbb{K}^n,$$
$$K_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : V \oplus \mathbb{K}^m \to V \oplus \mathbb{K}^m.$$

These two operators have finite dimensional range and hence they are compact. □

Proposition 5.4.31. *If $T : X \to Y$ is Fredholm and $K : X \to Y$ is compact, then $T + K$ is Fredholm and*

$$\operatorname{ind}(T + K) = \operatorname{ind}(T).$$

Proof. Let S, K_1, K_2 be as in Proposition 5.4.30. Then since SK is compact,

$$S(T + K) = I - K_1 + SK = I + K_3$$

for some compact operator K_3. Similarly, since KS is compact,

$$(T + K)S = I - K_2 + KS = I + K_4,$$

where K_4 is compact. By Proposition 5.4.30, $T + K$ is Fredholm, and as before, $t \mapsto \operatorname{ind}(T + tK)$ is constant, and so $\operatorname{ind}(T + K) = \operatorname{ind}(T)$. □

The last fact we will prove about Fredholm operators is that the index is additive under composition.

Proposition 5.4.32. *If $S : X \to Y$ and $T : Y \to Z$ are Fredholm operators, then TS is also Fredholm, and*

$$\operatorname{ind}(TS) = \operatorname{ind}(S) + \operatorname{ind}(T).$$

Lemma 5.4.33. *If $R : X \to Y$ is Fredholm and $\operatorname{ind}(R) = 0$, then there exists a compact operator $K : X \to Y$ such that $R + K$ is invertible.*

Proof. If R is in normal form,

$$R = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : V \oplus \mathbb{K}^m \to V \oplus \mathbb{K}^n,$$

then we can take

$$K = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

□

Definition 5.4.34. For every $R \in \mathcal{L}(X, Y)$ and $m, n \in \mathbb{N}_0$, let $R_{m,n} : X \oplus \mathbb{K}^m \to Y \oplus \mathbb{K}^n$ denote the operator $(x, v) \mapsto (Rx, 0)$.

Clearly, $R_{m,n}$ is Fredholm precisely when R is Fredholm, and in this case,

$$\operatorname{ind}(R_{m,n}) = \operatorname{ind}(R) + m - n.$$

Proof of Proposition 5.4.32. *Case 1*: Assume $\operatorname{ind}(S) = \operatorname{ind}(T) = 0$. Then by Lemma 5.4.33, we can choose compact operators $K_1 : X \to Y$, $K_2 : Y \to Z$ such that $S + K_1$ and $T + K_2$ are invertible. Then $R := (T + K_2)(S + K_1)$ is invertible. Multiplying out, $R = TS + K_3$ for K_3 compact, so $TS = R - K_3$ is Fredholm, and

$$\operatorname{ind}(TS) = \operatorname{ind}(R) = 0.$$

Case 2: In general, choose $l, m, n \in \mathbb{N}_0$ such that $\operatorname{ind}(S_{l,m}) = \operatorname{ind}(T_{m,n}) = 0$. Then

$$T_{m,n}S_{l,m} = (TS)_{l,n}$$

is Fredholm of index 0. Hence TS is Fredholm, and

$$\operatorname{ind}(TS) = n - l = (n - m) + (m - l) = \operatorname{ind}(T) + \operatorname{ind}(S).$$

□

5.5 Fourier series

For $n \in \mathbb{N}$, define $\mathbb{T}^n := \mathbb{R}^n / 2\pi\mathbb{Z}^n$ with the quotient topology.

Definition 5.5.1. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called *periodic* if for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$ one has $f(x + 2\pi k) = f(x)$.

We can identify periodic functions on $\mathbb{R}^n$ with functions on $\mathbb{T}^n$. For any $k \in \mathbb{Z}^n$, let $e_k : \mathbb{T}^n \to \mathbb{C}$ be the function on $\mathbb{T}^n$ corresponding to the periodic function $\mathbb{R}^n \to \mathbb{C}$ given by $(2\pi)^{-n/2} \exp(2\pi i x \cdot k)$, where $x \cdot k = \sum_{j=1}^n x_j k_j$. One easily checks that

$$\langle e_k, e_l \rangle = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases},$$

where

$$\langle f, g \rangle := \int_{\mathbb{T}^n} f \overline{g}$$

for $f, g : \mathbb{T} \to \mathbb{C}$. Note that the Lebesgue measure on $\mathbb{T}^n$ is obtained by identifying $\mathbb{T}^n$ with $[0, 2\pi)^n$

Proposition 5.5.2. *The set $\{e_k\}_{k \in \mathbb{Z}^n}$ is an orthonormal basis for $L^2(\mathbb{T}^n)$ in the sense that the span of the e_k is dense in $L^2(\mathbb{T}^n)$.*

Proof. One uses the Stone–Weierstraß. See [Fol99]. □

Definition 5.5.3. For any $f \in L^1(\mathbb{T}^n)$, $k \in \mathbb{Z}^n$, define the *Fourier coefficient* by

$$\hat{f} := \langle f, e_k \rangle = \int_{\mathbb{T}^n} f e_{-k} = (2\pi)^{-n/2} \int f(x) e^{-ik\cdot x}\, dx.$$

Let

$$\ell^2(\mathbb{Z}^n) = \{g : \mathbb{Z}^n \to \mathbb{Z} \mid \sum_{k \in \mathbb{Z}^n} |g(k)|^2 < \infty\},$$

with norm

$$\|g\|_{\ell^2} = \left(\sum_{k \in \mathbb{Z}^n} |g(k)|^2 \right)^{1/2}.$$

The *Fourier transform*

$$\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n),$$
$$f \mapsto \hat{f}.$$

is a bijective isometry. In particular, for $f, g \in L^2(\mathbb{T}^n)$, one has

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)},$$
$$\|f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^n} \left|\hat{f}(k)\right|^2.$$

Let $v_1, \dots, v_n$ be the standard basis for $\mathbb{R}^n$. For $f : \mathbb{T}^n \to \mathbb{C}$, set

$$(\partial_j f)(x) := \lim_{t\to 0} \frac{1}{t}(f(x + tv_j) - f(x)),$$

where by tv_j, we mean the image of tv_j in $\mathbb{T}^n$.

Proposition 5.5.4. *For $f \in C^1(\mathbb{T}^n)$, $k \in \mathbb{Z}^n$, $j = 1, \dots n$, one has*

$$\left(\frac{1}{i}\partial_j f\right)^{\hat{}}(k) = k_j \hat{f}(k).$$

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Lemma 5.5.5. *For every $y \in C^1(\mathbb{T}^n)$ and every j, one has*

$$\int_{\mathbb{T}^n} \partial_j y = 0.$$

Proof. Let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ be the quotient map, and let $h := g \circ \pi : \mathbb{R}^n \to \mathbb{C}$. Then by Fubini's theorem

$$\int_{\mathbb{T}^n} \partial_j g = \int_{[0,2\pi]^n} \partial_j h = \int_0^{2\pi} \cdots \int_0^{2\pi} \partial_j h(x_1, \dots, x_n) dx_j dx_1 \cdots \widehat{dx_j} \cdots dx_n = 0,$$

since h is periodic. □

Proof of Proposition 5.5.4. We have

$$\partial_j e^{ik\cdot x} = ik_j e^{ik\cdot x},$$

and hence $\partial_j e_k = ik_j e_k$. This implies that

$$0 = \int_{\mathbb{T}^n} \partial_j (f e_{-k}) = \int_{\mathbb{T}^n} (\partial_j f \cdot e_{-k} - ik_j f e_{-k})$$

and the Fourier coefficient becomes

$$\left(\frac{1}{i}\partial_j f\right)^{\hat{}}(k) = \int_{\mathbb{T}^n} \frac{1}{i}\partial_j f \cdot e_{-k} = \int_{\mathbb{T}^n} k_j f e_{-k} = k_j \hat{f}(k).$$

□

Recall that if $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{N}_0$, we define

$$D^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n},\ D_\alpha = \left(\frac{1}{i}\partial_1\right)^{\alpha_1} \cdots \left(\frac{1}{i}\partial_n\right)^{\alpha_n},\ \xi^\alpha := \xi^{\alpha_1} \cdots \xi^{\alpha_n},$$

for $\xi = (\xi_1, \dots, \xi_n) < in \mathbb{R}^n$. For every polynomial $P(\xi) = \sum_{|\alpha|\le r} a_\alpha \xi^\alpha$, $a_\alpha \in \mathbb{C}$, we set $P(D) := \sum_{|\alpha|\le r} a_\alpha D_\alpha$.

Proposition 5.5.6. *For any $f \in C^r(\mathbb{T}^n)$ and P as above, one has*

$$(P(D)f)\hat{}(k) = P(k)\hat{f}(k),$$

i.e. $(P(D)f)\hat{} = P\hat{f}$.

Proof. This follows from linearity of the Fourier transform and repeated use of Proposition 5.5.4. □

5.6 Sobolev embedding on the torus

Let $m \in \mathbb{N}_0$, $f \in C^m(\mathbb{T}^n)$. Define

$$\begin{aligned} \|f\|'_{(m)} &:= \left(\sum_{|\alpha|\le m} \int_{\mathbb{T}^n} |D^\alpha f|^2\right)^{1/2} = \left(\sum_{|\alpha|\le m} \sum_{k\in\mathbb{Z}^n} \left|k^\alpha \hat{f}(k)\right|^2\right)^{1/2} \\ &= \left(\sum_{k\in\mathbb{Z}^m} \left(\sum_{|\alpha|\le m} (k^\alpha)^2\right) \left|\hat{f}(k)\right|^2\right)^{1/2}. \end{aligned}$$

Definition 5.6.1. For $\xi \in \mathbb{R}^n$, set $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

Lemma 5.6.2. *For every $m \in \mathbb{N}_0$, there are constants $C_1, C_2 > 0$, such that for every $\xi \in \mathbb{R}^n$, one has*

$$C_1 \langle \xi \rangle^{2m} \le \sum_{|\alpha|\le m} (\xi^\alpha)^2 \le C_2 \langle \xi \rangle^{2m}.$$

Proof. For every j, we have $|\xi_j| \le |\xi| \le \langle \xi \rangle$, which implies that such a C_2 exists. For the other one, note that

$$\langle \xi \rangle^{2m} = (1 + \xi_1^2 + \cdots + \xi_n^2)^m \le C_3 \sum_{|\alpha|\le m} (\xi^\alpha)^2$$

for some constant $C_3 > 0$. □

Definition 5.6.3. For any $s \in \mathbb{R}$, let

$$h_s(\mathbb{Z}^n) := \{g : \mathbb{Z}^n \to \mathbb{C} \mid \|g\|_{(s)} < \infty\},$$

where

$$\|g\|_{(s)} := \left(\sum_{k\in\mathbb{Z}^n} \langle k \rangle^{2s} |g(k)|^2\right)^{1/2}.$$

Note that $h_s(\mathbb{Z}^n)$ is a Hilbert space with inner product

$$\langle g_1, g_2 \rangle_{(s)} := \sum_{k\in\mathbb{Z}^n} \langle k \rangle^{2s} g_1(k)\overline{g_2(k)}$$

and we have a surjective isometry

$$\begin{aligned} \varphi : h_s(\mathbb{Z}^n) &\xrightarrow{\cong} \ell^2(\mathbb{Z}^n) \\ g &\mapsto \langle \cdot \rangle^s g. \end{aligned}$$

The Fourier transform $\mathcal{F}$ maps $C^\infty(\mathbb{T}^n)$ to $h_s(\mathbb{Z}^n)$, and we can make the following definition.

Definition 5.6.4. For $f, g \in C^\infty(\mathbb{T}^n)$ and $s \in \mathbb{R}$, set

$$\langle f, g\rangle_{(s)} := \langle \hat{f}, \hat{g}\rangle_{(s)}, \ \|f\|_{(s)} := \langle f, f\rangle_{(s)}^{1/2} = \left\|\hat{f}\right\|_{(s)}.$$

For any $m \in \mathbb{N}_0$, the two norms $\|\cdot\|_{(m)}$ and $\|\cdot\|'_{(m)}$ on $C^\infty(\mathbb{T}^n)$ are equivalent by the lemma.

Definition 5.6.5. For any $s \in \mathbb{R}$, let $H_s(\mathbb{T}^n)$ be the completion of the inner product space $(C^\infty(\mathbb{T}^n), \langle\cdot,\cdot\rangle_{(s)})$.

The Fourier transform $\mathcal{F} : C^\infty(\mathbb{T}^n) \to h_s(\mathbb{Z}^n)$ has dense image and therefore extends to a surjective isometry $\mathcal{F} : H_s(\mathbb{T}^n) \xrightarrow{\cong} h_s(\mathbb{Z}^n)$: In general, if X and Y are two metric spaces, and $A \subseteq X$ is dense, then we can ask when a $f : A \to Y$ extends to $f : X \to Y$. In general, this will not be the case, but if f is uniformly continuous, there is a continuous extension.

For $s \le t$, one has $\|\cdot\|_{(s)} \le \|\cdot\|_{(t)}$, because $\langle\xi\rangle^s \le \langle\xi\rangle^t$, and hence $h_t(\mathbb{Z}^n)$ is a dense subset of $h_s(\mathbb{Z}^n)$, so there is an injective bounded operator

$$H_t(\mathbb{T}^n) \to H_s(\mathbb{T}^n),$$

which is the identity on $C^\infty(\mathbb{T}^n)$. Thus for $s \le t$, we have a commutative diagram

$$\begin{array}{ccc} H_t(\mathbb{T}^n) & \xrightarrow{\cong} & h_t(\mathbb{Z}^n) \\ \downarrow & & \downarrow \\ H_s(\mathbb{T}^n) & \xrightarrow{\cong} & h_s(\mathbb{Z}^n) \end{array}$$

and we will usually consider H_t as a dense subspace of H_s. Note that $H_0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$, because for $f \in C^\infty(\mathbb{T}^n)$,

$$\|f\|_{(0)} = \left\|\langle\cdot\rangle^0 \hat{f}\right\|_{\ell^2} = \left\|\hat{f}\right\|_{\ell^2} = \|f\|_{L^2},$$

and $C^\infty \subseteq L^2$ is dense. Therefore, $H_s(\mathbb{T}^n) \subseteq L^2(\mathbb{T}^n)$ for $s \ge 0$. For $s < 0$, H_s consists of distributions.

Proposition 5.6.6. *For $f, g \in C^\infty(\mathbb{T}^n)$ and $s \in \mathbb{R}$, we have*

$$\left|\int_{\mathbb{T}^n} f\overline{g}\right| \le \|f\|_{(s)} \|g\|_{(-s)}.$$

Proof. We find

$$\int_{\mathbb{T}^n} f\overline{g} = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\overline{\hat{g}(k)} = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s \hat{f}(k) \cdot \langle k\rangle^{-s}\overline{\hat{g}(k)},$$

so by the Schwarz inequality,

$$\left|\int_{\mathbb{T}^n} f\overline{g}\right| \le \left\|\hat{f}\right\|_{(s)} \|\hat{g}\|_{(-s)} = \|f\|_{(s)} \|g\|_{(-s)}.$$

□

From now on, write $H_s = H_s(\mathbb{T}^n)$.

Proposition 5.6.7. *For $s \in \mathbb{R}$, let $B : H_s \times H_{-s} \to \mathbb{C}$ be the continuous bilinear map such that*

$$B(f, g) = \int_{\mathbb{T}^n} f\overline{g},$$

for $f, g \in C^\infty(\mathbb{T}^n)$ (an extension exists by Proposition 5.6.6). Then

$$H_{-s} \to (H_s)^*, \quad g \mapsto B(\cdot, g)$$

is a surjective, antilinear isometry.

Proof. The composition

$$H_s \times H_{-s} \overset{\mathcal{F}\times\mathcal{F}}{\to} h_s \times h_{-s} \overset{\cong}{\to} \ell^2 \times \ell^2 \overset{\langle\cdot,\cdot\rangle}{\to} \mathbb{C}$$

is exactly B, and the map $\ell^2 \to (\ell^2)^*$, $h \mapsto \langle \cdot, h\rangle$, is a surjective antilinear isometry. □

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Proposition 5.6.8. *For $s < t$, the canonical map*

$$H_t(\mathbb{T}^n) \to H_s(\mathbb{T}^n)$$

is a compact operator.

Proof. Recall that we have a commutative diagram

$$\begin{array}{ccc} H_t & \xrightarrow{\cong} & \ell^2(\mathbb{Z}^n) \\ \downarrow & & \downarrow \varphi \\ H_s & \xrightarrow{\cong} & \ell^2(\mathbb{Z}^n). \end{array}$$

Here, the upper map is $f \mapsto \langle\cdot\rangle^t \hat{f}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ for $\xi \in \mathbb{R}^n$. The lower map is $f \mapsto \langle\cdot\rangle^s \hat{f}$. Now, because $\langle k \rangle^{s-t} \to 0$ as $|k| \to \infty$, the map φ is compact. □

Definition 5.6.9. For $m \geq 0$, let $C^m(\mathbb{T}^n)$ be the vector space of m times continuously differentiable maps $\mathbb{T}^n \to \mathbb{C}$. For $f \in C^\infty(\mathbb{T}^n)$, let

$$\|f\|_{C^m} := \max_{|\alpha| \leq m} \|D^\alpha f\|_\infty ,$$

where $\|g\|_\infty = \sup_x |g(x)|$ for a continuous function g.

The space $(C^m(\mathbb{T}^n), \|\cdot\|_{C^m})$ is a Banach space (exercise).

Proposition 5.6.10. *Let $s \in \mathbb{R}, m \in \mathbb{N}_0$ with $s - \frac{n}{2} > m$. Then there is a constant $C > 0$ such that for all $f \in C^m(\mathbb{T}^n)$,*

$$\|f\|_{C^m} \leq C \|f\|_{(s)} .$$

The proof rests on the following lemma.

Lemma 5.6.11. *For $s \in \mathbb{R}$, one has that $s > n$ if and only if*

$$\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{-s} < \infty.$$

Proof. The sum is finite if and only if we have

$$\int_{\mathbb{R}^n} \langle x \rangle^{-s}\, dx < \infty,$$

which on the other hand happens precisely when

$$\int_{|x|\geq 1} |x|^{-s}\, dx < \infty.$$

Using polar coordinates, one finds that

$$\int_{|x|\geq 1} |x|^{-s}\, dx = \mathrm{Vol}(S^{n-1}) \int_1^\infty t^{n-s-1}\, dt,$$

which is finite if and only if $n - s - 1 < 1$, or, $n < s$. □

Proof of Proposition 5.6.10. Consider the Fourier transform

$$\left|(D_\alpha f)\hat{}(k)\right| = \left|k^\alpha \hat{f}(k)\right| \leq \langle k\rangle^{|\alpha|} \left|\hat{f}(k)\right|.$$

We find that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n} \left|(D_\alpha f)\hat{}(k)\right| &\leq \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{|\alpha|} \left|\hat{f}(k)\right| = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{|\alpha|-s}\langle k\rangle^{s} \left|\hat{f}(k)\right| \\ &\leq \left(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2(|\alpha|-s)}\right)^{1/2} \|f\|_{(s)} =: C_{|\alpha|} \|f\|_{(s)}.\end{aligned}$$

By the Lemma, we see that $C_{|\alpha|} < \infty$ if and only if $2(s - |\alpha|) > n$, or, $s - \frac{n}{2} > \alpha$. Now, in L^2 one has

$$D_\alpha f = \sum_k (D_\alpha f)\hat{}(k) \cdot e_k.$$

If $|\alpha| < m$, then

$$\sum_k \left|(D_\alpha f)\hat{}(k)\right| \cdot \|e_k\|_\infty < \infty.$$

Hence,

$$\|D_\alpha f\|_\infty \leq \sum_k \left\|(D_\alpha f)\hat{}(k) e_k\right\| \leq (2\pi)^{-n/2} C_\alpha \|f\|_{(s)}.$$

□

Proposition 5.6.12 (Rellich)**.** *If $s \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $s - \frac{n}{2} > m$, then the inclusion $C^\infty(\mathbb{T}^n) \to C^m(\mathbb{T}^n)$ extends to a compact operator $H_s(\mathbb{T}^n) \to C^m(\mathbb{T}^n)$.*

Proof. By the last proposition, the inclusion $C^\infty \to C^m$ extends to a bounded operator $H_s \xrightarrow{T} C^m$. Choose t with $s > t > \frac{n}{2} + m$. Then the operator T is equal to the composition $H_s \to H_t \to C^m$, which is compact since $H_s \to H_t$ is compact, and $H_t \to C^m$ is bounded. □

Corollary 5.6.13. *We have*

$$\bigcap_{s\geq 0} H_s = C^\infty$$

as subspaces of L^2.

Let $P(D) = \sum_{|\alpha|\leq r} a_\alpha D_\alpha$ for $a_\alpha \in C^\infty(\mathbb{T}^n)$. Then we want to show that

$$\|P(D)f\|_{(s-r)} \leq C \|f\|_{(s)}.$$

Proposition 5.6.14. *For every multiindex α and $f \in C^\infty(\mathbb{T}^n)$, one has*

$$\|D_\alpha f\|_{(s)} \leq \|f\|_{(s+|\alpha|)}$$

for all $s \in \mathbb{R}$.

Proof. We have

$$\|D_\alpha f\|_{(s)}^2 = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2s} \left|(D_\alpha f)\hat{}(k)\right|^2 \leq \sum_{k\in\mathbb{Z}^n} \langle k\rangle^{2(s+|\alpha|)} \left|\hat{f}(k)\right|^2 = \|f\|_{(s+|\alpha|)}^2.$$

□

Lemma 5.6.15. *For every $m \in \mathbb{Z}$, there exists a constant $C = C(m,n)$ such that for every $a, f \in C^\infty(\mathbb{T}^n)$, one has*

$$\|af\|_{(m)} \le C\,\|a\|_{C^{|m|}}\,\|f\|_{(m)}$$

Remark 5.6.16. One could prove a similar result for $m \in \mathbb{R}$ using so-called interpolation techniques.

Proof. Assume that $m \ge 0$. Recall that for $g \in C^\infty(\mathbb{T}^n)$,

$$\|g\|'_{(m)} = \left(\sum_{|\alpha|\le m} \|D^\alpha g\|^2_{L^2}\right)^{1/2} \le \sum_{|\alpha|\le m} \|D^\alpha g\|_{L^2}.$$

For $g = af$,

$$D^\alpha(af) = \sum_{\beta+\gamma=\alpha} C_{\beta\gamma} D^\beta a \cdot D^\gamma f,$$

for $C_{\beta\gamma} \in \mathbb{N}_0$. For $|\alpha| \le m$, one has.

$$\|D^\alpha(af)\|_{L^2} \le \text{const} \cdot \|a\|_{C^m} \sum_{|\gamma|\le m} \|D^\gamma f\|_{L^2} \le \text{const} \cdot \|a\|_{C^m}\,\|f\|'_{(m)}.$$

It follows that

$$\|af\|'_{(m)} \le \text{const} \cdot \|a\|_{C^m}\,\|f\|'_{(m)}.$$

Recall that we have a map $H_s \times H_{-s} \to \mathbb{C}$, which defines a bijetive antilinear isomorphism $H_{-s} \to (H_s)^*$. We have

$$\|af\|_{(m)} = \sup_{g\in C^\infty(\mathbb{T}^n),\,\|g\|_{(m)}\le 1} \left|\int_{\mathbb{T}^n} af \cdot \overline{g}\right|.$$

Since

$$\left|\int af \cdot \overline{g}\right| \le \|f\|_{(-m)} \cdot \|\overline{a}g\|_{(m)} \le \|f\|_{(-m)} \cdot \text{const} \cdot \|a\|_{C^m} \cdot \|g\|_{(m)},$$

we have that

$$\|af\|_{(-m)} \le \text{const} \cdot \|a\|_{C^m} \cdot \|f\|_{(-m)}.$$

□

The last two results yield the following:

Proposition 5.6.17. *Let $P(D) = \sum_{|\alpha|\le r} a_\alpha D_\alpha$ with $a_\alpha \in C^\infty(\mathbb{T}^n)$. Then for any $m \in \mathbb{Z}$, there is a bounded operator*

$$P_m : H_m(\mathbb{T}^n) \to H_{m-r}(\mathbb{T}^n)$$

such that $P_m f = P(D)f$ for $f \in C^\infty$.

Definition 5.6.18. For $s \in \mathbb{R}$, $d \in \mathbb{N}$, let $H_s(\mathbb{T}^n;\mathbb{C}^d)$ be the completion of $C^\infty(\mathbb{T}^n;\mathbb{C}^d)$ with respect to the inner product

$$\langle f, g\rangle_{(s)} := \sum_{j=1}^d \langle f_j, g_j\rangle_{(s)},$$

where $f = (f_1, \dots, f_d)$, $g = (g_1, \dots, g_d)$.

The inclusion

$$C^\infty(\mathbb{T}^n;\mathbb{C}^d) \to \bigoplus_{j=1}^{d} H_s(\mathbb{T}^n)$$

extends to a surjective isometry

$$H_s(\mathbb{T}^n;\mathbb{C}^d) \overset{\cong}{\to} \bigoplus_{j=1}^{d} H_s(\mathbb{T}^n).$$

Proposition 5.6.19. *We have*

(i) $\left|\int_{\mathbb{T}^n}\langle f,g\rangle\right| \le \|f\|_{(s)}\,\|g\|_{(-s)}$, *for* $f,g\in C^\infty(\mathbb{T}^n;\mathbb{C}^d)$, $s\in\mathbb{R}$,

(ii) The map

$$H_{-s}(\mathbb{T};\mathbb{C}^d) \to (H_s(\mathbb{T}^n;\mathbb{C}^d))^*$$

given by $g\mapsto \int\langle\cdot,g\rangle$ *is an anti-linear surjective isometry.*

Proof. The proofs in the case $d=1$ carries on. □

Proposition 5.6.20. *Let* $P(\xi)=\sum_{|\alpha|\le r} a_\alpha\xi^\alpha$, *where* $a_\alpha\in M_{d_1,d_2}(\mathbb{C})$ *are* $d_1\times d_2$ *complex matrices. Then for* $f\in C^\infty(\mathbb{T}^n;\mathbb{C}^d)$, *one has*

$$(P(D)f)\hat{}=P\cdot\hat{f},$$

where $P(D):=\sum_{|\alpha|\le r} a_\alpha D_\alpha$, *and* $\hat{f}(k)=(\hat{f}_1(k),\dots,\hat{f}_d(k))\in\mathbb{C}^d$.

Proof. As in the case $d=1$. □

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Theorem 5.6.21. *Let* $P(D)=\sum_{|\alpha|\le r} a_\alpha D_\alpha$ *be an elliptic operator of degree* $r\ge 1$, *where* $a_\alpha\in M_d(\mathbb{C})$. *Then for every* $m\in\mathbb{Z}$, *the bounded operator*

$$H_m(\mathbb{T}^n;\mathbb{C}^d)\to H_{m-r}(\mathbb{T}^n;\mathbb{C}^d)$$

induced by $P(D)$ *is Fredholm of index* 0.

Proof. Set $Q(D)=\sum_{|\alpha|<r} a_\alpha D_\alpha$, $R(D)=\sum_{|\alpha|=r} a_\alpha D_\alpha$ so that $P(D)=Q(D)+R(D)$. Because the composite map

$$H_m \overset{Q(D)}{\to} H_{m-r+1}\subseteq H_{m-r}$$

is compact (since the inclusion is), we can ignore the lower order terms altogether, and it suffices to prove that $R(D):H_m\to H_{m-r}$ is Fredholm of index 0. Now we have a commutative diagram

$$\begin{array}{ccc} H_m & \xrightarrow{R(D)} & H_{m-r} \\ \mathcal{F}\downarrow\cong & & \cong\downarrow\mathcal{F} \\ h_m & \xrightarrow{M_R} & h_{m-r} \end{array}$$

where $M_R(f)(k)=R(k)\cdot f(k)$ for $f\in h_m$. Recall here that

$$h_m=\{g:\mathbb{Z}^n\to\mathbb{C}^d \mid \sum_{k\in\mathbb{Z}^n}\langle k\rangle^{2m}\,|g(k)|^2<\infty\}.$$

For every s, $h_s = V_s \oplus W$, where

$$W = \{f : \mathbb{Z}^n \to \mathbb{C}^d \mid f(k) = 0 \;\forall\, k \neq 0\} \cong \mathbb{C}^d,$$
$$V_s = \{f \in h_s \mid f(0) = 0\}.$$

Since $R(0) = 0$,

$$M_r = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} : V_m \oplus W \to V_{m-r} \oplus W.$$

We will show that $\varphi : V_m \to V_{m-r}$ is an isomorphism; this will complete the proof. Let

$$C := \sup_{\xi \in \mathbb{R}^n,\, |\xi|=1} \left\| R(\xi)^{-1} \right\| < \infty.$$

For all $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $R(\xi) = |\xi|^r R(\xi/|\xi|)$ and $R(\xi)^{-1} = |\xi|^{-r} R(\xi/|\xi|)^{-1}$, and so $\left\| R(\xi)^{-1} \right\| \leq C\, |\xi|^{-r}$. For all $\xi \in \mathbb{R}^n$ with $|\xi| \geq 1$,

$$\langle \xi \rangle = (1 + |\xi|^2)^{1/2} \leq \sqrt{2}\, |\xi|\,,$$

and so

$$\left\| R(\xi)^{-1} \right\| \leq 2^{r/2} C \langle \xi \rangle^{-r}.$$

For $f \in V_{m-r}$, $k \in \mathbb{Z}^n \setminus \{0\}$, set

$$(\psi f)(k) := R(k)^{-1} f(k).$$

By the estimate above, and noting that $f(0) = 0$,

$$\begin{aligned} \|\psi f\|_{(m)}^2 &= \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2m} \left|(\psi f)(k)\right|^2 = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \langle k \rangle^{2m} \left|(\psi f)(k)\right|^2 \\ &\leq 2^r C^2 \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{2(m-r)} \left|f(k)\right|^2 = 2^r C^2 \|f\|_{(m-r)}^2 < \infty. \end{aligned}$$

Hence $\psi : V_{m-r} \to V_m$ is bounded, and ψ and φ are inverse maps. □

5.7 L^2-Sobolev spaces on manifolds

Definition 5.7.1. For $U \subseteq \mathbb{T}^n$ open, $f \in C_c^\infty(U; \mathbb{C}^d)$, define $\tilde{f} : \mathbb{T}^n \to \mathbb{C}^d$ by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.7.2. *Let $U, V \subseteq \mathbb{T}^n$ be open, $K \subseteq V$ compact, $h : U \to V$ a smooth map, and $m \in \mathbb{Z}$. Then there exists $C > 0$ such that for all $f \in C_c^\infty(V)$ with* $\operatorname{supp}(f) \subseteq K$*, one has*

$$\left\| \widetilde{f \circ h} \right\|_{(m)} \leq C \, \|\tilde{f}\|_{(m)}.$$

Proof. We first prove the proposition for $m \geq 0$ by induction. For the case $m = 0$, note that $\|\cdot\|_{(0)} = \|\cdot\|_{L^2}$, and

$$\left\| \widetilde{f \circ h} \right\|_{L^2}^2 = \int_U |f \circ h|^2 = \int_V |f|^2 \cdot \left|\det D(h^{-1})\right| \leq \sup_{x \in K} \left|\det D(h^{-1})_x\right| \cdot \int_V |f|^2 = C \, \|\tilde{f}\|_{L^2}\,,$$

letting $C = \sup_{x \in K} \left|\det D(h^{-1})_x\right|$.

Now, let $m > 0$. For $j = 1, \dots, n$, we find

$$\partial_j(f \circ h) = \sum_{l=1}^{n} \partial_j h_l \cdot [(\partial_l f \circ h)],$$

where $h = (h_1, \dots, h_n)$. Set $K' = h^{-1}(K)$ and choose $\beta \in C^\infty(\mathbb{T}^n)$ such that $\operatorname{supp}(\beta) \subseteq U$ and $\beta = 1$ in a neighbourhood of K'.

For $|\alpha| \leq m-1$, there are constants $C_1, C_2, C_3 > 0$ independent of f, such that

$$\begin{aligned} \|D^\alpha \partial_j(f \circ h)\|_{L^2} \leq \|\partial_j(f \circ h)\|_{(m-1)} &\leq C_1 \|\beta \partial_j h_l\|_{C^{m-1}} \|\partial_l f \cdot h\| \\ &\leq \sum_{l=1}^{m} C_2 \|\partial_l f\|_{(m-1)} \leq C_3 \|f\|_{(m)}, \end{aligned}$$

which completes the case $m \geq 0$. For the case of negative m, consider again $m \geq 0$. Then

$$\|f \circ h\|_{(-m)} = \sup_{g \in C^\infty(\mathbb{T}^n, \mathbb{C}^d),\, \|g\|_{(m)} \leq 1} \left| \int_{\mathbb{T}^n} (f \circ h) \cdot g \right|.$$

As before, consider a cutoff function β. Then

$$\int_{\mathbb{T}^n} (f \circ h) \cdot g = \int_U (f \circ h) \cdot \beta g = \int_V f \cdot [(\beta g) \circ h^{-1}] \left|\det D(h^{-1})\right|.$$

Let $\varphi = [(\beta g) \circ h^{-1}] \left|\det D(h^{-1})\right|$. Then $\varphi \in C_c^\infty(V)$, and we have

$$\left| \int_{\mathbb{T}^n} (f \circ h) \cdot g \right| \leq \|f\|_{(-m)} \|\varphi\|_{(m)},$$

which finishes the proof since $\|\varphi\|_{(m)}$ does not depend on f and is bounded in the g considered: Namely, there exist constants $C_4, C_5, C_6 > 0$, independent of f and g, such that

$$\|\varphi\|_{(m)} \leq C_4 \left\|(\beta g) \circ h^{-1}\right\|_{(m)} \leq C_5 \|\beta g\|_{(m)} \leq C_6 \|g\|_{(m)} \leq C_6,$$

where the second inequality follows from the first part of the proof. This implies that

$$\|f \circ h\|_{(-m)} \leq C_6 \|f\|_{(-m)}.$$

□

Let X be a smooth n-manifold and $\pi : E \to X$ a rank d complex vector bundle. Let $U \subseteq X$ be open. A *presentation* of $E|_U$ is a triple $\delta = (U', \varphi, \rho)$, where

- $U' \subseteq \mathbb{T}^n$ is open,
- $\varphi : U \xrightarrow{\cong} U'$ is a diffeomorphism, and
- $\rho : E|_U \to \mathbb{C}^d$ is smooth, and $\rho_x := \rho|_{E_x}$ is a linear isomorphism for every $x \in U$.

For $s \in \Gamma(E)$, $\operatorname{supp}(s) \subseteq U$, define $s_\delta : \mathbb{T}^n \to \mathbb{C}^d$ by

$$s_\delta(x) := \begin{cases} \rho(s(\varphi^{-1}(x))), & x \in U', \\ 0, & \text{otherwise.} \end{cases}$$

22nd lecture, December 8th 2011

Let X be a closed n-manifold and $E \to X$ a rank d complex vector bundle. To define $H_m(X; E)$, $m \in \mathbb{Z}$, choose a finite family $\lambda = \{(U_j, \delta_j, \beta_j)\}_{j\in J}$, where

- $\{U_j\}$ is an open covering of X,
- $\delta_j(U'_j, \varphi_j, \rho_j)$ is a presentation of $E|_{U_j}$ (so $\varphi_j : U_j \xrightarrow{\cong} U'_j$), and
- $\beta_j \in C^\infty(X)$ satisfies $0 \le \beta_j \le 1$, $\mathrm{supp}(\beta_j) \subseteq U_j$, $\sum_{j\in J} \beta_j = 1$.

Define

$$\begin{aligned} T_m : \Gamma(E) &\to \bigoplus_{j\in J} H_m(\mathbb{T}^n; \mathbb{C}^d), \\ s &\mapsto \{(\beta_j s)_{\delta_j}\}_{j\in J}. \end{aligned}$$

Then T_m is injective and $\mathbb{C}$-linear. For $s, t \in \Gamma(E)$, define a scalar product on $\Gamma(E)$

$$\langle s, t\rangle_{(m)} := \langle T_m(s), T_m(t)\rangle = \sum_m = \sum_{j\in J} \langle (\beta_j s)_{\delta_j}, (\beta_j t)_{\delta_j}\rangle.$$

depending a priori on the choice of λ. By Proposition 5.7.2, the corresponding norm $\|\cdot\|_{(m)}$ is independent of λ up to equivalence.

Definition 5.7.3. Let $H_m(X; E)$ denote the completion of $(\Gamma(E), \langle\cdot,\cdot\rangle_{(m)})$.

The Sobolev embedding theorem carries over to this case.

Proposition 5.7.4 (Rellich, I)**.** *If $l, m \in \mathbb{Z}$, $l < m$, then the operator*

$$H_m(X; E) \to H_l(X; E)$$

is injective and compact.

Proof. We have a diagram which commutes for smooth sections of E, and therefore for every element of $H_m(X; E)$:

$$\begin{array}{ccc} H_m(X;E) & \xrightarrow{T_m} & \bigoplus_{j\in J} H_m(\mathbb{T}^n;\mathbb{C}^d) \\ \downarrow & & \downarrow \\ H_l(X;E) & \xrightarrow{T_l} & \bigoplus_{j\in J} H_l(\mathbb{T}^n;\mathbb{C}^d) \end{array}$$

The horizontal maps are isometries. The right vertical map is injective and compact, so its image is totally bounded, and then the left vertical map is injective totally bounded as well. That is, it takes bounded sets to pre-compact sets, hence it is bounded. □

Definition 5.7.5. For $k \in \mathbb{N}_0$, let $C^k(X; E)$ denote the space of sections of E of clas C^k. For $s \in C^k(X; E)$, set

$$\|s\|_{C^k} := \sum_j \left\|(\beta_j s)_{\delta_j}\right\|_{C^k}.$$

The space $(C^k(X; E), \|\cdot\|_{C^k})$ is a Banach space, and the norm is independent of λ up to equivalence.

Proposition 5.7.6 (Rellich, II)**.** *If $l, m \in \mathbb{N}_0$, $m - \frac{n}{2} > l$, then the inclusion $\Gamma(E) \subseteq C^k(X; E)$ extends to an injective compact operator $H_m(X; E) \to C^l(X; E)$.*

Proof. The proof is the same as before. We have a diagram

$$\begin{array}{ccc} H_m(X;E) & \longrightarrow & \bigoplus_{j\in J} H_m(\mathbb{T}^n;\mathbb{C}^d) \\ \downarrow & & \downarrow \\ C^l(X;E) & \longrightarrow & \bigoplus_{j\in J} C^l(\mathbb{T}^n;\mathbb{C}^d) \end{array}$$

The vertical maps, extending the inclusions of smooth functions into C^l, exist since the C^l-norms bound the H_m-norms. The right vertical map is injective and compact by the corresponding result by on $\mathbb{T}^n$. As before, the horizontal are isometries and by the same arguments as before, the left vertical map satisfies the conditions of the proposition. □

Let now X be a Riemannian manifold and $E \to X$ a Hermitian vector bundle.

Proposition 5.7.7. *For any $m \in \mathbb{Z}$ there exists $C > 0$ such that for all $s, t \in \Gamma(E)$, we have*

$$\left| \int_X \langle s, t\rangle \right| \leq C \left\|s\right\|_{(m)} \left\|t\right\|_{(-m)}$$

Proof. This follows from the corresponding result for $\mathbb{T}^n$. □

Proposition 5.7.8. *The operator*

$$H_{-m}(X;E) \to (H_m(X;E))^*$$
$$t \mapsto \int_X \langle \cdot, t\rangle$$

is a complex antilinear homeomorphism for every $m \in \mathbb{Z}$.

Proof. Exercise. □

Lemma 5.7.9. *Let $P(D) = \sum_{|\alpha|\leq r} a_\alpha D_\alpha$, $a_\alpha \in M_d(\mathbb{C})$. If $P(D)$ is elliptic of degree r, then*

$$Q := P(D)^* P(D) + I : H_m(\mathbb{T}^n;\mathbb{C}^d) \to H_{m-2r}(\mathbb{T}^n;\mathbb{C}^d)$$

is an isomorphism for all $m \in \mathbb{Z}$

Proof. Note that $P(D) = \sum_{|\alpha|\leq r} a_\alpha^* D_\alpha$. We already know that Q is Fredholm of index 0, so it suffices to show that it is injective, or equivalently, that $(P^*P+I)u \neq 0$, where $0 \neq u \in h_m(\mathbb{Z}^n;\mathbb{C}^d)$ and $P(\xi) = \sum_{|\alpha|\leq r} a_\alpha \xi^\alpha$. Let $k \in \mathbb{Z}^n$, $A := P(k) \in M_d(\mathbb{C})$, $0 \neq v \in \mathbb{C}^d$. Then

$$\langle (A^*A + I)v, v\rangle = |Av|^2 + |v|^2 \neq 0$$

implies that $P^*(k)P(k) + I$ is invertible for all $k \in \mathbb{Z}^n$. □

Lemma 5.7.10. *Let $E, F \to X$ be complex vector bundles of rank d and $P : \Gamma(E) \to \Gamma(F)$ an elliptic operator of degree $r \geq 1$. Let $x_0 \in X$, $m \in \mathbb{Z}$, $N > 0$. Then there exist*

- *an open neighbourhood U of x_0,*
- *presentations $\delta = (U', \varphi, \rho)$ of E_U and $\varepsilon = (U', \varphi, \sigma)$ of $F|_U$, and*
- *a differential operator $Q : C^\infty(\mathbb{T}^n;\mathbb{C}^d) \to C^\infty(\mathbb{T}^n;\mathbb{C}^d)$*

such that for all $s \in \Gamma(E)$,

$$Q(s_\delta) = (Ps)_\varepsilon$$

and such that $Q^*Q + I$ induces an isomorphism

$$H_l(\mathbb{T}^n;\mathbb{C}^d) \to H_{l-2r}(\mathbb{T}^n;\mathbb{C}^d)$$

for $|l| \leq N$.

Proof. The proof uses the method of "freezing coefficients". The idea is that on the torus, an elliptic operator with constant coefficients induces a Fredholm operator of index 0. The idea is that if we have an elliptic operator on the manifold, then globally, the coefficients are "almost constant".

The problem is local, so we can assume that $X = \mathbb{R}^n$, $x_0 = 0$, $E = F = X \times \mathbb{C}^d$. Let $P = \sum_{|\alpha| \leq r} a_\alpha D_\alpha$, $a_\alpha : \mathbb{R}^n \to M_d(\mathbb{C})$. For $t > 0$, let $\chi_t : \mathbb{R}^n \to \mathbb{R}^n$ be the map $x \mapsto tx$. For $f : \mathbb{R}^n \to \mathbb{C}^d$, set

$$P_t f := t^r [P(f \circ \chi_{-t})] \circ \chi_t.$$

As usual, write $\partial_j = \frac{\partial}{\partial_j}$. Then

$$\partial_j f(t^{-1}x) = t^{-1}(\partial_j f)(t^{-1}x),$$

and so

$$D_\alpha(f \circ \chi_{t^{-1}}) = t^{-|\alpha|}(D_\alpha f) \circ \chi_{t^{-1}}.$$

This means that

$$P(f \circ \chi_{t^{-1}}) = \sum_{|\alpha| \leq r} a_\alpha D_\alpha (f \circ \chi_{-t}) = \sum_{|\alpha| \leq r} t^{-|\alpha|} a_\alpha [(D_\alpha f) \circ \chi_{-t}],$$

and finally,

$$P_t f = \sum_{|\alpha| \leq r} t^{r-|\alpha|} (a_\alpha \circ \chi_t) D_\alpha f.$$

Define $P_0 f := \sum_{|\alpha|=r} a_\alpha(0) D_\alpha$. Observe now that if $q : \mathbb{R}^n \to \mathbb{C}$ is smooth, then $q \circ \chi_t \to_{t \to 0} q(0)$ in C^∞ over compact subsets of $\mathbb{R}^n$. I.e. for $h_t(x) := q(tx) - q(0)$, one has for any $R > 0$ and multi-index α, that

$$\sup_{|x| \leq R} |D^\alpha h_t(x)| \to 0$$

as $t \to 0$. We want to apply this to the coefficients of P_t. Choose $\beta \in C^\infty(\mathbb{R}^n)$ with

$$\beta(x) = \begin{cases} 0, & \text{for } \max_j |x_j| \geq 3 \\ 1, & \text{for } \max_j |x_j| \leq 2 \end{cases}$$

and set

$$L_t := \beta P_t + (1 - \beta) P_0.$$

Consider the operator

$$\begin{aligned} \sum_{|\alpha| \leq r} b_{\alpha,t} D_\alpha : &= L_t - P_0 = \beta(P_t - P_0) \\ &= \beta \sum_{|\alpha|=r} (a_\alpha \circ \chi_t - a_\alpha(0)) D_\alpha + \beta \sum_{|<|r} t^{r-|\alpha|} (a_\alpha \circ \chi_t) D_\alpha. \end{aligned}$$

Thus, for all α one has that $b_{\alpha,t} \to 0$ in C^∞ when $t \to 0$. We identify $[-3,3]^n$ with its image in $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. By trivially extending the coefficients of L_t, we obtain an operator $Q_t = \sum_{|\alpha| \leq r} \tilde{b}_{\alpha,t} D_\alpha$ on $\mathbb{T}^n$. Consider

$$Q_t^* Q_t - P_0^* P_0 = \sum_{|\alpha| \leq 2r} c_{\alpha,t} D_\alpha,$$

where, for all α, $c_{\alpha,t} \to 0$ in C^∞ as $t \to 0$. For $f \in C^\infty(\mathbb{T}^n, \mathbb{C}^d)$,

$$\|(Q_t^* Q_t - P_0^* P_0) f\|_{(m-2r)} \leq \text{const.} \cdot \sum_{|\alpha| \leq 2r} \|c_{\alpha,t}\|_{C^{|m-2r|}} \|f\|_{(m)} .$$

By Lemma 5.7.9, the operator $P_0^* P_0 + I : H_m \to H_{m-2r}$ is an isomorphism for all $m \in \mathbb{Z}$. Hence, for any $m \in \mathbb{Z}$, we have that

$$Q_t^* Q_t + I : H_m \to H_{m-2r}$$

is an isomorphism for sufficiently small $t > 0$ (depending on m). □

23rd lecture, December 13th 2011

Theorem 5.7.11. *Let X be a closed n-manifold, let $E \to X$, $F \to X$ be complex vector bundles, and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order $r \geq 1$. Let $d := \operatorname{rank} E = \operatorname{rank} F$. Let $l, m \in \mathbb{Z}$, $l < m$. Then:*

(i) (Elliptic inequality) There exists a constant $C > 0$ such that for all $s \in H_m(X; E)$,

$$\|s\|_{(m)} \leq C(\|Ps\|_{(m-r)} + \|s\|_{(l)}).$$

(ii) (Elliptic regularity) If $s \in H_l(X; E)$ and $Ps \in H_{l-r}$ and $Ps \in H_{m-r}(X; F)$ (note that a priori, $Ps \in H_{l-r}$), then $s \in H_m(X; E)$. In particular if $Ps \in \Gamma(F)$, then $s \in \Gamma(E)$.

Proof. By Lemma 5.7.10, we can find

- a finite open covering $\{U_j\}_{j \in J}$ of X,
- for each j a presentation $\delta_j = (U_j', \varphi_j, \rho_j)$ of $E|_U$, $\varepsilon_j = (U_j', \varphi_j, \sigma_j)$ of $F|_U$,
- for each j a $\beta_j \in C^\infty(X)$ with $\operatorname{supp}(\beta_j) \subseteq U_j$ such that $\sum_j \beta_j = 1$,
- for each j a partial differential operator $Q_j : C^\infty(\mathbb{T}^n; \mathbb{C}^d) \to C^\infty(\mathbb{T}^n; \mathbb{C}^d)$ of order r such that for some open neighbourhood V_j of $\operatorname{supp}(\beta_j)$ in U_j, one has $Q_j s_{\delta_j} = (Ps)_{\varepsilon_j}$ for all $s \in \Gamma(E|_{V_j})$ and such that

$$Q_j^* Q_j + I : H_k \to H_{k-2r}$$

is an isomorphism for $k = l, \ldots, m$.

To see (i), observe first that for every $f \in H_m(\mathbb{T}^n; \mathbb{C}^d)$, we have

$$\|f\|_{(m)} C_1 \left\|Q_j^* Q_j f + f\right\|_{(m-2r)} \leq C_2(\|Q_j f\|_{(m-r)} + \|f\|_{(m-1)}).$$

It is enough to prove the claim for $s \in \Gamma(E)$. to see this, pick $t \in \Gamma(E)$ with $\|t - s\|_{(m)} \leq \frac{\tilde{C}}{3} \|s\|_{(m)}$ for suitably small $\tilde{C}$. Then

$$\begin{aligned} \|s\|_{(m)} &\leq \|t\|_{(m)} + \|t - s\|_{(m)} \leq C(\|Pt\|_{(m-r)} + \|t\|_{(l)}) + \frac{\tilde{C}}{3} \|s\|_{(m)} \\ &\leq C(\|Ps\|_{(m-r)} + C' \|P(t-s)\|_{(m-r)} + \|t\|_{(l)} + \|t - s\|_{(l)}) + \tilde{C} \|s\|_{(m)} \end{aligned}$$

For $s \in \Gamma(E)$, we have

$$\begin{aligned} \|s\| &\leq \sum_j \left\|(\beta_j s)_{\delta_j}\right\|_{(m)} \leq C_2 \sum_j (\left\|Q_j (\beta_j s)_{\delta_j}\right\|_{(m-r)} + \left\|(\beta_j s)_{\delta_j}\right\|_{(m-1)}) \\ &\leq C_3 \left(\sum_j \|P(\beta_j s)\|_{(m-r)} + \|s\|_{(m-1)} \right). \end{aligned}$$

Now $P(\beta_j s) = [P, \beta_j]s + Ps$, and $[P, \beta_j]$ is of order $\leq r - 1$, so

$$\|P(\beta_j s)\|_{(m-r)} \leq \|[P, \beta_j]s\|_{(m-r)} + \|\beta_j Ps\| \leq C_4(\|s\|_{(m-1)} + \|Ps\|_{(m-r)}).$$

So altogether,

$$\|s\|_{(m)} \leq C_5(\|Ps\|_{(m-r)} + \|s\|_{(m-1)}),$$

so we have proved (i) for $l = m - 1$. Now using this for $m \to m - 1$, we have

$$\|s\|_{(m-1)} \leq C_6(\|Ps\|_{(m-r-1)} + \|s\|_{(m-2)}),$$

and so

$$\|s\|_{(m)} \leq C_7(\|Ps\|_{(m-r)} + \|s\|_{(m-2)}),$$

which proves (i) by induction.

It suffices to prove (ii) for $l = m - 1$. Let $s \in H_{m-1}(X; E)$ with $Ps \in H_{m-r}(X; F)$. Since $s = \sum_j \beta_j s$, it suffices to prove that $\beta_j s \in H_m(X; E)$ for every j. Since as before $P(\beta_j s) = [P, \beta_j]s + \beta_j Ps$, $[P, \beta_j]$ has order $\leq r - 1$, and since $s \in H_{m-1}$, we have $P(\beta_j s) \in H_{m-r}$. Fix j and write $\beta = \beta_j$, $U = U_j$, etc. Choose a sequence $\{\tilde{p}_\nu\}$ in $\Gamma(F)$ with

$$\|\tilde{p}_\nu - P(\beta s)\|_{(m-r)} \to 0$$

as $\nu \to \infty$. Choose $\gamma \in C^\infty(X)$ such that $\mathrm{supp}(\gamma) \subseteq U$ and $\gamma = 1$ in a neighbourhood of $\mathrm{supp}(\beta)$. Then

$$\gamma P(\beta s) = [\gamma, P]\beta s + P(\gamma\beta s) = 0 + P(\beta s) = P(\beta s).$$

Set $p_\nu := \gamma\tilde{p}_\nu$. Then

$$\|p_\nu - P(\beta s)\|_{(m-r)} = \|\gamma(\tilde{p}_\nu - P(\beta s))\|_{(m-r)} \leq \text{const.} \cdot \|\gamma\|_{C^{|m-r|}} \|\tilde{p}_\nu - P(\beta s)\|_{(m-r)} \to 0,$$

as $\nu \to \infty$. Choose a sequence $\{t_\nu\}$ in $\Gamma(E)$ with $\|t_\nu - s\|_{(m-1)} \to 0$ as $\nu \to \infty$. Then $\{(\beta t_\nu)_\delta\}$ is a Cuachy sequence in $H_{m-1}(\mathbb{T}^n; \mathbb{C}^d)$, and similarly $\{(p_\nu)_\varepsilon\}$ is a Cauchy sequence in H_{m-r}. Set $f := \lim_{\nu\to\infty}(\beta t_\nu)_\delta \in H_{m-1}$, $g := \lim_{\nu\to\infty}(p_\nu)_\varepsilon \in H_{m-r}$. Now, observe that

$$\|Q(\beta t_\nu)_\delta - (p_\nu)_\varepsilon\|_{(m-r-1)} = \|(P\beta t_\nu - p_\nu)_\varepsilon\|_{m-r-1} \leq C_1 \|P\beta t_\nu - p_\nu\|_{(m-r-1)} \to 0$$

as $\nu \to \infty$. This implies that $Qf = g$, and then $(Q^*Q + I)f = Q^*g + f \in H_{m-2r}$. Since $Q^*Q + I : H_m \to H_{m-2r}$ is an isomorphism, in particular it is surjective, so there is an $f' \in H_m$ with $(Q^*Q + I)(f' - f) = 0$. Now $f' - f \in H_{m-1}$, and $Q^*Q + I : H_{m-1} \to H_{m-2r-1}$ is injective, so we must have $f = f'$. Now choose a compact subset $K \subseteq U'$ and a sequence $\{f_\nu\}$ in $C^\infty(\mathbb{T}^n; \mathbb{C}^d)$ such that $\mathrm{supp}(f_\nu) \subseteq K$ for all ν and $\|f_\nu - f\|_{(m)} \to 0$ as $\nu \to \infty$. Let $q_\nu \in \Gamma(E)$ be the section with $\mathrm{supp}(q_\nu) \subseteq U$ and $(q_\nu)_\delta = f_\nu$. Then

$$\|q_\nu - \beta t_\nu\|_{(m-1)} \leq C_2 \|f_\nu - (\beta t_\nu)_\delta\|_{(m-1)} \to 0$$

as $\nu \to \infty$. That implies that $\beta s = \lim_\nu q_\nu$ in H_{m-1}. Since $\{q_\nu\}$ converges in H_m, we have $\beta s \in H_m$. $\square$

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