Conformal Field Theory

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Lecture overview

Disclaimer

These are notes from a course given by Benjamin Himpel, Hans-Christian Herbig, and Johan Martens during the spring of 2012. More precisely, Hans-Christian covered lectures 1–4 and 17, Benjamin lectures 5–10 and 19, Johan lectures 11–16, 18, and 20. They have been written and TeX'ed during the lecture and have not been proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturers. Also, I've made these notes primarily to be able to look back on what happened with ease. That being said, feel very free to send any comments and/or corrections to `fuglede@qgm.au.dk`.

The most recent version of these notes is available at `http://home.imf.au.dk/pred`.

1st lecture, February 8th 2012

1 Aim of the course

The plan of the course is to work through the first chapters of [Uen08]. There are several approaches to the topic; one uses vertex algebras, which we will not discuss. Another one is a highly technical thing called chiral algebras, which is a more coordinate free approach to vertex algebras. In this course, we will mostly be dealing with conformal blocks, and the most important object will be the Knizhnik–Zamolodchikov equations.

2 Central extensions of Lie algebras

2.1 Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra with base field $\mathbb{C}$ (which is not important at present) and let V be a $\mathfrak{g}$-module. Then we can write down a cochain complex

$$C^n(\mathfrak{g}, V) = \mathrm{Alt}^n(\mathfrak{g}, V),$$

the space of alternating n-multilinear maps from $\mathfrak{g}$ to V, with differential δ given by

$$\begin{aligned}(\delta\omega)(X_0, \dots, X_n) = &\sum_{i=0}^{n} (-1)^i X_i \dot{\omega}(X_0, \dots, \hat{X}_i, \dots, X_n) \\ &+ \sum_{0 \le i < j \le n} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_n).\end{aligned}$$

This is a very well-known complex, whose cohomology depends on both $\mathfrak{g}$ and V, and which has been computed for many Lie algebras.

Consider the special case where V is the trivial $\mathfrak{g}$-module, and $n = 2$. Write Z and B for the spaces of cocycles and boundaries. We have $\gamma \in Z^2(\mathfrak{g}, V)$ if and only if $\gamma(X, [Y, Z]) + \gamma(Y, [Z, X]) + \gamma(Z, [X, Y]) = 0$. Also, we have $\gamma \in B^2(\mathfrak{g}, V)$ if and only if $\gamma(X, Y) = \alpha([X, Y])$ for some $\alpha \in C^1(\mathfrak{g}, V)$.

Definition 2.1.1. A *central extension* $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by V is a short exact sequence of Lie algebras

$$0 \to V \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

such that $[V, \tilde{\mathfrak{g}}] = 0$.

We can always split a central extension as $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$ (as vector spaces), so the issue here is that we might have something like

$$[X, Y]^{\sim} = [X, Y] + \gamma(X, Y),$$

where $\gamma(X, Y) \in V$. Now γ is a 2-cocycle if and only if $[\ ,\ ]^{\sim}$ satisfies the Jacobi identity (check!).

Definition 2.1.2. Two central extensions $\tilde{\mathfrak{g}}_1, \tilde{\mathfrak{g}}_2$ of $\mathfrak{g}$ by V are *equivalent* if there exists a Lie algebra isomorphism $\varphi : \tilde{\mathfrak{g}}_1 \to \tilde{\mathfrak{g}}_2$ such that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & \tilde{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{=} & & \downarrow{\varphi} & & \downarrow{=} & & \\ 0 & \longrightarrow & V & \longrightarrow & \tilde{\mathfrak{g}}_2 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0. \end{array}$$

Exercise 2.1.3. Show that there is a 1-1 correspondence between central extensions of $\mathfrak{g}$ by V and elements of $H^2(\mathfrak{g}, V)$. Show also that all statements can be extended to non-trivial V.

2.2 Examples of central extensions

Let $\mathfrak{g}$ be a Lie algebra (which will not play the role of $\mathfrak{g}$ in the previous setup) over $\mathbb{C}$. Consider the *formal loop algebra* $L\mathfrak{g} = \mathfrak{g}((\xi))$. If $\dim \mathfrak{g} < \infty$, then $\mathfrak{g}((\xi)) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((\xi))$ which will be a Lie algebra (we could also consider the *loop algebra* $\mathfrak{g}[\xi^{-1}, \xi]$, but $L\mathfrak{g}$ will be the natural object for us). Assume that $\mathfrak{g}$ has an invariant scalar product $(\ ,\ )$, i.e. $([X, Y], Z) = (X, [Y, Z])$ (for simple Lie algebras, this could be the Cartan–Killing form). Let V be a 1-dimensional vector space, $V = \mathbb{C} \cdot c$ and define

$$\gamma(X \otimes f(\xi), Y \otimes g(\xi)) = c \cdot (X, Y) \operatorname{Res}_{\xi=0}(g(\xi)\, df(\xi)),$$

that is, $\operatorname{Res}_{\xi=0}(g(\xi)\, df(\xi))$ is the coefficient of ξ^{-1}. We will often drop the dependence on ξ in the notation.

Exercise 2.2.1. Let $\varphi(f, g) = \operatorname{Res}_{\xi=0}(g\, df)$. Show that

1. $\varphi(f, g) = -\varphi(g, f)$, that
2. $\varphi(fg, h) + \varphi(gh, f) + \varphi(h, fg) = 0$, and that
3. γ is a 2-cocycle.

This exercise allows us to extend $L\mathfrak{g}$ by the cocycle γ to $\hat{\mathfrak{g}} : L\mathfrak{g} \oplus \mathbb{C} \cdot c$. If $\mathfrak{g}$ is (semi)simple, $\hat{\mathfrak{g}}$ is called the *affine Kac–Moody algebra associated with* $\mathfrak{g}$.

We fix now some notation. By the *positive part* of $\mathfrak{g}$ we mean $\mathfrak{g}_+ := \mathfrak{g} \otimes \xi \mathbb{C}[\![\xi]\!]$, and the *negative part* of $\mathfrak{g}$ is $\mathfrak{g}_- := \mathfrak{g} \otimes \xi^{-1}\mathbb{C}[\xi^{-1}]$. We have a triangular decomposition

$$\hat{g} = \hat{g}_+ \oplus \mathfrak{g} \oplus \mathbb{C}c \oplus \hat{g}_-.$$

We also write $X(n) = X \otimes \xi^n$ for $X \in \mathfrak{g}, n \in \mathbb{Z}$ and $X(0) = X = X \otimes 1$.

Remark 2.2.2. One problem here is that people use the term “affine Kac–Moody algebra” for different objects. One alternative definition of the affine Kac–Moody algebra is the following: Let $\mathfrak{g}_{\text{aff}} = \mathfrak{g}[\xi^{-1}, \xi] \oplus \mathbb{C} \cdot c \subseteq \hat{\mathfrak{g}}$, $\mathfrak{g}_+ = \xi\mathfrak{g}[\xi]$, and $\mathfrak{g}_- = \xi^{-1}\mathfrak{g}[\xi^{-1}]$ with a triangular decomposition as before. Another one is the following: Let $D = \xi\frac{\partial}{\partial\xi}$ be the Euler derivative of $\mathbb{C}(\!(\xi)\!)$. This extends to a derivation of $\hat{\mathfrak{g}}$. Then the *extended Kac–Moody algebra* is given by the semi-direct product

$$\hat{\mathfrak{g}}^e := \hat{g} \rtimes \mathbb{C} \cdot D.$$

The derived algebra of this is the affine Lie algebra from before,

$$[\hat{g}^e, \hat{g}^e] = \hat{g}.$$

There is also a Laurent polynomial version of $\hat{g}^e$. A reference for all of this is [Kac90].

Another prominent example of a central extension is the *Virasoro algebra* denoted Vir. Consider Laurent series $\mathbb{C}(\!(\xi)\!)$. The derivations of this ring form a free module of dimension 1,

$$\mathrm{Der}(\mathbb{C}(\!(\xi)\!)) = \mathbb{C}(\!(\xi)\!)\partial_\xi,$$

which furthermore has the structure of a Lie algebra. We would like to centrally extend this,

$$0 \to \mathbb{C} \cdot c \to \mathrm{Vir} \to \mathrm{Der}(\mathbb{C}(\!(\xi)\!)) \to 0.$$

Instead of doing this through 2-cocycles, we simply give the Lie bracket by

$$[f\partial_\xi, g\partial_\xi] = (fg' - f'g)\partial_\xi - \frac{1}{12}\mathrm{Res}_{\xi=0}(fg'''\,d\xi).$$

Exercise 2.2.3. We have

$$\mathrm{Res}(fg'''\,d\xi) = -\mathrm{Res}(f'''g\,d\xi) = -\frac{1}{2}\mathrm{Res}_{\xi=0}\left(\begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix}\,d\xi\right).$$

2.3 Universal enveloping algebras

If V is a vector space over $\mathbb{C}$, let $T^nV = V \otimes \cdots \otimes V$, and write $T^0V = \mathbb{C} = \mathbb{C} \cdot 1$. This defines the *tensor algebra*

$$TV = \bigoplus_{n \geq 0} T^nV,$$

which is a graded associative unital algebra with multiplication $\mu_\otimes T^nV \otimes T^mV \to T^{n+m}V$.

Remark 2.3.1. The tensor algebra TV is a bialgebra, furthermore having the structure of a *Hopf algebra*, i.e. there is a map $\Delta : TV \to TV \otimes TV$, called the comultiplication and satisfying $\Delta(v) = 1 \otimes v + v \otimes 1$ for $v \in V = T^1(V)$, a map $\varepsilon : TV \to \mathbb{C}$, called the counit, and a map $s : TV \to TV$ called the antipode and satisfying $s(v) = -v$, all of these maps being compatible in a certain way that we will not elaborate on.

The universal enveloping algebra is constructed as a certain quotient of TV. Assume now that we are in the special where $V = \mathfrak{g}$ is a Lie algebra. In this case, we define a two-sided ideal

$$I = \langle x \otimes y - y \otimes x - [x, y] \cdot 1 \mid x, y \in \mathfrak{g} \rangle$$

in TV. The *universal enveloping algebra* is

$$U\mathfrak{g} = T\mathfrak{g}/I.$$

Remark 2.3.2. The universal enveloping algebra inherits the Hopf structure from $T\mathfrak{g}$.

If $\mathfrak{g}$ is abelian, then $U\mathfrak{g} = S\mathfrak{g}$, the symmetric algebra over $\mathfrak{g}$.

For the free Lie algebra $\mathrm{Free\,Lie}(V) \subseteq TV$ consisting of all commutators of tensor multiplications, we have $U(\mathrm{Free\,Lie}(V)) = TV$.

The universal enveloping algebra is a *filtered algebra*: Let $F_n U\mathfrak{g}$ be the image of $\bigoplus_{i=0}^n T^i\mathfrak{g}$ under the canonical projection $T\mathfrak{g} \to U\mathfrak{g}$, and we have a sequence

$$0 \subseteq F_0 U\mathfrak{g} = \mathbb{C}\cdot 1 \subseteq \cdots \subseteq F_n U\mathfrak{g} \subseteq F_{n+1}U\mathfrak{g} \subseteq \cdots,$$

with $\bigcup_{n\geq 0} F_n U\mathfrak{g} = U\mathfrak{g}$, and $F_n U\mathfrak{g} F_m U\mathfrak{g} \subseteq F_{n\cdot m}U\mathfrak{g}$.

Fact 2.3.3. *The map $\mathfrak{g} = T^1\mathfrak{g} \to U\mathfrak{g}$ is injective, so we will not distinguish in the notation but write $x \in U\mathfrak{g}$ for $x \in \mathfrak{g}$. Moreover, if $(x_i)_{i\in I}$ is a totally ordered basis for $\mathfrak{g}$, then the elements $X_{i_1}\cdots X_{i_m} \in U\mathfrak{g}$, $i_1 \leq \cdots \leq i_m$, m being arbitrary, form a vector space basis for $U\mathfrak{g}$. This is the Poincaré–Birkhoff–Witt theorem.*

Some of the consequences of the above are the following: The map $S^n\mathfrak{g} \to F_n U\mathfrak{g} \subseteq U\mathfrak{g}$, called the *PBW map* given by

$$x_1\cdots x_n \mapsto \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}\cdots x_{\sigma(n)},$$

induces an algebra isomorphism $S\mathfrak{g} \to \mathrm{gr}U\mathfrak{g}$, where $\mathrm{gr}_n U\mathfrak{g} = F_n U\mathfrak{g}/F_{n-1}U\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a decomposition into Lie subalgebras, it follows that $U\mathfrak{g} = U\mathfrak{g}_1 \otimes U\mathfrak{g}_2$ as a vector space and as a left $U\mathfrak{g}_1$-module and right $U\mathfrak{g}_2$-module. This is important in the construction of Verma modules.

Fact 2.3.4. *There is a 1-1 correspondence between $\mathfrak{g}$-modules and $U\mathfrak{g}$-modules. This implies that*

$$H^\bullet(\mathfrak{g}, V) = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{C}, V).$$

2nd lecture, February 10th 2012

2.4 Simple Lie algebras

Today we will define Verma modules for affine Kac–Moody algebras. Before we do this, we will recall some basic material about simple Lie algebras over $\mathbb{C}$.

By a *simple Lie algebra* we mean one with no non-trivial ideals. We use the *Killing form* $X, Y \mapsto \mathrm{tr}(\mathrm{ad}_X\,\mathrm{ad}_Y)$. We recall some basic facts.

- If $\mathfrak{g}$ is simple, every invariant bilinear form is a multiple of the Killing form.
- If $\mathfrak{g}$ is simple, the Killing form is non-degenerate and definite.

Recall that the classification of simple Lie algebras gives us a list of four infinite families of simple Lie algebras, as well as some exceptional cases. We write $A_n = \mathfrak{sl}(n+1)$, where the n is the dimension of the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ or the number of vertices in the Dynkin diagram. Similarly, $B_n = \mathfrak{o}(2n+1)$, for $n \geq 3$, $C_n = \mathfrak{sp}(2n)$, $n \geq 2$, $D_n = \mathfrak{o}(2n)$, $n \geq 4$. The exceptional cases are E_6, E_7, E_8, F_4, G_2.

Without the restrictions on n above, we have $\mathfrak{sl}(2) = \mathfrak{o}(3)$, $\mathfrak{o}(4) = \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$, $\mathfrak{o}(5) = \mathfrak{sp}(4)$, $\mathfrak{o}(6) = \mathfrak{sl}(4)$.

Recall that we can reconstruct a simple Lie algebra from its root system Δ in Euclidean space. We have $\mathfrak{h}^*_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}(\Delta)$ with complexification $\mathfrak{h}^*$. Recall also that we have a root system decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where every $\mathfrak{g}_\alpha$ is 1-dimensional. We can also decompose the root system into positive and negative roots as $\Delta = \Delta_+ \sqcup \Delta_-$. With this notation, we have the triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha.$$

By a simple root $\alpha \in \Delta_+$ we mean one that does not decompose into positive roots.

We use the following normalization conventions: There is an object called the *longest root* (or *highest root*) $\theta \in \Delta_+$ defined by the condition that there is no simple root α_i such that $\theta + \alpha_i \in \Delta$.

Example 2.4.1. For A_n, the longest root is $\theta = \sum_i \alpha_i$, the sum being over all simple roots. For D_n, $\theta = \alpha_1 + 2\sum_{i=2}^{\mathrm{rk}-2} \alpha_i + \alpha_{n-1} + \alpha_n$.

Choose an invariant bilinear form so that $(\theta, \theta)2$. For each $\lambda \in \mathfrak{h}^*$, there exists a unique $H_\lambda \in \mathfrak{h}$ such that $\lambda(H) = (H_\lambda, H)$ for all $H \in \mathfrak{h}$. We obtain an inner product on $\mathfrak{h}$ by $(H_\lambda, H_\mu) := (\lambda, \mu)$. For $\alpha \in \Delta$, H_α is called the *root vector*.

Example 2.4.2. For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, we have a basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now, $[H, E] = 2E = \alpha(H)$, $[H, F] = -2F$, $[E, F] = H$, $\Delta = \{\alpha, -\alpha\}$, $\mathfrak{g}_\alpha = \mathbb{C}E$, and $\mathfrak{g}_{-\alpha} = \mathbb{C}F$. The scalar product is $(X, Y) = \mathrm{tr}(X, Y)$.

3 Integrable highest weight representations

3.1 Preliminaries

For $\lambda \in \mathfrak{h}^*$, an irreducible left $\mathfrak{g}$-module is called a *highest weight module* with *highest weight* λ if there exists a vector $e \in V_\lambda$, called the *highest weight vector*, such that

$$He = \lambda(H)e, \quad Xe = 0,$$

for all $H \in \mathfrak{h}$ and all $X \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_+$. A weight $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is called an *integral weight* if

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} =: \langle \lambda, \alpha \rangle \in \mathbb{Z},$$

and it is called a *dominant weight* if $w(\lambda) \leq \lambda$ for all elements $w \in W$ of the Weyl group and the standard partial ordering $\leq$ on the roots. We have *fundamental weights* $\omega_1, \ldots, \omega_n \in \mathfrak{h}^*$ such that

$$\langle \omega_i, \alpha_j \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

Fact 3.1.1. *A weight λ is dominant if and only if it is a positive linear combination of the fundamental weights.*

Fact 3.1.2. *A weight λ is a highest weight of an irreducible finite-dimensional $\mathfrak{g}$-module if and only if it is integral and dominant. Furthermore, θ is the highest weight of* ad. *We use the notation P_+ for the set of integral and dominant weights (here the notation P comes from the French "poid" meaning weight).*

Pick an orthonormal basis $J^1, \ldots, J^n$ for $\mathfrak{g}$ and define the *Casimir element*

$$\Omega := \sum_{a=1}^{n} J^a J^a = \sum g_{ab} X^a X^b \in U\mathfrak{g},$$

where g_{ab} is the Killing form and X^a an arbitrary basis.

Fact 3.1.3. *We have the following:*

1. *$\Omega \in U\mathfrak{g}$ is central.*

2. *Ω acts on the highest weight module V_λ by $((\lambda,\lambda)+2(\lambda,\rho))\mathrm{id}$, where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$.*

The *dual Coxeter number* is $g^* = 1+(\theta,\rho)$. A corollary of the second of the above facts is that

$$\Omega.X = \sum_a [J^a[J^a,X]] = 2g^* X.$$

This follows since

$$\Omega.X = ((\theta,\theta)+2(\theta,\rho))X = (2+2(\theta,\rho))X.$$

Proof of Fact 3.1.3 (2). Since Ω is central, by Schur's Lemma it acts diagonally on V_λ. Therefore it is enough to look at $\Omega.|\lambda\rangle$, where $|\lambda\rangle$ is the highest weight vector. Choose some $0\neq E_\alpha\in\mathfrak{g}_\alpha$, $0\neq E_{-\alpha}\in\mathfrak{g}_{-\alpha}$. Then for $H\in\mathfrak{h}$,

$$([E_\alpha,E_{-\alpha}],H) = -(E_{-\alpha},[E_\alpha,H]) = \alpha(H)(E_\alpha,E_{-\alpha}) = (H_\alpha,H)(E_\alpha,E_{-\alpha}).$$

It follows from this that $[E_\alpha,E_{-\alpha}] = (E_\alpha,E_{-\alpha})H_\alpha$. Another fact we will need is that $(\mathfrak{g}_\alpha,\mathfrak{g}_\beta)=0$ for $\alpha+\beta\neq 0$. Choose an orthonormal basis $H_1,\dots,H_n$ for $\mathfrak{h}$. Then

$$\Omega = \sum_{i=1}^n H_iH_i + \sum_{\alpha\in\Delta}\frac{1}{(E_\alpha,E_{-\alpha})}E_\alpha E_{-\alpha}.$$

Writing $H_\lambda = \sum\lambda(H_i)H_i$, we find

$$\sum H_iH_i|\lambda\rangle = \sum\lambda(H_i)^2|\lambda\rangle = (H_\lambda,H_\lambda)|\lambda\rangle = (\lambda,\lambda)|\lambda\rangle.$$

This gives the first summand in the claim. For the other one, note that $E_{-\alpha}E_\alpha|\lambda\rangle = 0$,

$$\begin{aligned} E_\alpha E_{-\alpha}|\lambda\rangle &= [E_\alpha,E_{-\alpha}]|\lambda\rangle = (E_\alpha,E_{-\alpha})H_\alpha|\lambda\rangle \\ &= (E_\alpha,E_{-\alpha})\lambda(H_\alpha)|\lambda\rangle = (E_\alpha,E_{-\alpha})(\lambda,\alpha)|\lambda\rangle, \end{aligned}$$

and

$$\sum_{\alpha\in\Delta}\frac{1}{(E_\alpha,E_{-\alpha})}E_\alpha E_{-\alpha} = \sum_{\alpha\in\Delta_+}\frac{1}{(E_\alpha,E_{-\alpha})}(E_\alpha E_{-\alpha}+E_{-\alpha}E_\alpha).$$

Putting this together, we find that

$$\sum_{\alpha\in\Delta}\frac{1}{(E_\alpha,E_{-\alpha})}E_\alpha E_{-\alpha} = \sum_{\alpha\in\Delta_+}\langle\lambda,\alpha\rangle|\lambda\rangle = 2(\rho,\lambda)|\lambda\rangle = 2(\rho,\lambda)|\lambda\rangle.$$

□

We now define certain irreducible representations $\mathcal{H}_\lambda$ of the affine Kac–Moody algebra. Let $l\in\mathbb{N}$ be a *level*, put

$$P_l = \{\lambda\in P_+ \mid 0\leq(\theta,\lambda)\leq l\},$$

and write

$$\hat{\mathfrak{p}}_+ = \hat{g}_+\oplus(\mathfrak{g}\oplus\mathbb{C}\cdot c)\subseteq L\mathfrak{g}\oplus\mathbb{C}\cdot c = \hat{\mathfrak{g}}.$$

Let λ be a highest weight with highest weight representaiton V_λ. We define an action of $\hat{\mathfrak{p}}_+$ as follows: For all $v\in V_\lambda$, $a\in\hat{\mathfrak{g}}_+$

$$c\cdot v = l\cdot v,\quad a\cdot v = 0.$$

Using the PBW theorem, define the *Verma module* (or *induced representation*)

$$\mathcal{M}_\lambda := U(\hat{\mathfrak{g}}) \otimes_{U\hat{\mathfrak{p}}_+} V_\lambda = \mathrm{Ind}_{\hat{\mathfrak{p}}_+}^{\hat{\mathfrak{g}}} V_\lambda.$$

Note that the Verma module is not irreducible, but it contains a maximal proper submodule $\mathcal{J}_\lambda$, and we define an irreducible $\hat{g}$-module

$$\mathcal{H}_\lambda := \mathcal{M}_\lambda / \mathcal{J}_\lambda.$$

Theorem 3.1.4 ([Kac90]). *For each $\lambda \in P_l$, $\mathcal{H}_\lambda$ is the unique left $\hat{\mathfrak{g}}$-module such that*

1. *$V_\lambda = \{|v\rangle \in \mathcal{H}_\lambda \mid \hat{\mathfrak{g}}_+|v\rangle = 0\}$, the irreducible $\mathfrak{g}$-module with highest weight λ,*
2. *c acts on $\mathcal{H}_\lambda$ by $l \cdot \mathrm{id}$, and*
3. *$\mathcal{H}_\lambda$ is generated over $\hat{\mathfrak{g}}_-$ with the only relation*

$$|J_\lambda\rangle := (X_\theta \otimes \xi^{-1})^{l-(\theta,\lambda)+1}|\lambda\rangle = 0,$$

where X_θ is the element of $\mathfrak{g}$ dual to θ, and $|\lambda\rangle$ is the highest weight vector, and $\mathcal{J}_\lambda = U(\hat{\mathfrak{p}}_-)|J_\lambda\rangle$.

3rd lecture, February 15th 2012

We recall the definition of the representations $\mathcal{H}_\lambda$. Let l be the level, let $\mathfrak{g}$ be a simple Lie algebra, let

$$P_l = \{\lambda \in P_+ \mid 0 \leq (\theta, \lambda) \leq l\},$$

and let $\hat{\mathfrak{p}_+} = \hat{\mathfrak{g}}_+ \oplus \mathfrak{g} \oplus \mathbb{C} \cdot c$, let V_λ be a highest weight module. We have an action on V_λ given by $c \cdot v = l \cdot v$ and $a \cdot v = 0$ for all $v \in V_\lambda, a \in \hat{\mathfrak{g}}_+$ and define the Verma module

$$\mathcal{M}_\lambda = U(\hat{\mathfrak{g}}) \otimes_{U\hat{\mathfrak{p}}_+} V_\lambda$$

with maximal proper submodule $\mathcal{J}_\lambda$. We then define $\mathcal{H}_\lambda = \mathcal{M}_\lambda / \mathcal{J}_\lambda$. Its properties are summarized in Theorem 3.1.4.

3.2 Segal–Sugawara construction

In physics, if you have relativistic field theory, you have the Lorentz group acting as a symmetry group. If you have such a thing, associated there is a energy-momentum tensor. If you enlarge the isometric symmetry to a conformal symmetry, there is also an energy-momentum tensor, which is what gives rise to the following rather unmotivated construction, which could be motivated better by using vertex algebras; we will not do that.

Let V be a $\hat{\mathfrak{g}}$ module, e.g. $\mathcal{H}_\lambda$. Define $X(n) = X \otimes \in \xi^n \in \hat{\mathfrak{g}}$ for $X \in \mathfrak{g}$, $n \in \mathbb{Z}$, and define the "generating function"

$$X(z) = \sum_{n \in \mathbb{Z}} X(n) z^{-n-1} \in \mathrm{End}_V[\![z^{-1}, z]\!].$$

The next unmotivated thing is the *normal ordered product*

$$: X(n)Y(m) : = \begin{cases} X(n)Y(m) & n < m, \\ \frac{1}{2}(X(n)Y(m) + Y(m)X(n)) & m = n, \\ Y(m)X(n) & n > m. \end{cases}$$

In the special case where X and Y coincide, then for $n > m$,

$$\begin{aligned} :X(n)Y(m): &= [X(m), X(n)] + X(n)X(m) = (X,X) \cdot c \operatorname{Res}_{\xi=0}(\xi^m \, d\xi^n) + X(n)X(m) \\ &= n(X,X)\delta_{n+m,0} \cdot c + X(n)X(m). \end{aligned}$$

Let J^a be an orthonormal basis, and let $\Omega = \sum J^a J^a \in U\mathfrak{g}$ as before. Define the *energy-momentum tensor* by

$$\begin{aligned} T(z) &= \frac{1}{2(g^*+l)} \sum_{a=1}^{\dim \mathfrak{g}} :J^a(z)J^a(z): := \frac{1}{2(g^*+l)} \sum_{n,m\in\mathbb{Z}} \sum_a :J^a(n)J^a(m): \, z^{-n-m-2} \\ &= \sum_{k\in\mathbb{Z}} L_k z^{-k-2}, \end{aligned}$$

where the L_k are the *Virasoro operators*

$$L_k = \frac{1}{2(g^*+l)} \sum_{n\in\mathbb{Z}} :J^a(n)J^a(k-n): .$$

Some special cases are the following: For $k \neq 0$,

$$L_k = \frac{1}{(2g^*+l)} \sum_{n\in\mathbb{Z}} \sum_a J^a(n)J^a(k-n).$$

For $k = 0$, note first that (using Einstein summation convention to drop the sums over a)

$$\begin{aligned} \sum_{k\in\mathbb{Z}} :J^a(m)J^a(-m): &= \sum_{m>0} :J^a(m)J^a(-m): + \sum_{m>0} :J^a(-m)J^a(m): + :J^a(0)J^a(0): \\ &= 2\sum_{m>0} J^a(-m)J^a(m) + J^aJ^a. \end{aligned}$$

It follows that

$$L_0 = \frac{1}{2(g^*+l)} \sum_{a=1}^{\dim \mathfrak{g}} J^aJ^a + \frac{1}{g^*+l} \sum_{m>0} \sum_a J^a(-m)J^a(m),$$

and it turns out that this operator will operate on $\mathcal{H}_\lambda$ which will be important later, whereas, if we had not used the normal ordering,

$$\frac{1}{2(g^*+l)} \sum_{a=1}^{\dim \mathfrak{g}} J^a(m)J^a(-m)$$

would not.

Lemma 3.2.1. *We have*

$$\sum_{a=1}^{\dim \mathfrak{g}} [X, J^a](m)J^a(n) + J^a(m)[X, J^a](n) = 0.$$

Proof. Since the J^a form an orthonormal basis, we have

$$[X, J^a](m)J^a(n) = ([X, J^a], J^b)J^b(m)J^a(n) = -(J^a, [X, J^b])J^b(m)J^a(n) = -J^b(m)[X, J^b](n).$$

□

The real deal is the next theorem, which we will prove later.

Theorem 3.2.2. *Assume that c acts by multiplication by l. We have*

$$[L_n, X(m)] = -mX(n+m),$$
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0},$$

where c_V is the Virasoro charge *defined by*

$$c_V = \frac{l \cdot \dim \mathfrak{g}}{g^* + l}.$$

Remark 3.2.3. Recall that for $f, g \in \mathbb{C}((\xi))$,

$$\left[f(\xi)\frac{\partial}{\partial \xi}, g(\xi)\frac{\partial}{\partial \xi}\right] = (f'g - g'f)\frac{\partial}{\partial \xi} - \frac{c_V}{12}\operatorname{Res}_{\xi=0}(fg'''\,d\xi),$$

so with $L_n = -\xi^{n+1}\frac{\partial}{\partial \xi}$, we recover the second statement of the theorem.

3.3 Calculations with generating functions

Lemma 3.3.1. *We have*

$$[L_n, X(z)] = z^n(z\frac{d}{dz} + n + 1)X(z).$$

Proof. By Theorem 3.2.2, the left hand side is

$$\begin{aligned}[L_n, X(z)] &= \sum_{m\in\mathbb{Z}}[L_n, X(m)]z^{-m-1} = \sum_{m\in\mathbb{Z}} -mX(n+m)z^{-m-1} \\ &= \sum_{k\in\mathbb{Z}}(n-k)X(k)z^{n-k-1} = \sum_{m\in\mathbb{Z}} X(m)z^n(z\frac{d}{dz} + n + 1)z^{-m-1} \\ &= \sum_{m\in\mathbb{Z}} X(m)z^n((-m-1)z^{-m-1} + (n+1)z^{-m-1}) = \sum_{m\in\mathbb{Z}} X(m)(n-m)z^{n-m-1}.\end{aligned}$$

□

More sophisticated than the above is the following: Let $f \in \mathbb{C}((z))$ and consider a derivation $\underline{l} = l(z)\frac{d}{dz} \in \mathbb{C}((z))\frac{d}{dz} = \operatorname{Der}(\mathbb{C}((z)))$. Introduce the pairings

$$X[f] = \operatorname{Res}_{z=0}(X(z)f(z)\,dz),$$
$$T[\underline{l}] = \operatorname{Res}_{z=0}(T(z)l(z)\,dz).$$

Proposition 3.3.2. *Both $X[f]$ and $T[\underline{l}]$ act on $\mathcal{H}_\lambda$, and for the module $V = \mathcal{H}_\lambda$, we have*

$$L_0 = T[z\frac{d}{dz}],$$
$$[T[\underline{l}], X[f]] = -[X[\underline{l}(f)]],$$
$$[T[\underline{l}], T\underline{m}] = T[[\underline{l}, \underline{m}]] + \frac{c_V}{12}\operatorname{Res}_{z=0}(l'''m\,dz).$$

Remark 3.3.3. The action of $\hat{\mathfrak{g}}$ on $\mathcal{H}_\lambda$ extends to the space $\hat{g}^e$ from Remark 2.2.2 with $D = L_0$.

3.4 Filtration on the highest weight modules

Let λ be a highest weight representation. Define

$$\mathcal{H}_\lambda(d) = \{|v\rangle \in \mathcal{H}_\lambda \mid L_0|v\rangle = (d+\Delta_\lambda)|v\rangle\},$$

where $d \in \mathbb{Z}$, and

$$\Delta_\lambda = \frac{(\lambda,\lambda)+2(\rho,\lambda)}{2(g^*+l)}.$$

Note that for $|v\rangle \in V_\lambda$, by Theorem 3.2.2, we have

$$\begin{aligned} L_0|v\rangle &= \frac{1}{2(g^*+l)}\sum_a J^a J^a|v\rangle = \Delta_\lambda|v\rangle, \\ L_0X(-m)|v\rangle &= [L_0, X(-m)]|v\rangle + X(-m)L_0|v\rangle = mX(-m)|v\rangle + \Delta_\lambda X(-m)|v\rangle \\ &= (\Delta_\lambda + m)X(-m)|v\rangle. \end{aligned}$$

This implies that $X(-m)|v\rangle \in \mathcal{H}_\lambda(m)$. Continuing with this reasoning,

$$L_0X(-m_1)\cdots X(-m_k)|v\rangle = \left(\Delta_\lambda + \sum_{i=1}^k m_i\right) X(-m_1)\cdots X(-m_k)|v\rangle \in \mathcal{H}_\lambda(d),$$

where $d = \sum_i m_i$. It follows that for every d, $\mathcal{H}_\lambda(d)$ is finite dimensional, and if $d<0$, $\mathcal{H}_\lambda(d) = 0$.

Out of all of this, we make an ascending filtration

$$F_p\mathcal{H}_\lambda := \sum_{d=0}^p \mathcal{H}_\lambda(d),$$

and so $\bigcup_{p\geq} F_p\mathcal{H}_\lambda = \mathcal{H}_\lambda$, and

$$F_0\mathcal{H}_\lambda = V_\lambda \subseteq \cdots \subseteq F_p\mathcal{H}_\lambda \subseteq F_{p+1}\mathcal{H}_\lambda \subseteq \cdots.$$

The dual spaces satisfy

$$\begin{aligned} \mathcal{H}_\lambda^\dagger(d) &:= \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda(d), \mathbb{C}), \\ \mathcal{H}_\lambda^\dagger &:= \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda, \mathbb{C}) = \prod_{d=0}^\infty \mathcal{H}_\lambda^\dagger(\mathbb{C}), \end{aligned}$$

and $\mathcal{H}_\lambda^\dagger$ has a canonical descending filtration

$$F^p\mathcal{H}_\lambda^\dagger = \prod_{d\geq p}\mathcal{H}_\lambda^\dagger(d) = \mathrm{Ann}(F_{p-1}\mathcal{H}_\lambda \subset \mathcal{H}_\lambda),$$

so we have $F^0\mathcal{H}_\lambda = \mathcal{H}_\lambda^\dagger, \bigcap F^p\mathcal{H}_\lambda^\dagger = 0$, and inclusions

$$\cdots \supseteq F^p\mathcal{H}_\lambda^\dagger \supseteq F^{p+1}\mathcal{H}_\lambda^\dagger \supseteq \cdots.$$

The filtrations $\{F_p\}$ and $\{F^p\}$ generate topologies on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$ respectively. The latter is complete and Hausdorff.

4th lecture, February 17th 2012

We have introduced finite-dimensional subspaces

$$\mathcal{H}_\lambda(d) = \{|v\rangle \in \mathcal{H}_\lambda \mid L_0|v\rangle = (d+\Delta_\lambda)|v\rangle\} \subseteq \mathcal{H}_\lambda,$$

where here

$$\Delta_\lambda = \frac{(\lambda,\lambda)+2(\rho,\lambda)}{2(g^*+l)}, \quad \rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha,$$

and we introduced an ascending filtration $F_p\mathcal{H}_\lambda$. We also considered the dual spaces $\mathcal{H}_\lambda^\dagger(d) = \mathrm{Hom}_\mathbb{C}(\mathcal{H}_\lambda(d),\mathbb{C})$, $\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_\mathbb{C}(\mathcal{H}_\lambda,\mathbb{C}) = \prod_{d\geq 0}\mathcal{H}_\lambda^\dagger(d)$ and introduced a descending filtration $F^p\mathcal{H}_\lambda^\dagger = \mathrm{Ann}(F_{p-1}\mathcal{H}_\lambda)$, and they generate a complete Hausdorff topology.

We have a dual pairing

$$\langle\,|\,\rangle : \mathcal{H}_\lambda^\dagger \times \mathcal{H}_\lambda \to \mathbb{C},$$

so from our left $\hat{\mathfrak{g}}$-action on $\mathcal{H}_\lambda$, we get a right $\hat{g}$-action on $\mathcal{H}_\lambda^\dagger$, given by

$$\langle ua|v\rangle = \langle u|av\rangle$$

for $a \in \hat{\mathfrak{g}}$. We consider the dual space of V_λ given by

$$V_\lambda^\dagger = \{\langle v| \in \mathcal{H}_\lambda^\dagger \mid \langle v|\hat{\mathfrak{g}}_- = 0\}.$$

This is an irreducible lowest weight $\hat{g}$-right module with lowest weight λ generated by $V_\lambda^\dagger$over $\hat{\mathfrak{g}}_+$ with the only relation $\langle\lambda|(X_{-\theta}\otimes\xi)^{l-(\theta,\lambda)+1}$.

Lemma 3.4.1. *We have*

$$X(m)\mathcal{H}_\lambda(d) \subseteq \mathcal{H}_\lambda(d-m), \quad L_m\mathcal{H}_\lambda(d) \subseteq \mathcal{H}_\lambda(d-m)$$

and, dually,

$$\mathcal{H}_\lambda^\dagger(d)X(m) \subseteq \mathcal{H}_\lambda^\dagger(d+m), \quad \mathcal{H}_\lambda^\dagger(d)L_m \subseteq \mathcal{H}_\lambda^+(d+m)$$

Proposition 3.4.2. *For any root vector $X_\alpha \in \mathfrak{g}_\alpha \subseteq \mathfrak{g}$ and any $f \in \mathbb{C}((\lambda))$, the action of $X_\alpha\otimes f(\xi)$ on $\mathcal{H}-\lambda$ is locally nilpotent, meaning that if we pick a vector in $\mathcal{H}_\lambda$, then a finite power of $X_\alpha\otimes f(\xi)$ will annihilate it.*

We define a left action on the dual space by

$$X(n)\langle\varphi| = -\langle\varphi|X(-n),$$

for $\langle\varphi| \in \mathcal{H}_\lambda^\dagger$.

We now want to go between highest weights and lowest weights using the Weyl group. Let $w \in W$ be the longest element of the Weyl group, uniquely determined by requiring that $w(\Delta_+) = \Delta_-$. Then for any $\lambda \in P_l$, we define $\lambda^\dagger := -w(\lambda)$ (so that $-\lambda^\dagger$ is a lowest weight of V_λ).

Example 3.4.3. For $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda^\dagger = \lambda$.

Lemma 3.4.4. *There exists a bilinear pairing $(\,|\,) : \mathcal{H}_\lambda\times\mathcal{H}_\lambda^\dagger \to \mathbb{C}$, unique up to a constant multiple such that*

$$(X(n)u|v) + (u|X(-n)v) = 0,$$

for any $u \in \mathcal{H}_\lambda$, $v \in \mathcal{H}_\lambda^\dagger$, and $X \in \mathfrak{g}$.

Some properties of this pairing are rthat $(\mathcal{H}_\lambda(d)|\mathcal{H}_{\lambda^\dagger}(d')) = 0$ for $d \neq d'$, and $\mathcal{H}_\lambda^\dagger \cong \widehat{\mathcal{H}_{\lambda^\dagger}}$, the completion with respect to $\{F_p\}$.

In the rest of this lecture, we will prove some parts of Theorem 3.2.2.

Proof of Theorem 3.2.2 for some n and m. We want to prove that $[L_n, X(m)] = -mX(n+m)$ in a given representation with $c = l$. Introduce the shorthand $\alpha = 1/(2g^* + l)$, so that

$$L_n = \alpha \sum_{j\in\mathbb{Z}} : J^a(-j)J^a(n+j) :,$$

where we sum over repeated indices. Recall that $X(m) = X \otimes \xi^m$. We find that

$$\begin{aligned}[X(m), \alpha^{-1}L_n] &= \left[X(m), \sum_{j\geq -n/2} J^a(-j)J^a(n+j)\right] + \left[X(m), \sum_{j<-n/2} J^a(n+j)J^a(-j)\right] \\ &=: (*) + (**),\end{aligned}$$

and by the Leibniz rule,

$$(*) = \sum_{j\geq -n/2} [X(m), J^a(-j)]J^a(n+j) + J^a(-j)[X(m), J^a(n+j)]. \tag{1}$$

By definition,

$$\begin{aligned}[X(m), J^a(-j)] &= [X, J^a](m-j) + c(X, J^a)\operatorname{Res}_{\xi=0}(\xi^{-j}\, d\xi^m) \\ &= [X, J^a](m-j) + cm \cdot (X, J^a)\delta_{m,j}.\end{aligned}$$

On the other hand,

$$\begin{aligned}[X(m), J^a(n+j)] &= [X, J^a](m+n+j) + c \cdot (X, J^a)\operatorname{Res}_{\xi=0}(\xi^{n+j}\, d\xi^m) \\ &= [X, , J^a](m+n+j) + cn \cdot (X, J^a)\delta_{m+n+j,0}.\end{aligned}$$

Note also that $(X, J^a)J^a = X$. Plugging all of this into (1) and reordering the terms, we obtain

$$\begin{aligned}(*) = &\sum_{j\geq -n/2} [X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j) \\ &+ \sum_{j\geq -n/2} cm(X(n+j)\delta_{m,j} + X(.j)\delta_{m+n+j,0}).\end{aligned}$$

For the term (**), we find

$$(**) = \sum_{j<-n/2} [X(m), J^a(n+j)]J^a(-j) + J^a(n+j)[X(m), J^a(-j)].$$

As before, we write

$$\begin{aligned}[X(m), J^a(n+j)] &= [X, J^a](m+n+j) + cm\delta_{m+n+j,0}(X, J^a), \\ [X(m), J^a(-j)] &= [X, J^a](m-j) + cm(X, J^a)\delta_{m,j}.\end{aligned}$$

This implies that

$$\begin{aligned}(**) = &\sum_{j<-n/2} [X, J^a](m+n+j)J^a(-j) + J^a(n+j)[X, J^a](m-j) \\ &+ \sum_{j<-n/2} cm(\delta_{m+n+j,0}X(-j) + \delta_{m,j}X(n+j)).\end{aligned}$$

Changing indices to $j' = -n - j$, the first term above becomes

$$\sum_{j<-n/2} [X, J^a](m+n+j)J^a(-j) + J^a(n+j)[X, J^a](m-j)$$
$$= \sum_{j'>-n/2} [X, J^a](m-j')J^a(n+j') + J^a(-j')[X, J^a](m+n+j').$$

Now, adding together the expressions for (*) and (**), we obtain

$$\begin{aligned}(*) + (**) = 2 &\sum_{j>-n/2} \left([X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j)\right) \\ &+ [X, J^a](m + \tfrac{n}{2})J^a(\tfrac{n}{2}) + J^a(\tfrac{n}{2})[X, J^a](m + \tfrac{n}{2}) \\ &+ \sum_{j\in\mathbb{Z}} cm(\delta_{m+n+j,0}X(-j) + \delta_{m,j}X(n+j)).\end{aligned}$$

Here, the second line only appears when n is even. In this case, the entire line should equal 0 – that this is the case (which may or may not be true ...) is an exercise. In general, the last line becomes $2mcX(n+m)$.

We now consider the first line above in the case $m \geq 0$. Recall the identity

$$[X, J^a](m)J^a(n) + J^a(m)[X, J^a](n) = 0.$$

Note now that $([X, J^a], J^a) = -(X, [J^a, J^a]) = 0$ for all a, so

$$\begin{aligned}[[X, J^a](m-k), J^a(n+k)] &= [[X, J^a], J^a](m+n) + c([X, J^a], J^a) \cdot \text{something} \\ &= [[X, J^a], J^a](m+n).\end{aligned}$$

Applying these formulas and letting $j' = j - m$, $k = j' + m$, we obtain

$$\begin{aligned}&2 \sum_{j>-n/2} \left([X, J^a](m-j)J^a(n+j) + J^a(-j)[X, J^a](m+n+j)\right) \\ &= 2 \sum_{-n/2 \geq j' > -m-n/2} [X, J^a](j')J^a(n+m+j') \\ &= 2 \sum_{m-n/2 \geq k > -n/2} [X, J^a](m-k)J^a(n+k) \\ &= \sum_{m-n/2 \geq k > -n/2} [[X, J^a], J^a](n+m) + J^a(n+k)[X, J^a](m-k) + [X, J^a](m-k)J^a(n+k) \\ &= \sum_{m-n/2 \geq k > -n/2} [[X, J^a], J^a](n+m) = m[J^a, [J^a, X]](n+m) = m2g^*X(m+n)\end{aligned}$$

In the above, we have used that $J^a(n+k)[X, J^a](m-k) + [X, J^a](m-k)J^a(n+k) = 0$, which may or may not be true, and which may or may not be related to the problem with the second line above. Putting everything together,

$$[X(m), \alpha^{-1}L_n] = m2(g^* + l)X(n+m),$$

modulo the two problems encountered above. □

5th lecture, February 22nd 2012

4 Conformal blocks

4.1 Stable pointed curves

Today, we begin our discussion of conformal blocks.

Definition 4.1.1. A *stable N-pointed smooth curve* is a tuple $\mathfrak{X} = (C; Q_1, \dots, Q_N)$, where C is a compact genus g Riemann surface, and $Q_1, \dots, Q_N$ are points on C, and $N \geq 3$.

Later on, we will be able to define conformal blocks without any restriction on N.

Definition 4.1.2. The *n'th infinitesimal neighbourhood* of C at Q is a choice of isomorphism

$$s^{(n)} : \mathcal{O}_{C,Q}/m_Q^{n+1} \xrightarrow{\cong} \mathbb{C}[\![\xi]\!]/(\xi^{n+1}),$$

where m_Q is the maximal ideal of $\mathcal{O}_{C,Q}$ of germs of holomorphic functions vanishing at Q.

Definition 4.1.3. A *formal neighbourhood* of C at Q is the limit isomorphism

$$s^{(\infty)} : \widehat{\mathcal{O}_{C,Q}} \xrightarrow{\cong} \mathbb{C}[\![\xi]\!],$$

where $\widehat{\mathcal{O}_{C,Q}}$ is the completion of the direct limit of $\mathcal{O}_{C,Q}/m_Q^{n+1}$.

Definition 4.1.4. A *stable N-pointed smooth curve of genus g with (n-th) formal neighbourhoods* is the collection

$$\mathfrak{X} = (C; Q_1, \dots, Q_N; s_1, \dots, s_N) = (C; Q_1, \dots, Q_N; s_1^{(n)}, \dots, s_N^{(n)})$$

4.2 The space of conformal blocks

Definition 4.2.1. Define the *generalized affine Lie algebra*

$$\hat{\mathfrak{g}}_N = \bigoplus_{j=1}^{N} \mathfrak{g} \otimes \mathbb{C}(\!(\xi_j)\!) \oplus \mathbb{C}c,$$

with commutator given by

$$(X_j \otimes f_j), (Y_j \otimes g_j)] = ([X_j, Y_j] \otimes f_j g_j) + c \sum_{i=1}^{N} (X_i, Y_i) \operatorname{Res}(g_i \, df_i),$$

where we use the notation $(a_j) = (a_1, \dots, a_N)$, and $c \in \operatorname{Center}(\hat{\mathfrak{g}}_N)$.

Also, write

$$\hat{\mathfrak{g}}(\mathfrak{X}) = \mathfrak{g} \otimes H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N} Q_J)),$$

where $\mathcal{O}_C(* \sum_{j=1}^{N} Q_j)$ is the sheaf of meromorphic functions on C with singularities at $Q_1, \dots, Q_N$ of arbitrarily high order. By Laurent expansion, we get an embedding

$$t : H^0(C, \mathcal{O}_C(* \sum_{j=1}^{N} Q_j)) \hookrightarrow \bigoplus_{j=1}^{N} \mathbb{C}(\!(\xi_j)\!).$$

We have that $\hat{\mathfrak{g}}(\mathfrak{X}) \subseteq \hat{\mathfrak{g}}_N$ is a Lie subalgebra and it acts on

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N}$$

from the left. Recall here, that $\mathcal{H}_\lambda = \mathcal{M}_\lambda / \mathcal{J}_\lambda$ by definition, where $\mathcal{M}_\lambda$ is the Verma module, and $\mathcal{J}_\lambda$ is a maximal proper submodule. Similarly, we have a right action on the dual spaces

$$\mathcal{H}^\dagger_{\vec{\lambda}} = \mathcal{H}^\dagger_{\lambda_1} \otimes \cdots \otimes \mathcal{H}^\dagger_{\lambda_N}.$$

Definition 4.2.2. Define the *space of covacua* as the "space of coinvariants"

$$\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) = \mathcal{H}_{\vec{\lambda}}/\hat{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}},$$

and the *conformal block* or *space of vacua* by

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) = \mathrm{Hom}(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}), \mathbb{C}).$$

We will show that these spaces are finite dimensional, that we have an isomorphism for equivalent formal coordinates, that we have a propagation of conformal blocks telling us how they change as we add points, and lastly we will discuss correlation functions.

4.3 Finite dimensionality of the conformal blocks

First, we discuss the action of $\hat{\mathfrak{g}}(\mathfrak{X})$ in greater detail. Let $X_j \in \mathfrak{g}$, $f(\xi_j) \in \mathbb{C}((\xi_j))$. Recall that by definition,

$$\mathcal{M}_\lambda = U(\hat{\mathfrak{g}}) \otimes_{\hat{\mathfrak{p}}_+} V_\lambda,$$

which is a left $\hat{\mathfrak{g}}$-module. Therefore, the elements

$$X_j \otimes f(\xi_j) =: X_j[f_j] \in \hat{\mathfrak{g}}$$

act on $\mathcal{H}_\lambda = \mathcal{M}_\lambda/\mathcal{J}_\lambda$ from the left by

$$\rho_j(X_j[f_j])|v_1 \otimes \cdots \otimes v_n\rangle = |v_1 \otimes \cdots \otimes (X_j[f_j])v_j \otimes \cdots \otimes v_N\rangle,$$

where we write $|v_1 \otimes \cdots \otimes v_n\rangle = |v_1\rangle \otimes \cdots \otimes |v_N\rangle$, and we let

$$(X_1 \otimes f_1, \ldots, X_N \otimes f_N)|v_1 \otimes \cdots \otimes v_N\rangle := \sum_{j=1}^{N} \rho_j(X_j[f_j])|v_1 \otimes \cdots \otimes v_N\rangle.$$

This defines the action of $\hat{g}(\mathfrak{X})$ on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}^{\dagger}_{\vec{\lambda}}$.

Recall that we have a perfect pairing $\langle\,|\,\rangle : \mathcal{H}^{\dagger}_{\lambda} \times \mathcal{H}_\lambda \to \mathbb{C}$, which defines a pairing $\langle\,|\,\rangle : \mathcal{H}^{\dagger}_{\vec{\lambda}} \times \mathcal{H}_{\vec{\lambda}}$.

Lemma 4.3.1. *We have*

$$\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) = \{\langle\psi| \in \mathcal{H}^{\dagger}_{\vec{\lambda}} \mid \langle\psi|\hat{\mathfrak{g}}(\mathfrak{X}) = 0\}.$$

This gives us a pairing $\langle\,|\,\rangle : \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \to \mathbb{C}$.

Theorem 4.3.2. *The spaces $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$ are finite dimensional.*

We prove this theorem a bit later. Define the subspaces

$$\mathfrak{g}[1] = \{(X \otimes 1, \ldots, X \otimes 1) \mid X \in \mathfrak{g}\} \subseteq \hat{\mathfrak{g}}_N,$$
$$\hat{\mathfrak{g}}^{(m)}_N = \mathfrak{g} \otimes \bigoplus_{j=1}^{N} \mathbb{C}[\xi_j^{-1}]\xi_j^{-m} \subseteq \hat{\mathfrak{g}}_N,$$

for $m = 0, 1, \ldots$. The spaces $\mathfrak{g}[1] \oplus \hat{\mathfrak{g}}^{(m)}_N$ are subalgebras of $\hat{g}_N$ for $m \geq 1$.

Proposition 4.3.3. *If $W_m = \mathcal{H}_{\vec{\lambda}}/(\mathfrak{g}[1] \oplus \hat{\mathfrak{g}}^{(m)}_N)\mathcal{H}_{\vec{\lambda}}$ is finite-dimensional for some $m \geq 1$, then Theorem 4.3.2 is true.*

Proof. Consider the filtration $\{F_\bullet\}$ of $\hat{\mathfrak{g}}_N$ defined by

$$F_p\hat{\mathfrak{g}}_N = \begin{cases} \mathfrak{g}\otimes\bigoplus_{j=1}^N \mathbb{C}[\![\xi]\!]\xi_j^{-p}\oplus\mathbb{C}\cdot c & \text{for } p\geq 0 \\ \mathfrak{g}\otimes\bigoplus_{j=1}^N \mathbb{C}[\![\xi]\!]\xi_j^{-p} & \text{for } p<0 \end{cases}.$$

From this we obtain a filtration on the universal enveloping algebra

$$F_pU(\hat{\mathfrak{g}}_N) = \sum_{\sum p_j<p} F_{p_1}\hat{\mathfrak{g}}_N\otimes\cdots\otimes F_{p_N}\hat{g}_N,$$

which we can use to get a filtration on $\mathcal{H}_\lambda$,

$$F_P\mathcal{H}_{\vec{\lambda}} = F_pU(\hat{\mathfrak{g}}_N)V_{\vec{\lambda}},$$

where $V_{\vec{\lambda}} = V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_N}$. By Theorem 3.1.4, $V_{\vec{\lambda}}$ generates $\mathcal{H}_{\vec{\lambda}}$ as a $U(\mathfrak{g}\otimes\bigoplus_{j=1}^N\mathbb{C}[\xi_j^{-1}])$-module. Then $F_pU(\hat{\mathfrak{g}}_N)\cdot F_q\mathcal{H}_{\vec{\lambda}}\subseteq F_{p+q}\mathcal{H}_{\vec{\lambda}}$, and that $F_p(\mathcal{H}_{\vec{\lambda}}) = \{0\}$ for $p<0$. The filtration gives us a grading $\mathrm{gr}^F_\bullet$, and we get an induced filtration and grading for $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ and W_m. We obtain short exact sequences

$$0\to\mathrm{gr}^F_\bullet(\hat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}})\to\mathrm{gr}^F_\bullet(\mathcal{H}_{\vec{\lambda}})\to\mathrm{gr}^F_\bullet(\mathcal{V}_{\vec{\lambda}}(\mathfrak{X}))\to 0,$$
$$0\to\mathrm{gr}^F_\bullet(\hat{\mathfrak{g}}_N^{(m)}\oplus\mathfrak{g}[1]\mathcal{H}_{\vec{\lambda}})\to\mathrm{gr}^F_\bullet(\mathcal{H}_{\vec{\lambda}})\to\mathrm{gr}^F_\bullet(W_M)\to 0.$$

Now the Riemann–Roch theorem tells us that

$$\dim H^0(C,\mathcal{O}_C(mQ)) - \dim H^1(C,\mathcal{O}_C(mQ)) = 1-g+\deg(\mathcal{O}_C(mQ)) = 1-g+m.$$

It is well-known that $H^1(C,\mathcal{O}_C(mQ)) = \{0\}$ if $m\geq 2g-1$. From this we get that

$$\mathrm{gr}^F_\bullet(\hat{\mathfrak{g}}_N^{(m)}\oplus\mathfrak{g}[1]\mathcal{H}_{\vec{\lambda}})\subseteq\mathrm{gr}^F_\bullet(\hat{\mathfrak{g}}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}),$$

so we get a surjective map $\mathrm{gr}^F_\bullet W_m\twoheadrightarrow\mathrm{gr}^F_\bullet\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$, which proves that $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is finite dimensional. □

6th lecture, February 24th 2012

We begin today by noting that $\mathfrak{g}(\mathfrak{X})$ is a subalgebra of $\hat{\mathfrak{g}}_N$ since the Res-part of the bracket vanishes by the Residue theorem.

The main idea for the proof of the finite dimensionality of $\mathcal{V}_\lambda(\mathfrak{X})$ in the case $N=1$ (so we write $\lambda=\vec{\lambda}$) is the following: First we introduce a filtration on $\mathcal{H}_\lambda$ by

$$G_k\mathcal{H}_\lambda = \begin{cases} \{0\}, & k<0, \\ V_\lambda, & k=0, \\ G_{k-1}\mathcal{H}_\lambda+\hat{\mathfrak{g}}G_{k-1}\mathcal{H}_\lambda, & k>0. \end{cases}$$

This is in some sense really just the filtration induced by the filtration on the tensor algebra on $\mathfrak{g}$. From this we obtain a filtration on $\mathcal{V}_\lambda(\mathfrak{X})$, and we need to show two things.

1. That $\dim G_kV_\lambda(\mathfrak{X})<\infty$ for all $k\in\mathbb{Z}$,
2. and that $\dim G_k\mathcal{V}_\lambda(\mathfrak{X}) = \dim G_{k+1}\mathcal{V}(\mathfrak{X})$ for k sufficiently large.

Instead of the second statement above, we will see that we can prove that for all $X\in\mathfrak{g}$ and $n\in\mathbb{Z}$ there exists $l\in\mathbb{Z}$ such that for all $k\in\mathbb{Z}$ and $|u\rangle\in G_{k-l}\mathcal{H}_\lambda$, we have

$$(X\otimes\xi^n)^l|u\rangle\equiv 0 \mod \hat{g}(\mathfrak{X})\mathcal{H}_\lambda+G_{k-1}\mathcal{H}_\lambda.$$

For the root vectors $X_\lambda, X_{-\alpha}$ this has already been mentioned in a previous lecture. TO show it for $H_\alpha = X_\alpha, X_{-\alpha}]$, we need the trick of transfering powers of H over to X_α, $X_{-\alpha}$. Recall from last time that we introduced spaces $W_m = \mathcal{H}_{\vec{\lambda}}/(\mathfrak{g}[1]\oplus\hat{g}_N^{(m)})\mathcal{H}_\lambda$, and we showed that if W_m is finite dimensional for some m, then $\mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$ is finite dimensional.

Proposition 4.3.4. *For $m \geq 2g-1$, W_m is finite dimensional.*

Proof. Set $W = W_m$ and

$$\overline{g} = \bigoplus_{j=1}^{N} (\mathfrak{g} \otimes (\bigoplus_{n=0}^{m-1} \mathbb{C}\xi^{-n})),$$

for fixed m. Since $\hat{\mathfrak{g}}_N^{(m)}$ is an ideal of $\mathfrak{g}_N^{(0)}$ and since we have an isomorphism $\overline{\mathfrak{g}} \to \mathfrak{g}_N^{(m)}/\mathfrak{g}_N^{(0)}$ by projecting, we have a Lie algebra structure on $\overline{\mathfrak{g}}$. Then define $U = \mathcal{H}_{\vec{\lambda}}/\hat{\mathfrak{g}}_N^{(m)}\mathcal{H}_{\vec{\lambda}}$. This is a $U(\overline{\mathfrak{g}})$-module. Let $\overline{V} = \mathrm{Im}(V_\lambda \to U)$. Then $\overline{V}$ is finite dimensional and

$$U = U(\overline{\mathfrak{g}}) \cdot \overline{V}. \tag{2}$$

Consider the filtration on $U(\hat{g}_N^{(0)})$ induced by the filtration on the tensor algebra. From this we get a filtration $G_k\mathcal{H}_{\vec{\lambda}} = G_k U(\hat{g}_N^{(0)}) \cdot V_\lambda$. Some easy facts about this are

$$[G_k U(\hat{\mathfrak{g}}_N^{(0)}), G_l U(\hat{g}_N^{(0)})] \subseteq G_{k+l-1} U(\hat{\mathfrak{g}}_N^{(0)}),$$
$$G_k U(\hat{<} g_N^{(0)}) G_l \mathcal{H}_{\vec{\lambda}} \subseteq G_{k+l}\mathcal{H}_{\vec{\lambda}}.$$

From this we get filtrations $G_k U(\overline{\mathfrak{g}})$, $G_k W_1$,$G_k U$.

Thanks to Jens-Kristian Egsgaard for covering the rest of the lecture for me: Then

$$gr^G_{\cdot} U(\overline{\mathfrak{g}}) \cong S_{\cdot}(\overline{\mathfrak{g}})$$

where $S_{\cdot}$ is the graded symmetric algebra.

By (2) we have that

$$gr^G_{\cdot} \cong S_{\cdot}(\overline{\mathfrak{g}}) \cdot \overline{V}$$

Lemma 4.3.5. *we have the following commuting diagram*

$$\begin{array}{ccccc} G_k U & \xrightarrow[r]{} & S_k(\overline{\mathfrak{g}}) \downarrow & & \overline{V} \downarrow \varphi \\ G_k W & \longrightarrow & & gr^G_K W & \end{array}$$

where all maps a surjective (and comes from projections.) Since $\overline{V}$ and $\overline{\mathfrak{g}}$ are finite dimensional, we have that $gr^K_k W$ is finite dimensional. So to complete the proof of the main theorem, it is enough to show:

Lemma 4.3.6. $\exists K \forall k \geq K : gr^G_k W = 0$

Proof. for this proof, it is sufficient to prove this ☐

Lemma 4.3.7. $\forall X \in \mathfrak{g} : \exists L > 0 : \forall \geq L, j = 1, \cdots, N, n = 0, \cdots, m-1, |y> \in G_{k-l}$: $\pi_k(\rho_j(X \otimes \xi_j^{-n})^l |u>) = 0$ $\pi_k = \varphi \circ r$

since $a \in S(\overline{\mathfrak{g}})$ is of the form

$$a = \prod_{i=1}^{\dim \mathfrak{g}} \prod_{j=1}^{N} \prod_{n=1}^{m-1} (J^i \otimes \xi_j^{-n})^{k_{i,j,n}}$$

choose $k \geq K > LN \dim \mathfrak{g} m$, then there is at least one $k_{i,j,n} > L$. then for all $|v> \in \overline{V}$ we have $\varphi_k(a|v>) = 0$. Then $gr^G_k W = \varphi_k(S_k(\overline{\mathfrak{g}})\overline{V}) = 0$. This is from 4.3.6 to 4.3.7.

proof of lemma 4.3.7. fix j, n and let

$$E = X_\alpha, F = X_{-\alpha}, H = [X_\alpha, X_{-\alpha}]$$

$E \otimes \xi_j^{-n}$ and $F \otimes \xi_j^{-n}$ acts nilpotently on $\mathcal{H}_{\vec{\lambda}}$ and $S_{k-l}(\bar{\mathfrak{g}})\overline{V}$ finite dim. for $l = 0, \cdots, k$ implies $\exists L : \forall l \geq L, j = 1, \cdots, N, n = 0, \cdots, m-1, |u> \in G_{k-l}U : \pi_k \left(\rho_j(E \otimes \xi_j^{-n})^l\right) |u> = \pi_k \left(\rho_j(F \otimes \xi_j^{-n})^l\right) |u> = 0$ □

for H we have a trick, that starts like this: for $s \geq 1, t \geq 1$, $|u> \in G_{k-s-t}U$ let

$$|u> \in \rho_j((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1})|u>$$

then for $E[1] = \oplus_{i=1}^N E \otimes 1$

$$E[1]|u> = [\rho_j(E \otimes 1), \rho_j((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1})|u> + \rho_j(H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}]E[1]|u>$$

call the last term form $*$. then by the def. of W:

$$\pi_k(E[1]|u>) = 0$$

then

$$0 = -2(s-1)\pi_k\rho_j((H \otimes \xi_j^{-n})^{s-2}(F \otimes \xi_j^{-n})^{t+1}(E \otimes \xi_j^{-n}))|u> + (t+1)\pi_k\rho_j(H \otimes \xi_j^{-n})^s(F \otimes \xi_j^{-n})^t E[1]|u> + \pi_k(*)$$

proof by induction: $[F, H] = 2F$ and $[E, F] = H$. term in above sum vanish becasue $\pi_k((k-1)st\text{part})$. Residue part vanish because $E \otimes 1$(there is no residue). therefore we can decrease s by increasin t. This shows the lemma is true for H, and that finishes the lemma.

□

7th lecture, February 29th 2012

4.4 Conformal blocks in equivalent formal neighbourhoods

Today we will see which conformal isomorphisms give isomorphic conformal blocks. Let $\mathcal{D} = \mathrm{Aut}\mathbb{C}((\xi))$. We will think of these as automorphisms of $\mathbb{C}[[\xi]]$ instead (which might require a bit of thought). We have

$$\mathcal{D} \cong \left\{ \sum_{n=0}^{\infty} a_n \xi^{n+1} \mid a_0 \neq 0 \right\}$$

given by $\mathfrak{h} \mapsto h(\xi)$, where $h \circ g$ gets mapped to $h(g(\xi))$. Let

$$\mathcal{D}^p = \{h \in \mathcal{D} \mid h(\xi) = \xi + a_p \xi^{p+1} + \cdots\}$$

for $p > 0$. We have a filtration

$$\mathcal{D} = \mathcal{D}^0 + \supset \mathcal{D}^1 \supset \ldots.$$

Now let

$$\underline{d} = \mathbb{C}[[\xi]]\xi \frac{d}{d\xi}, \quad \underline{d}^p = \mathbb{C}[[\xi]]\xi^{p+1}\frac{d}{d\xi}, \, p = 0, 1, \ldots.$$

We then have another filtration $\underline{d} = \underline{d}^+ \supset \underline{d}^1 \supset \ldots$, and for $\underline{l} \in \underline{d}$ and $f(\xi) \in \mathbb{C}[[\xi]]$, we define

$$\exp(\underline{l})(f(\xi)) = \sum_{k=0}^{\infty} \frac{1}{k!}(\underline{l}^k f(\xi)),$$

which is well-defined because $\underline{d}$ is nilpotent. Then if we set

$$\mathcal{D}_+^0 = \{h \in \mathcal{D} \mid h(\xi) = a\xi + a_1\xi^2 + \ldots, a > 0\},$$
$$\underline{d}_+^0 = \{l(\xi)\frac{d}{d\xi} \mid l(\xi) = \alpha\xi + \alpha_1\xi^2 + \ldots, \alpha \in \mathbb{R}\},$$

we have the following result, which we will not prove.

Lemma 4.4.1. *The map* $\exp : \underline{d} \to \mathcal{D}$ *is surjective and* $\exp : \underline{d}_+^+ \to \mathcal{D}_+^0$ *is an isomorphism.*

In general, exp is not injective since $\exp(2\pi i\xi\frac{d}{d\xi}) = \mathrm{id}$. Recall the energy momentum tensor

$$T(z) = \frac{1}{2(g^* + l)}\sum_{d=1}^{\dim \mathfrak{g}} : J^a(z)J^a(z) := \frac{1}{2(g^* + l)}\sum_{n,m\in\mathbb{Z}}\sum_{d} : J^a(n)J^a(m) : z^{-n-m-2}$$
$$= \sum_{k\in\mathbb{Z}} L_k z^{-k-2},$$

and we now let

$$T[\underline{l}] = \mathrm{Res}_{z=0}(T(z)l(z)\,dz),$$

so e.g. $T(\xi\,d\xi) = L_0$. Then

$$\exp(T[\underline{l}]) := \sum_{k=0}^{\infty}\frac{1}{k!}T[\underline{l}]^k$$

acts on $\mathcal{H}_\lambda$ from the left and on $\mathcal{H}_\lambda^\dagger$ from the right. By the lemma, there is for $h \in \mathcal{D}_+^0$ a unique $l \in \underline{d}_+^0$ with $\exp(\underline{l}) = h$, and so we can define

$$G[h] = \exp(-T[\underline{l}]),$$

which operates on $\mathcal{H}_\lambda$ and $\mathcal{H}_\lambda^\dagger$. The following theorem follows from a direct but tedious computation.

Theorem 4.4.2. *For* $h \in \mathcal{D}_+^0$, $f(\xi)\,d\xi \in \mathbb{C}((\xi))\,d\xi$, $g(\xi) \in \mathbb{C}((\xi))$ *and* $\underline{l} = l(\xi)\frac{d}{\xi}$, *where* $l \in \mathbb{C}((\xi))$, *we have*

1. $G[h](X \otimes f(\xi))G[h]^{-1} = X \otimes f(h(\xi))$,
2. $G[h_1 \circ h_2] = G[h_1]G[h_2]$ *for* $h_1, h_2 \in \mathcal{D}^p$, $p \geq 1$, *and*
3. $G[h]T[\underline{l}]G[h]^{-1} = T[\mathrm{ad}(h)(\underline{l})] + \frac{c_0}{12}\mathrm{Res}_{\xi=0}(\{h(\xi)_1;\xi\}l(\xi)\,d\xi)$, *where here we use the Schwarzian derivative* $\{h(\xi);\xi\} = \frac{f'''(\xi)}{f'(\xi)} - \frac{3}{2}\left(\frac{f''(\xi)}{f'(\xi)}\right)^2$.

From this we get our desired description of how the conformal blocks change.

Proposition 4.4.3. *For* $h_j \in \mathcal{D}_+^0$, $j = 1, \ldots, N$, *let*

$$\mathfrak{X} = (C; Q_1, \ldots, Q_N, \xi_1, \ldots, \xi_N),$$
$$\mathfrak{X}_{(h)} = (C; Q_1, \ldots, Q_N, h_1(\xi_1), \ldots, h_N(\xi_N)).$$

Then the map

$$G[h_1]\hat{\otimes}\cdots\hat{\otimes}G[h_N] : \mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1}^\dagger\hat{\otimes}\cdots\hat{\otimes}\mathcal{H}_{\lambda_N}^\dagger \to \mathcal{H}_{\vec{\lambda}}$$

given by

$$\langle\varphi_1\hat{\otimes}\cdots\hat{\otimes}\varphi_N| \mapsto \langle\varphi_1 G[h_1]\hat{\otimes}\cdots\hat{\otimes}\varphi_N G[h_N]|$$

induces a canonical isomorphism

$$\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}) \to \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}_{(h)}).$$

Corollary 4.4.4. *If* $\mathfrak{X}$ *and* $\mathfrak{X}'$ *have the same first infinitesimal neighbourhood, then* $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}) \cong \mathcal{V}_{\vec{\lambda}}^\dagger(\mathfrak{X}')$.

4.5 Propagation of vacua

We now consider the following: What happens if we change $\mathfrak{X} = (C; Q_1, \dots, Q_N; \eta_1, \dots, \eta_N)$ to $\tilde{\mathfrak{X}} = (C; Q_1, \dots, Q_{N+1}; \eta_1, \dots, \eta_{N+1})$? (Note here the shift in notation from ξ to η.) We have a map $i : \mathcal{H}_{\vec{\lambda}} \to \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$, where $\mathcal{H}_0$ corresponds to the highest weight 0, given by $|v\rangle \mapsto |v\rangle \otimes |0\rangle$ for fixed $|0\rangle \in \mathcal{H}_0$. From this we get a surjection $\hat{i}^* : \mathcal{H}^\dagger_{\vec{\lambda}} \hat{\otimes} \mathcal{H}^\dagger_0 \to \mathcal{H}^\dagger_{\vec{\lambda}}$.

Theorem 4.5.1. *The map $\hat{i}^*$ induces a map $\mathcal{V}^\dagger_{\vec{\lambda},0}(\tilde{\mathfrak{X}}) \cong \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$.*

Proof. We first prove the existence of the map $\hat{i}^*$. Recall that

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) = \{\langle\psi| \in \mathcal{H}^\dagger_{\vec{\lambda}} \mid \langle\psi|\hat{\mathfrak{g}}(\mathfrak{X}) = 0\}.$$

Then we have a map $\hat{i}^* : \mathcal{V}^\dagger_{\vec{\lambda},0}(\mathfrak{X}) \to \mathcal{H}^\dagger_{\vec{\lambda}}$. Consider $\langle\psi| = \hat{i}^*(\langle\tilde{\psi}|) \in \mathcal{H}^\dagger_{\vec{\lambda}}$ for $\langle\tilde{\psi}| \in \mathcal{V}^\dagger_{\vec{\lambda},0}(\mathfrak{X})$. Let $X \in \mathfrak{g}$, let $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^N Q_j))$, and let $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$. Then we have

$$\sum_{j=1}^N \langle\psi|\rho_j(X[f])|u\rangle = \sum_{j=1}^N \langle\tilde{\psi}|\rho_j(X[f])|u \otimes 0\rangle.$$

Now since f is holomorphic at the point $P = Q_{N+1}$ and 0 is the highest weight at P, we have

$$\langle\tilde{\psi}|\rho_{N01}(X[f])|u \otimes 0\rangle = 0.$$

Since $\langle\tilde{\psi}|\hat{\mathfrak{g}}(\mathfrak{X}) = 0$, then $\langle\psi|\hat{\mathfrak{g}}(\mathfrak{X}) = 0$. Therefore we have a linear map

$$\hat{i}^* : \mathcal{V}^\dagger_{\vec{\lambda},0} : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}).$$

We turn now to the injectivity of $\hat{i}^*$: Assume that $\langle\psi| = i^*(\langle\tilde{\psi}|) = 0$. Recall that the filtration $\{\mathcal{F}_\bullet\}$ defined by

$$F_p\hat{\mathfrak{g}} = \begin{cases} \bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\![\xi_j]\!]\xi_j^{-p} \oplus \mathbb{C} \cdot c, & p \geq 0, \\ \bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\![\xi_j]\!]\xi_j^{-p}, & p < 0, \end{cases}$$

gives us a filtration $\mathcal{F}_p\mathcal{H}_{\vec{\lambda}} = \mathcal{F}_p U(\hat{g}_N) \cdot V_{\vec{\lambda}}$, and we proceed by induction on this filtration. For $p = 0$ we have

$$\langle\psi|u \otimes 0\rangle = \langle\psi|u\rangle = 0.$$

For the induction step, note that elements in $F_{p+1}\mathcal{H}_0$ are if the form $X(m)|v\rangle$, where $|v\rangle \in \mathcal{F}_p\mathcal{H}_0$. Choose $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+1} Q_j))$ and $M > 0$ such that at $P = Q_{N+1}$, we have $f \equiv \eta^M \bmod (\eta^M)$, where $X \otimes \eta^k|v\rangle = 0$ for $k \geq M$. This is possible by the Mittag–Leffler theorem which says that on every non-compact surface, there exists a meromorphic function with given principal points. Then in $\mathcal{V}^\dagger_{\vec{\lambda},0}(\mathfrak{X})$,

$$\langle\tilde{\psi}|u \otimes X(m)|v\rangle = \langle\tilde{\psi}|u \otimes X[f]|v\rangle = -\sum_{j=1}^N \langle\tilde{\psi}|\rho_j(X[f])|u \otimes v\rangle = 0,$$

for all u by the induction hypothesis. In particular $\langle\tilde{\psi}|u \otimes v\rangle = 0$ for all $u \otimes v \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0$, and therefore $\langle\tilde{\psi}| = 0$.

We consider now the surjectivity of $\hat{i}^*$. Let $\langle\psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. We will define inductively an element $\langle\tilde{\psi}| \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_0, \mathbb{C})$, show that it descends to $\mathcal{H}_\lambda \otimes \mathcal{M}_0$, and that it satisfies the condition $\langle\tilde{\psi}|\hat{g}(\mathfrak{X}) = 0$.

First, define $\langle\tilde{\psi}|u \otimes 0\rangle = \langle\psi|u\rangle$ for $u \in \mathcal{H}_{\vec{\lambda}}$. Then

$$\sum_{j=1}^{N}\langle\tilde{\psi}|\rho_j(X[g])|u \otimes 0\rangle = \sum_{j=1}^{N}\langle\psi|\rho_j(X[g])|u\rangle = 0$$

for $g \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. Suppose that we have defined $\langle\psi|$ on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{F}_p\mathcal{M}_0$ and that it satisfies $\sum_{j=1}^{N}\langle\tilde{\psi}|\rho_j(X[g])|u\otimes v\rangle = 0$ for all $|u \otimes v\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes F_p\mathcal{M}_0$ and $g \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N} Q_j))$. We then define $\langle\psi|$ on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{F}_{p+1}\mathcal{M}_0$ by

$$\langle\tilde{\psi}|u \otimes X(m)|v\rangle := -\sum_{j=1}^{N}\langle\psi|\rho_j(X[f])|u \otimes v\rangle$$

for any $u \in \mathcal{H}_{\vec{\lambda}}$, where $X(m)|v\rangle \in \mathcal{F}_{p+1}\mathcal{M}_0$, $|v\rangle \in \mathcal{F}_p\mathcal{M}_0$, and the meromorphic function f is chosen as in the proof of injectivity. What one now has to do is to prove that this is independent of f, which is not difficult, and that

$$\sum_{j=1}^{N+1}\langle\psi|\rho_j(X[g]) = 0$$

on $\mathcal{H}_{\vec{\lambda}} \otimes \mathcal{F}_{p+1}\mathcal{M}_0$ for all $g \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+1} Q_j))$. One then proves that $\langle\tilde{\psi}|$ is linear using commutation relations. We will not do this. □

8th lecture, March 3rd 2012

Last time, we began proving the following theorem.

Theorem 4.5.2. *We have a canonical isomorphism* $\mathcal{V}^{\dagger}_{\vec{\lambda},0}(\tilde{\mathfrak{X}}) \cong \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$.

Corollary 4.5.3. *We have a canonical isomorphism* $\mathcal{V}_{\vec{\lambda},0}(\tilde{\mathfrak{X}}) \cong \mathcal{V}_{\vec{\lambda}}(\mathfrak{X})$.

4.6 Correlation functions

Let C be a smooth curve. Let ω_C be the canonical line bundle over C. Consider $C^M = C \times \cdots \times C$ for $M \in \mathbb{N}$ with canonical line bundle ω_{C^M}. One can show that

$$\omega_{C^M} = \omega_C^{\boxtimes M} := \pi_1^*\omega_C \otimes \cdots \otimes \pi_M^*\omega_C,$$

where π_i is projection onto the i'th factor. Let $\mathfrak{X} = (C; Q_1, \ldots, Q_N; \eta_1, \ldots, \eta_N)$.

Theorem 4.6.1. *Fix* $\langle\psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X})$. *For each* $M \geq 0$, $M \in \mathbb{Z}$, $X_1, \ldots, X_M \in \mathfrak{g}$, $|\varphi\rangle \in \mathcal{H}_{\vec{\lambda}}$, *we can define a meromorphic form* F *and use the notation*

$$F = \langle\psi|X_1(P_1)\cdots X_M(P_M)|\varphi\rangle dP_1 \ldots dP_M \in H^0(C^M, \omega_C^{\boxtimes M}(\sum_{1\leq i\leq j\leq M} *\Delta_{ij} + \sum_{i=1}^{M}\sum_{j=1}^{N} *\pi_i^{-1}(Q_J))),$$

where here $\Delta_{ij} = \{(P_1, \ldots, P_M) \mid P_i = P_j\}$, *such that* F *satisfies the following properties:*

(0) For $M = 0$, $F = \langle\psi, \varphi\rangle$ *is the canonical pairing* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathfrak{X}) \times \mathcal{V}_{\vec{\lambda}}(\mathfrak{X}) \to \mathbb{C}$.

(1) F *is linear with respect to* $|\varphi\rangle$ *and multilinear with respect to* X_i.

(2) For $\sigma \in S_M$,

$$F = \langle\psi|X_{\sigma(1)}(P_{\sigma(1)})\cdots X_{\sigma(M)}(P_{\sigma(M)})|\varphi\rangle dP_1 \ldots dP_M.$$

Proof. Choose $M+1$ nonsingular points $P_1, \dots P_M, P$ of C and formal neighbourhoods $\eta_{N+1}, \dots, \eta_{N+M+1}$. Let

$$\hat{\mathfrak{X}} = (C; Q_1, \dots, Q_N, Q_{N+1}, \dots, Q_{N+M+1}; \eta_1, \dots, \eta_{N+M+1}),$$
$$\tilde{\mathfrak{X}} = (C; Q_1, \dots, Q_N, Q_{N+1}, \dots, Q_{N+M}; \eta_1, \dots, \eta_{N+M}).$$

By propagation, we have isomorphisms

$$i_M : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \cong \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0_M}}(\tilde{\mathfrak{X}}),$$
$$i_{M+1} \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \cong \mathcal{V}^\dagger_{\vec{\lambda}, \vec{0_{M+1}}}(\hat{\mathfrak{X}}),$$

where $\vec{0_k} = (0, \dots, 0)$. The proof of the theorem follows from three claims.

Claim 4.6.2. *For $|\tilde{u}\rangle \in \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{H}_{\vec{0_M}}$ and $X \in \mathfrak{g}$, the expression*

$$\langle \tilde{\psi} | \tilde{u} \otimes X(-1)|0\rangle \, d\eta := \langle \tilde{\psi} | (|\tilde{u}\rangle \otimes (X(-1)|0\rangle)) \, d\eta$$

defines an element in $T^*_P C$, were $\eta = \eta_{M+N+1}$.

Claim 4.6.3. *For $j = 1, \dots, N+M$, put*

$$\omega_j = \sum_{n \in \mathbb{Z}} \langle \tilde{\psi} | \rho_j(X(n)) | \tilde{u} \rangle \xi_j^{-n-1} d\xi_j.$$

Then there is a meromorphic 1-form on $\omega \in H^0(C, \omega_C(* \sum_{j=1}^{N+M} Q_j))$ with

$$t(\omega) = (\omega_1, \dots, \omega_{N+M}),$$

where $t : H^0(C, \omega_C(* \sum_{j=1}^{N+M} Q_j)) \to \bigoplus_{j=1}^{N+M} \mathbb{C}(\!(\xi_j)\!) \, d\xi_j$.

Claim 4.6.4. *We have $\langle \hat{\psi} | \tilde{u} \otimes X(-1)|0\rangle d\eta = \omega_{Q_{N+M+1}}$.*

From these three claims we can construct F: If $|u\rangle = |u\rangle \otimes X_1(-1)|0\rangle \otimes \cdots \otimes X_M(-1)|0\rangle$, then

$$\begin{aligned}\langle \tilde{\psi} | \tilde{u} \rangle d\eta_1 \dots d\eta_M &= \langle \tilde{\psi} | u \otimes (X_1(-1)|0\rangle \otimes \cdots \otimes X_M(-1)|0\rangle) d\eta_1 \dots d\eta_M \\ &\in T^*_{P_1} C \otimes \cdots T^*_{P_M} C\end{aligned}$$

if $P_k \neq Q_j$ and $P_j \neq P_k$, $j \neq k$. Since the set of poles has codimension greater than 2 (and the complement is connected), then by Hartog's theorem, we can view it as an element of

$$H^0(C^m, \omega_C^{\boxtimes M}(\sum_{i<j} * \Delta_{ij} + \sum_{i=1}^{M} \sum_{j=1}^{N} * \pi_i^{-1} Q_j))$$

which is denoted by

$$\langle \psi | X_1(P_1) \cdots X_M(P_M) | u \rangle \, dP_1 \dots dP_M.$$

Parts (0) and (1) of the theorem follows. So does (2). □

Proof of Claim 4.6.2. By the Mittag–Leffler theorem, choose $f \in H^0(C, \mathcal{O}_C(*(P + Q_1)))$ on C such that $f = \eta^{-1} +$ something regular at P, and such that $f \equiv 0 \bmod (\xi_j^{n_j})$ at Q_j, $j \neq 1$, where $\eta = \eta^{-1}_{M+1}(\xi)$, $\xi_j = \eta_j^{-1}(\xi)$ and $n_j \gg 0$, $\rho_j(X[f])|\tilde{u}\rangle = 0$, and such that f is holomorphic at Q_j, $j \neq 1$, and $j \neq N+M+1$. Then

$$\langle \tilde{\psi} | \tilde{u} \otimes X(-1)|0\rangle = \langle \tilde{\psi} | \tilde{u} \otimes (X[f])|0\rangle = -\langle \tilde{\psi} | \rho_1(X[f]) | \tilde{u} \otimes 0\rangle.$$

If we change formal neighbourhoods from η_{M+1} to $\tilde{\eta}_{M+1}$, then

$$\tilde{\eta} = \tilde{\eta}_{M+1}^{-1}(\xi) = a_1\eta + a_2 + \cdots ,$$

where $a_1 \neq 0$. Then

$$\langle\tilde{\psi}|\tilde{u} \otimes X(-1)|0\rangle_{\tilde{\eta}} = a_1^{-1}\langle\tilde{\psi}|\tilde{u} \otimes (-1)|0\rangle_{\eta}.$$

Thus $\langle\tilde{\psi}|\tilde{u}\otimes X(-1)|0\rangle$ depends only on the first order neighbourhood and so $\langle\tilde{\psi}|\tilde{u}\otimes X(-1)|0\rangle_{\eta}d\eta \in T_P^*C$ is independent of η. □

9th lecture, March 7th 2012

We first remark on the notation from last time that we write

$$X_i(P_i) = X_i(-1)|0\rangle_{P_i}$$

for $X_i \in \mathfrak{g}$ and fixed $|0\rangle_{P_i} \in V_0$ associated to P_i. This expression a priori depends on the formal neighbourhood chosen, even if the correlation functions defined last time did not: Recall that we defined

$$\begin{aligned} F &= \langle\psi|X_1(P_1)\cdots X_M(P_M)|u\rangle dP_1 \ldots dP_M \\ &:= \langle\tilde{\psi}|(u \otimes X_1(-1)|0\rangle_{P_1} \cdots \otimes X_M(-1)|0\rangle_{P_M})dP_1 \ldots dP_M \end{aligned}$$

makes sense, where $|\tilde{\psi}\rangle u \otimes X_i(-1)|0\rangle_{P_i}\, dP_i \in T^*_{P_i}C$, and where $u \in \mathcal{H}_{\vec{\lambda}}$. Recall for the notation also that we defined $i_M : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \cong \mathcal{V}^\dagger_{\vec{\lambda},\vec{0_M}}(\tilde{\mathfrak{X}})$ and $i_{M+1} : \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \cong \mathcal{V}^\dagger_{\vec{\lambda},\vec{0_{M+1}}}(\hat{\mathfrak{X}})$, where in $\tilde{\mathfrak{X}}$, we added points $Q_{N+1}, \ldots, Q_{N+M}$, and in $\hat{\mathfrak{X}}$, we further added Q_{N+M+1} and wrote $\tilde{\psi} = i_M(\psi)$, $\hat{\psi} = i_{M+1}(\psi)$.

We now continue our discussion on correlation functions and prove the following.

Theorem 4.6.5. *Under the residue pairing*

$$\bigoplus_{j=1}^N \mathbb{C}(\!(\xi_j)\!) \times \bigoplus_{j=1}^N \mathbb{C}(\!(\xi_j)\!)\, d\xi_j \to \mathbb{C},$$

$$(f_1(\xi_1), \ldots, f_N(\xi_N), g_1(\xi_1)\, d\xi_1, \ldots, g_N(\xi_N)\, d\xi_N) \mapsto \sum_{j=1}^N \mathrm{Res}_{\xi_j=0}(f_j(\xi_j)g_j(\xi_j)\, d\xi_j),$$

the vector spaces $H^0(C, \mathcal{O}(*\sum_{j=1}^N Q_j))$ and $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$ are annihilators of each other, recalling that we have an embedding

$$t : H^0(C, \mathcal{O}(*\sum_{j=1}^N Q_j)) \to \bigoplus_{j=1}^N \mathbb{C}(\!(\xi_j)\!),$$

and similarly for the meromorphic forms.

Proof. Since the residue is independent of local coordinates, we can use formal coordinates at Q_j. For $m, n \in \mathbb{Z}$, $m, n > 0$, we have an exact sequence

$$0 \to \mathcal{O}_C(-m\sum_{j=1}^N Q_j) \to \mathcal{O}_C(n\sum_{j=1}^N Q_j) \to \bigoplus_{j=N}^{n} \bigoplus_{j=-n}^{m-1} \mathbb{C}\xi_j^k \to 0,$$

where the third map is the map t where we let coefficients of ξ^k, $k \geq m$ be 0. From this sequence, we get a long exact sequence

$$0 \to H^0(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \to H^0(C, \mathcal{O}_C(\sum_{j=1}^N Q_j)) \xrightarrow{p} \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^*$$
$$\xrightarrow{c} H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \to H^1(C, \mathcal{O}_C(\sum_{j=1}^N Q_j)) \to 0.$$

By Serre duality, $H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \cong H^0(C, \omega_C(m\sum_{j=1}^N Q_j))$. We will show that under the (Serre) dual pairing

$$\langle\, ,\, \rangle : H^1(C, \mathcal{O}_C(-m\sum_{j=1}^N Q_j)) \to H^0(C, \omega_C(m\sum_{j=1}^N Q_j)) \to \mathbb{C},$$

we will have

$$\langle c\left(\bigoplus_{j=1}^N g_j(\xi_j)\right), \tau\rangle = \sum_{j=1}^N \mathrm{Res}(g_j(\xi_j)\tau_j), \tag{3}$$

where $\tau_j$is the Laurent expansion of τ around Q_j, and $g_i(\xi_j) \sum \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k$. Then by the long exact sequence above and Serre duality, we get that $\bigoplus_{j=1}^N g_j(\xi_j) \in \mathrm{Im} p$ if and only if $\sum_{j=1}^N \mathrm{Res}(g_j(\xi_j)\tau_j) = 0$ for all $\tau \in H^0(C, \omega_C(m\sum Q_j))$.

Suppose now that $\bigoplus_{j=1}^N f_j(\xi_j) \in \bigoplus_{j=1}^N \mathbb{C}(\!(\xi_j)\!)$ is annihilated by $H^0(C, \omega_C(*\sum_{j=1}^N Q_j))$. We want to show that $\bigoplus_{j=1}^N f_j(\xi_j) \in H^0(C, \mathcal{O}_C(*\sum Q_j))$. Fix $m \in \mathbb{Z}$ and $m > 0$. Consider $\bigoplus_{j=1}^N f_j(\xi_j) \bmod \xi_j^m$. If $f_j(\xi_j) = \sum_{k=-n_j}^{\infty} a_{j,k}\xi_j^k$, then $f_j, m(\xi_j) = \sum_{k=-n_j}^{m-1} a_{j,k}\xi_j^k$, so $\bigoplus_{j=1}^N f_{j,m}(\xi_j)$ is annihilated by $H^0(C, \omega_C(m\sum Q_j))$. Let $n = \max(n_j)$. Then $f_{j,m} = p(f^{(m)})$ for some $f^{(m)} \in H^0(C, \mathcal{O}_C(n\sum Q_j))$ by the long exact sequence from before.

We now claim that $f^{(m)} = \bigoplus_{j=1}^N f_j(\xi_j)$ for some m large enough (and therefore every $k \geq m$). Let $f^{(m)}(\xi_j)$ be the Laurent series expansion of $f^{(m)}$ around Q_j with formal parameter $\xi_j = \eta_j^{-1}(\xi)$. Suppose that $f^{(m)} = \bigoplus_{j=1}^N f_j(\xi_j)$ are not equal for any m. Then

$$g_j(\xi_j) := f_j(\xi_j) - f^{(m)}(\xi_j) \equiv 0 \bmod \xi_j^m$$

for all j, and there is a k such that $g_k(\xi_k) \neq 0$. Let $s \geq m$. Then

$$g_k(\xi_k) = \sum_{i=0}^{\infty} b_{s+i}\xi_k^{s+i},$$

where $b_s \neq 0$. Certainly, $\bigoplus g_j(\xi_j)$ is annihilated by $H^0(C, \omega_c(m\sum Q_j))$. Since $s \geq m > 0$, $H^0(C, \omega_C((s+1)Q_k)) \neq \{0\}$ by the residue theorem. Then

$$\sum_{j=1}^N \mathrm{Res}_{\xi_j=0}\, g_i(\xi_j)\omega_j(\xi_j) = \mathrm{Res}_{\xi_k=0}\, g_k(\xi_k)\omega_k(\xi_k) = b_s \neq 0,$$

which contradicts the annihilation by $H^0(C, \omega_C(m\sum Q_J))$, and thus we have proved the claim. This shows that the annihilator of $H^0(C, \omega_C(*\sum Q_J))$ is $H^0(C, \mathcal{O}_C(*\sum Q_j))$.

For the other part, we start with the exact sequence

$$0 \to \omega_C(-m\sum Q_j) \to \omega_C(n\sum Q_j) \to \bigoplus_{j=1}^N \bigoplus_{k=-n}^{m-1} \mathbb{C}\xi_j^k\, d\xi_j,$$

and the proof is the same.

Thus it remains to show (3). For $\varepsilon > 0$, let $D_\varepsilon^{(i)} = \{\xi_j \mid |\xi_j| < \varepsilon\}$ around Q_j. Let $C_\varepsilon = C \setminus \bigcup_{j=1}^N D_\varepsilon^{(j)}$. Cut C_ε along a closed curve such that the cut surface R_ε is simply-connected (see [Uen08, p. 23] for a figure showing this curve; this step may not be necessary). Consider a covering $\{U_i\}_{i=1}^t$ of C for some $t \in \mathbb{Z}$ such that $Q_j \in U_j$, $j = 1, \dots, N$ and $Q_j \notin U_i$ for $i \neq j$. Choose U_i small enough that there exist $h_j \in H^0(C, \mathcal{O}_C(m \sum_{i=1}^N Q_i))$, $j = 1, \dots, N$ with $p(h_j) = g_j(\xi_j)$, where p is the one from the long exact sequence from before. The existence of such h_j follows from the Mittag–Leffler theorem.

Let $h_i = 0$ for $i \geq N+1$ and

$$h_{ab} := h_b - h_a \in H^0(U_a \cap U_b, \mathcal{O}_C(-n \sum_{j=1}^N Q_j)).$$

Then $\{h_{ab}\}$ defines a Čech 1-cocycle representing

$$c\left(\bigoplus_{j=1}^N g_j(\xi_j)\right) \in H^1(C, \mathcal{O}_C(-n \sum Q_j)).$$

Serre duality is given by now given by

$$\begin{aligned} H^1(C, \mathcal{O}_C(-m \sum Q_j)) \times H^0(C, \omega_C(n \sum Q_j)) &\to \mathbb{C}, \\ (\omega, \tau) &\mapsto \frac{1}{2\pi i} \int_C \omega \wedge \tau. \end{aligned}$$

By the Dolbeault lemma, $H^1(C, \mathcal{O}_C(-m \sum Q_j)) \cong H^{0,1}_{\overline{\partial}}(C, \mathcal{O}_C(-m \sum Q_j))$. Let $\mathcal{D}_C^{(0,k)}$ be the sheaf of germs of type $(0,k)$ forms on C and set

$$\mathcal{D}_C^{(0,i)}(-n \sum Q_j) = \mathcal{O}_C(-n \sum Q_j) \otimes_{\mathcal{O}_C} \mathcal{D}_C^{(0,i)}.$$

Then we have that

$$H^{(0,1)}_{\overline{\partial}}(C, \mathcal{O}_C(-m \sum Q_j)) \cong \frac{\ker(\overline{\partial} : H^0(C, \mathcal{D}_C^{(0,1)}(-m \sum Q_j)) \to H^0(C, \mathcal{D}_C^{(0,2)}(-m \sum Q_j))}{\overline{\partial}(H^0(C, \mathcal{D}_C^{(0,0)}(-m \sum Q_j)))},$$

and $[\{h_{ab}\}]$ are given by $h_{ab} = s_b - s_a$ where $s_i \in H^0(U_i, \mathcal{D}_C^{(0,0)}(-n \sum Q_j))$. Then $h_a - s_a = h_b - s_b$ for all a, b. This gives us a global function h on C with poles at Q_j, given by $h = h_a - s_a$ on U_a, and $[\overline{\partial} h]$ is a non-zero Dolbeault cohomology class associated to $\{h_{ab}\}$. We have $\overline{\partial} h = -\overline{\partial} s_q$ on U_q and find

$$\begin{aligned} \langle c(\bigoplus g_j(\xi_j)), \tau \rangle &= \frac{1}{2\pi i} \int_C \overline{\partial} h \wedge \tau = \frac{1}{2\pi i} \int_C d(h\tau) \\ &= \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_\varepsilon} d(h\tau) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\partial C} h\tau \\ &= \lim_{\varepsilon \to 0} \sum_{j=1}^N \frac{1}{2\pi i} \int_{\partial D_\varepsilon^{(j)}} (h_j - s_j)\tau \\ &= \lim_{\varepsilon \to 0} \sum_{j=1}^N \frac{1}{2\pi i} \left(\int_{\partial D_\varepsilon^{(j)}} h_j \tau - \int_{\partial D_\varepsilon^{(j)}} s_j \tau \right) \\ &= \sum_{j=1}^N \mathrm{Res}_{Q_j}(h_j \tau) - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{D_\varepsilon^{(j)}} d(s_j \tau) \\ &= \mathrm{Res}_{\xi_j = 0}(g_j(\xi_j)\tau_j), \end{aligned}$$

where in the last equality we used that $s_j \tau \in C^\infty$ in a neighbourhood of Q_j so the limit vanishes. □

10th lecture, March 9th 2012

Today we prove Claim 4.6.3 and Claim 4.6.4.

Proof of Claim 4.6.3. For a meromorphic function $f \in H^0(C, \mathcal{O}_C(*\sum_{j=1}^{N+M} Q_j))$, let $f_j(\xi_j) = \sum a_n^{(j)} \xi_j^n$ be the formal Laurent expansion at Q_j in the formal coordinates $\xi_j = \eta_j^{-1}(\xi)$. Then $t(f) = (f_1(\xi_1), \dots, f_{N+M}(\xi_{N+M}))$ by definition, and

$$\begin{aligned}\sum_{j=1}^{N+M} \mathrm{Res}_{\xi_j=0}(f_j(\xi_j)\omega_j) &= \sum_{j=1}^{N+M} \sum_{n\in\mathbb{Z}} \langle\tilde{\psi}|\rho_j(X(n))|\tilde{u}\rangle a_n^{(j)} \\ &= \langle\tilde{\psi}|X \otimes t(f)|\tilde{u}\rangle = 0,\end{aligned}$$

where $X(n) = X \otimes \xi_j^n$ and we have used that $\rho_j(X(n))a_n^{(j)} = \rho_j(X \otimes a_n^{(j)} \xi_j^n)$. By the Theorem 4.6.5, there exists $\omega \in H^0(C, \omega_c(*\sum_{j=1}^{N+M} Q_j))$ with $t(\omega) = (\omega_1, \dots, \omega_{N+M})$. □

Proof of Claim 4.6.4. Since $\langle\hat{\psi}|\tilde{u} \otimes X(-1)|0\rangle d\eta \in T_P^* C$, we may assume that η is a local holomorphic cooridnate. Fix $0 \le i \le N+M$ and write $\eta = \eta_{N+M+1}^{-1}(\xi)$. Let $f \in H^0(C, \mathcal{O}_C(*(P \oplus Q_i)))$ on C such that $f = \eta^{-1} + \text{reg.}$ at P and $f \equiv 0 \bmod(\xi_j^{n_j})$ at Q_j, $j \ne i$, $1 \le j \le N+M$, where $n_j > 0$ is large enough that $\rho_j(X \otimes \xi_j^{n_j})|\tilde{u}\rangle = 0$, and such that $f\omega$ holomorphic at Q_j, $j \ne i$, $1 \le j \le N+M$. Then

$$\begin{aligned}\langle\hat{\psi}|\tilde{u} \otimes X(-1)|0\rangle_P &= -\sum_{k=1}^{N+M} \langle\tilde{\psi}|\rho_k(X[f])|\tilde{u}\rangle \\ &= -\langle\tilde{\psi}|\rho_i(X[f])|\tilde{u}\rangle.\end{aligned}$$

Also, by Claim 4.6.3,

$$\begin{aligned}\mathrm{Res}_P(\frac{1}{\eta}\omega) = \mathrm{Res}_P(f\omega) &= -\sum_{k=1} \mathrm{Res}_{Q_k}(f\omega) = -\mathrm{Res}_{Q_i}(f\omega) \\ &= -\mathrm{Res}_{\xi_i=0}(f_i(\xi_i) \sum_{n\in\mathbb{Z}} \langle\tilde{\psi}|\rho_i(X(n))|\tilde{u}\rangle \xi_i^{n-1}\, d\xi_i) \\ &= -\langle\tilde{\psi}|\rho_i(X[f])|\tilde{u}\rangle = \langle\hat{\psi}|\tilde{u} \otimes X(-1)|0\rangle_P.\end{aligned}$$

□

This completes the proof of Theorem 4.6.1. We now add some further properties.

Theorem 4.6.6. *We have the following:*

(3) *For $k = 1, \dots, N$ and $\xi_k = \eta_k^{-1}(\xi)$ if ξ_k are holomorphic coordinates, then*

$$\begin{aligned}&\mathrm{Res}_{\xi_k=0}(\xi_k^n \langle\psi|X(\xi_k)X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\varphi\rangle d\xi_k) \\ &\quad = \langle\psi|X_1(P_1)X_2(P_2)\cdots X_M(P_M)(\rho_k(X(n))|\varphi\rangle),\end{aligned}$$

where $X(P) = X(-1)|0\rangle_P$, and we forget about the ˜ and ˆ from before. In other words we have an expansion

$$\langle\psi|X(\xi_k)X_1(P_1)\cdots X_M(P_M)|\varphi\rangle d\xi_k = \sum_{n\in\mathbb{Z}} \langle\psi|X_1(P_1)\cdots X_M(P_M)(\rho_k(X(n))|\varphi\rangle).$$

Proof. By Claim 4.6.3, we have

$$\begin{aligned}
&\langle\psi|X(\xi_k)X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\varphi\rangle d\xi_k \\
&\qquad = \langle\tilde{\psi}X(\xi_k|\otimes X_1(-1)|0\rangle_{P_1}\otimes\cdots\otimes X_M(-1)|0\rangle_{P_M}\otimes|\varphi\rangle d\xi_k \\
&\qquad = \sum_{n\in\mathbb{Z}}\langle\tilde{\psi}|(X(-1))|0\rangle_{P_1}\otimes\cdots\otimes X_M(-1)|0\rangle_{P_M}(\rho_k(X(n))|\varphi\rangle)\xi_k^{-n-1}d\xi_k \\
&\qquad = \sum_{n\in\mathbb{Z}}\langle\psi|X_1(P_1)\cdots X_M(P_M)(\rho_k(X(n))|\varphi\rangle)\xi_k^{-n-1}d\xi_k.
\end{aligned}$$

□

Theorem 4.6.7. *(4) For local holomorphic coordinates z at $P\in C$, P' in the coordinate neighbourhood,*

$$\begin{aligned}
&\langle\psi|X(P)Y(P')X_1(P_1)X_2(P_2)\cdots X_M(P_M)|\varphi\rangle \\
&\qquad = \frac{l\cdot(X,Y)}{(z(P)-z(P'))^2}\langle\psi|X_1(P_1)\dots X_M(P_M)|\varphi\rangle \\
&\qquad\quad + \frac{1}{z(P)-z(P')}\langle\psi|[X,Y](P')(P')X_1(P_1)\cdots X_M(P_M)|\varphi\rangle \\
&\qquad\quad + \textit{part which is regular at } P=P'.
\end{aligned}$$

Proof. Let $P\in C$, U a coordinate neighbourhood of P with center at P. Let $P'\in U$ and let $w=z-z(P')$ be coordinates in a neighbourhood of P'. Let $f\in H^0(C,\mathcal{O}_C(*(P+\sum_{j=1}^{N+M}Q_j)))$ such that $f=1/z+\text{reg.}$ at P. Adjusting the holomorphic coordinates, we can assume $f=1/z$ around P. Then

$$f=\frac{1}{z-a+a}=\frac{1}{w+a}=\frac{1}{a(w+1/a)}=\frac{1}{a}\left(1-\frac{w}{a}+\frac{w^2}{a^2}-\cdots\right).$$

Let $\hat{\psi}^*=i_{M+2}(\psi)$ be another propagation of ψ, $Q_{N+M+2}=P'$, and consider

$$\langle\hat{\psi}^*|(X(-1)|0\rangle_P\otimes Y(-1)|0\rangle_P\otimes|\tilde{\Phi}\rangle)\,dz\,dw\in T_p^*(C)\otimes T_{P'}^*(C),$$

where here

$$|\tilde{\Phi}\rangle=X_1(-1)|0\rangle_{P_1}\otimes\cdots\otimes X_M(-1)|0\rangle_{P_M}\otimes|\Phi\rangle.$$

Recall that $\langle\tilde{\psi}^*|X[f]=0$, and in particular

$$\langle\tilde{\psi}^*|X[f](|0\rangle_P\otimes Y(-1)|0\rangle_{P'}\otimes\tilde{\Phi})=0.$$

It follows that

$$\begin{aligned}
\langle\hat{\psi}^*|X(-1)|0\rangle_P\otimes Y(-1)|0\rangle_{P'}\otimes|\tilde{\Phi}\rangle &= \langle\hat{\psi}^*|\rho_1(X[f])|0\rangle_P\otimes Y(-1)|0\rangle_{P'}\otimes|\tilde{\Phi}\rangle \\
&= -\sum_{j=1}^{N+M}\langle\hat{\psi}|Y(-1)|0\rangle_{P'}\otimes\rho_j(X[f])|\tilde{\Phi}\rangle \\
&\quad -\langle\hat{\psi}|\rho_{P'}(X[f])Y(-1)|0\rangle_{P'}\otimes|\tilde{\Phi}\rangle \\
&= -\sum_{j=1}^{N+M}\langle\tilde{\psi}|Y(P')(\rho_j(X[f])|\tilde{\Phi}\rangle) \\
&\quad -\langle\hat{\psi}|\rho_{P'}(X[f])Y(-1)|0\rangle_{P'}\otimes|\tilde{\Phi}\rangle.
\end{aligned}$$

Also,

$$\begin{aligned}&(X[f])Y(-1)|0\rangle_{P'}\\&= (X \otimes \frac{1}{w+a})(Y \otimes \frac{1}{w})|0\rangle_{P'} = [X \otimes \frac{1}{w+a}, Y \otimes \frac{1}{w}]|0\rangle'_P + (Y \otimes \frac{1}{w})(X \otimes \frac{1}{w+a})|0\rangle_{P'}.\end{aligned}$$

Here, $X \otimes \frac{1}{w+a} \in \hat{\mathfrak{g}}$ and thus the last term is 0. Therefore, since $\mathrm{Res}_{w=0}(d(\frac{1}{w+a})\frac{1}{w}) = -a^{-2}$, we find

$$(X[f])Y(-1)|0\rangle_{P'} = \left(\frac{[X,Y]}{a} \otimes w^{-1} - \frac{l(X,Y)}{a^2}\right)|0\rangle_{P'},$$

and since $a = z(P') = z(P') - z(P)$, we obtain the Theorem. □

We will not prove the following theorem.

Theorem 4.6.8. *(5) For local holomorphic coordinates z at Q_i and*

$$|v\rangle \in V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots V_{\lambda_N} \subseteq \mathcal{H}_{\vec{\lambda}},$$

we have

$$\langle\psi|X(P)X_1(P_1)\cdots X_M(P_M)|v\rangle = \frac{1}{z(P) - z(Q_i)}\langle\psi|X_1(P_1)\cdots X_M(P_M)(\rho_i(X)|v\rangle) + \textit{regular}.$$

Proposition 4.6.9. *Define*

$$\langle\psi|T(z)|\varphi\rangle dz^2 := \frac{1}{2(g^* + l)}\lim_{z\to w}\left(\sum_{d=1}^{\dim\mathfrak{g}}\langle\psi|J^a(z)J^a(w)|\varphi\rangle\, dz\, dw - \frac{l\dim\mathfrak{g}}{(z-w)^2}\langle\psi|\varphi\rangle\, dz\, dw\right),$$

where $J^1, \ldots, J^{\dim\mathfrak{g}}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the normalized Killing form $\langle\, ,\, \rangle$. Then

$$\mathrm{Res}_{\xi_k=0}(\xi_k^{n+1}\langle\psi|T(\xi_k)|\varphi\rangle)d\xi_k = \langle\psi|(\rho_k(L_n))|\varphi\rangle.$$

In other words, we have an expansion

$$\langle\psi|\varphi\rangle d\xi_k^2 = \sum_{n\in\mathbb{Z}}\langle\psi|(\rho_k(L_n))|\varphi\rangle\xi_k^{-n-2}d\xi_k^2.$$

11th lecture, March 12th 2012

An outline for the next weeks is the following:

- Factorization,
- Extra background we need, involving sheaf theory, deformation theory of stable curves,
- The sheaf of conformal blocks on $\overline{\mathcal{M}_{g,m}}$ (in particular the result that this is locally free),
- We will discuss how this result together with factorization gives the Verlinde formula.

4.7 Factorization

Everything we have seen so far is not just valid for compact Riemann surfaces with marked points but actually stable nodal curves with marked points, i.e. a marked curve with only nodal singularities such that the automorphism group is finite, e.g. any reduced component isomorphic to $\mathbb{P}^1$ needs to have at least three marked points.

Factorization deals with what happens when we degenerate a Riemann surface with marked points $Q_1, \ldots, Q_n$ to a curve $\mathfrak{X}$ with a node (see Figure **??**), and desinguralize it (by normalization) to obtain a non-singular stable curve $\tilde{\mathfrak{X}}$. We will be working under the assumption that the result is connected. On the normalization, we have two points P and P' over the node.

By propagation of the vacuum, we can always assume that every irreducible component of the curve has at least one marked point – this is referred to as condition (Q).

Before stating the factorization theorem, we need an aside on Lie algebras: Let $\mathfrak{g}$ be a (semi)simple Lie algebra. The positive Weyl chamber P_+ comes with a canonical involution $()^\dagger : \mu \mapsto \mu^\dagger := -w(\mu)$, where w is the longest element in the Weyl group. This preserves P_l. For simple $\mathfrak{g}$, this involution is the identity for $A_1, B_r, C_r, G_2, F_4, E_7, E_8, D_{4r}, D_{4r+1}, D_{4r+3}$. For A_r, it is given by flipping the Dynkin diagram. This is also true for E_6. For D_{4r+2} it flips the diagram vertically (i.e. switches the two special nodes of the diagram). If μ is a dominant weight with highest weight representation V_μ, then $V_{\mu^\dagger} \cong V_\mu^\dagger$.

Theorem 4.7.1 (Factorization)**.** *We have*

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X}) \cong \bigoplus_{\mu \in P_l} \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\tilde{\mathfrak{X}}),$$

where $\mu, \mu^\dagger$ are the weights associated to the points P, P' (note that the ordering does not matter).

Proof. The outline of the proof is the following:

- We will define $\iota_\mu : \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\tilde{\mathfrak{X}}) \to \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$ for every μ.
- We will show that ι_μ is injective,
- that $\iota := \bigoplus \iota_\mu$ is injective, and
- that ι is surjective.

Definition of ι: Suppose that μ is given. Then we can consider $V_\mu \otimes V_{\mu^\dagger}$, thought of as a single copy of $\mathfrak{g}$ acting diagonally by representations ρ_P and $\rho_{P'}$(rather than as a representation of $\mathfrak{g} \otimes \mathfrak{g}$). This representation is not irreducible. In particular the trivial $\mathfrak{g}$-representation occurs as a subrepresentation with multiplicity one. Pick a nonzero element within this subrepresentation, and denote it $|0_{\mu,\mu^\dagger}\rangle \in V_\mu \otimes V_{\mu^\dagger}$. For all $X \in \mathfrak{g}$,

$$\rho_P(X)|0_{\mu,\mu^\dagger}\rangle + \rho_{P'}(X)|0_{\mu,\mu^\dagger}\rangle = 0.$$

Remark that $\mathcal{H}_{\vec{\lambda}} \cong |0_{\mu,\mu^\dagger}\rangle \otimes \mathcal{H}_{\vec{\lambda}} \subseteq \mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$. Let $\langle\tilde{\Psi}| \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\tilde{\mathfrak{X}})$, and put $\iota_\mu(\langle\tilde{\Psi}|) = \langle\Psi|$ defined by

$$\langle\Psi|\Phi\rangle := \langle\tilde{\Psi}|0_{\mu,\mu^\dagger} \otimes \Phi\rangle.$$

for all $\Phi \in \mathcal{H}_{\vec{\lambda}}$. This tells us that $\langle\Psi| \in \mathcal{H}^\dagger_{\vec{\lambda}}$. What we need to check is the gauge condition to see that $\langle\Psi| \in \mathcal{V}^\dagger_{\vec{\lambda}}(\mathfrak{X})$. That is, we need to show that if $f \in H^0(C, \mathcal{O}(*\sum_{i=1}^N Q_j))$, then

$$\sum_{j=1}^N \langle\Psi|\rho_j(X[f])|\Phi\rangle = 0.$$

This is now a straight-forward check since we can view f as a function on $\tilde{C}$ which agrees on P and P', and we find

$$\begin{aligned}\sum_{j=1}^{N} \langle \Psi | \rho_j(X[f]) | \Phi \rangle &= \sum_{j=1}^{N} \langle \tilde{\Psi} | 0_{\mu,\mu^\dagger} \otimes \rho_j(X[f]) | \Phi \rangle \\ &= \sum_{j=1}^{N} \langle \tilde{\Psi} | 0_{\mu,\mu^\dagger} \otimes \rho_j(X[f]) | \Psi \rangle + \langle \tilde{\Psi} | (\rho_P(X[f]) | 0_{\mu,\mu^\dagger} \rangle) \otimes | \Phi \rangle + \langle \tilde{\Psi} | (\rho_{P'}(X[f]) | 0_{\mu,\mu^\dagger} \rangle) \otimes | \Phi \rangle,\end{aligned}$$

which is 0 since $\langle \tilde{\Psi} | \in \mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}$.

ι_μ **is injective**: Again we use the notation that $\iota_\mu(\langle \tilde{\Psi} |) = \langle \Psi |$. We claim that $\langle \Psi | X(P) | \Phi \rangle \, dP = \langle \tilde{|} \Psi X(P) | 0_{\mu,\mu^\dagger} \otimes \Phi \rangle \, dP$ for all $\tilde{\Psi}, \Phi, X$. This follows from Theorem 4.6.6 as

$$\langle \Psi | X(P) | \Phi \rangle \, dP0 \sum_{n \in \mathbb{Z}} \langle \Psi | \rho_j(X(n)) | \Phi \rangle \xi_j^{-n-1} \, d\xi_j,$$

whereas the right hand side becomes

$$\begin{aligned}\langle \tilde{|} \Psi X(P) | 0_{\mu,\mu^\dagger} \otimes \Phi \rangle \, dP &= \sum_{n \in \mathbb{Z}} \langle \tilde{\Psi} | \rho_j(X(n)) | 0_{\mu,\mu^\dagger} \Phi \rangle \xi_j^{-n-1} \, d\xi_j \\ &= \sum_{n \in \mathbb{Z}} \langle \tilde{\Psi} | 0_{\mu,\mu^\dagger} \otimes \rho_j(X(n)) | Phi \rangle \xi_j^{-n-1} \, d\xi_j \\ &= \sum_{n \in \mathbb{Z}} \langle \Psi | \rho_j(X(n)) | \Phi \rangle \xi_j^{-n-1} \, d\xi_j.\end{aligned}$$

Similarly,

$$\langle \Psi | X_1(P_1) \cdots X_M(P_M) | \Phi \rangle dP_1 \dots dP_M = \langle \tilde{\Psi} | X_1(P_1) \cdots X_M(P_M) | 0_{\mu,\mu^\dagger} \otimes \Phi \rangle dP_1 \dots dP_M$$

for any combination $X_i(P_i)$. Assume now that $\langle \Psi | = i_\mu \langle \tilde{\Psi} | = 0$ and let us show that $\langle \tilde{Psi} | = 0$. By the residue expression from Theorem 4.6.6,

$$\langle \tilde{\Psi} | X_2(P_2) \cdots X_M(P_M) | \rho_{P'}(X_1(n)) 0_{\mu,\mu^\dagger} \otimes \Phi \rangle = 0$$

for all X_i, Φ, and all n. This implies that

$$\begin{aligned}\langle \tilde{\Psi} | \rho_P(X_2(n_2) X_1(n_1)) 0_{\mu,\mu^\dagger} \Phi \rangle &= 0, \\ \langle \tilde{\Psi} | \rho_P(X_1(n_1)) \rho_{P'} X_2(n_2)) 0_{\mu,\mu^\dagger} \Phi \rangle &= 0, \\ \langle \tilde{\Psi} | \rho_{P'}(X_1(n_1) X_2(n_2)) 0_{\mu,\mu^\dagger} \Phi \rangle &= 0.\end{aligned}$$

This implies that $\langle \tilde{\Psi} | \tilde{\Phi} \rangle = 0$ for any $\tilde{\Phi} \in \mathcal{H}_{\mu,\mu^\dagger,\vec{\lambda}}$, since $\mathcal{H}_{\mu,\mu^\dagger}$ is an irreducible $\hat{\mathfrak{g}} \times \hat{\mathfrak{g}}$-module, and

$$X = \{ Y \in \mathcal{H}_{\mu,\mu^\dagger} \mid \langle \tilde{\Psi} | Y \otimes \Phi \rangle = 0 \}$$

for all $\tilde{\Psi}, \Phi$ is a subrepresentation which is therefore all of $\mathcal{H}_{\mu,\mu^\dagger}$

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ι **is injective**: We will not prove this, but one shows that the images have trivial intersection.

ι **is surjective**: Here one runs through a construction using formal manipulations of correlation functions. □

5 Sheaves of conformal blocks

The idea of the following is the following: Given a flat family C of stable pointed curves with formal neighbourhoods parametrized by a base B, i.e. we have the data $C \stackrel{\pi}{\to} B$ with sections $s_1, \dots, s_n$ and neighbourhoods $\eta_1, \dots, \eta_n$ and from this we will get a canonical sheaf of conformal blocks over B which is coherent and locally free.

5.1 Crash course on sheaf theory

The reference for the following is [Har77].

Definition 5.1.1. A *presheaf* $\mathcal{J}$ on a topological space $\mathcal{T}$ is contravariant functor $\mathcal{F}$ from the category of open sets of $\mathcal{T}$ with morphisms given by inclusions to the category Ab of abelian groups. I.e. if U is open in $\mathcal{T}$, then $\mathcal{F}(U)$ is an abelian group, and if $i : U \subseteq V$ is inclusion, we have a morphism $\mathcal{F}(V) \stackrel{\mathcal{F}(i)}{\to} \mathcal{F}(U)$.

Definition 5.1.2. A *sheaf* is a presheaf that further satisfies the following properties:

- If $x \in \mathcal{F}(U)$ and (V_i) is an open cover of U such that $x|_{V_i} := \mathcal{F}(V_i \hookrightarrow U)(x) = 0$ for all i, then $x = 0 \in \mathcal{F}(U)$.
- If for V_i covering U as above and we have $x_i \in \mathcal{F}(V_i)$ such that $x_i|_{V_i \cap V_j} = x_j|_{V_i \cap V_j}$, then there exists $x \in \mathcal{F}(U)$ such that $x|_{V_i} = x_i$.

Example 5.1.3. The main example of a sheaf is the following. Given X a holomorphic manifold (or algebraic variety or $\mathbb{C}$-analytic space), we have the *structure sheaf* $\mathcal{O}_X$ of holomorphic functions (or algebraic or $\mathbb{C}$-analytic functions), that is, $\mathcal{O}_X(U)$ is the set of holomorphic functions on U. Note here that $\mathcal{O}_X$ is actually a sheaf of algebras (rather than just one of abelian groups).

Definition 5.1.4. An $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, that is, for every open U, $\mathcal{F}(U)$ is a module over the algebra $\mathcal{O}_X(U)$ such that the module structure is compatible with restrictions.

Example 5.1.5. If $E \to X$ is a holomorphic vector bundle, the sheaf $\mathcal{O}_{X,E}$ of holomorphic sections of E is an $\mathcal{O}_X$-module.

Definition 5.1.6. An $\mathcal{O}_X$-module $\mathcal{F}$ is *locally free of rank r* if there exists an open cover V_i of X such that $\mathcal{F}|_{V_i} \cong \mathcal{O}_{V_i} \oplus \cdots \oplus \mathcal{O}_{V_i}$ with r summands (and here restriction and direct sum are defined in the obvious way).

Exercise 5.1.7. An $\mathcal{O}_X$-module $\mathcal{F}$ is locally free if and only if it is the sheaf of sections of a holomorphic vector bundle.

Example 5.1.8. If $Y \subseteq X$ is a holomorphic submanifold, we have the ideal sheaf $\mathcal{I}_Y$ where for $U \subseteq X$ open $\mathcal{I}_Y(U)$ is given by the set of holomorphic functions on U that vanish on $U \cap Y$. This is a $\mathcal{O}_X$-module which is not locally free. Also, as a $\mathcal{O}_X$-module, $\mathcal{I}_Y$ is torsion-free.

Example 5.1.9. For $p \in X$, the *skyscraper sheaf* S_P is the sheaf $S_P(U) = \mathbb{C}$ if $p \in U$ and $S_P(U) = 0$ if $p \notin U$. This is a torsion module.

Exercise 5.1.10. A *morphism* between two sheaves is given by a natural transformation between the functors.

The following propositions allows us to use the power of homological algebra for sheaves.

Proposition 5.1.11. *The category of $\mathcal{O}_X$-modules is an abelian category (that is, there are natural notions of kernels and cokernels of morphisms of sheaves, and these are again sheaves).*

Notice that for example, the category of vector bundles is not as well-behaved; kernels and cokernels of vector bundle morphisms will not generally be vector bundles again. In the above, kernels of morphisms will always be sheaves, but images and cokernels are generally only presheaves. In these cases one carries out what is called sheafification.

Definition 5.1.12. Given a presheaf $\mathcal{F}$, we can assign a sheaf $\mathcal{F}^+$ called the *sheafification* of $\mathcal{F}$ such that we have a morphism $\mathcal{F} \to \mathcal{F}^+$ of presheaves, and for any given morphism $\mathcal{F} \to \mathcal{G}$ of presheaves, where $\mathcal{G}$ is a sheaf, there is a unique morphism $\mathcal{F}^+ \to G$ making the resulting diagram commutative.

The sheafification is unique up to unique morphism.

Definition 5.1.13. An $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi-coherent* if locally it is isomorphic to a cokernel $\mathcal{O}^{n_1}|_U \to \mathcal{O}^{n_2}|_U$ (where n_1 and n_2 are allowed to be ∞, even uncountably), and $\mathcal{F}$ is called *coherent* if n_1 and n_2 are finite.

Proposition 5.1.14. *The categories of coherent and quasi-coherent sheaves are abelian.*

If $\mathcal{T}_1 \xrightarrow{f} \mathcal{T}_2$ is a morphism of topological spaces, if $\mathcal{F}$ is a sheaf on $\mathcal{T}_1$, and if $\mathcal{G}$ is a sheaf on $\mathcal{T}_2$, then we define the *direct image functor* f_* and the *pushforward sheaf* $f_*(\mathcal{F})$ on $\mathcal{T}_2$ by

$$f_*(\mathcal{F})(U) := \mathcal{F}(f^{-1}(U))$$

and the *inverse image sheaf* $f^{-1}(\mathcal{G})$ on $\mathcal{T}_1$ to be the sheafification of the presheaf defined by

$$f^{-1}(\mathcal{G})(U) = \lim_{V \supseteq f(U)} \mathcal{G}(V).$$

Now, if $X \xrightarrow{f} Y$ is a morphism between holomorphic manifolds, $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$- and $\mathcal{O}_Y$-modules repsectively, then $f_*(\mathcal{F})$ is an $\mathcal{O}_Y$-module, but $f^{-1}(\mathcal{G})$ is not necessarily an $\mathcal{O}_X$-module. Insteaf we define the following.

Definition 5.1.15. Define the *pullback module* by

$$f^*(\mathcal{G}) := f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

Exercise 5.1.16. If $\mathcal{G}$ is locally free, then f^* corresponds to pull-back of vector bundles.

Exercise 5.1.17. The functors f_* and f^* are adjoint, i.e.

$$\operatorname{Hom}(f_*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}(\mathcal{F}, f^*\mathcal{G}),$$

where Hom denotes the group of sheaf morphisms. Note that we also use $\mathcal{H}\mathrm{om}$ (or $\underline{\mathrm{Hom}}$ in [Uen08]) which is the sheaf such that

$$\mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}(\mathcal{F}(U), \mathcal{G}(U)).$$

Claim 5.1.18. *The category of $\mathcal{O}_X$-modules has enough injectives for the following to work.*

Suppose that we have a functor F from the category of $\mathcal{O}_X$-modules to some abelian category that is left-exact but not necessarily exact, i.e. if we have a short exact sequence of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

applying the functor, we obtain a sequence

$$0 \to F(\mathcal{F}_1) \to F(\mathcal{F}_2) \to F(\mathcal{F}_3).$$

There exists a sequence of functors from $\mathcal{O}_X$-modules to the same category as F, called (higher) *derived functors* denoted R^iF, $i \in \mathbb{N}$, that measures the failure of exactness, i.e. $R^0F = F$ and given a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

we get a long exact sequence

$$0 \to F(\mathcal{F}_1) \to F(\mathcal{F}_2) \to F(\mathcal{F}_3) \to R^1F(\mathcal{F}_1) \to R^1F(\mathcal{F}_2) \to R^1F(\mathcal{F}_3) \to R^2F(\mathcal{F}_1) \to \dots.$$

The claim about injectives goes into the machinery of constructing R^iF.

Example 5.1.19. If $f : X \to Y$ is a morphism and F_*, then $R^i f_*(\mathcal{F})$ are called the *higher direct images.*

Example 5.1.20. If $f : X \to \{\text{pt}\}$ is a morphism, the category of $\mathcal{O}_{\{\text{pt}\}}$ is the category of $\mathbb{C}$-vector spaces. Then f_* is the functor of global sections, and $R^i f_*(\mathcal{F}) =: H^i(\mathcal{F})$ is the i'th *sheaf cohomology* (which can also be defined as Čech cohomology). This way we can think of $R^i f_*$ for general $f : X \to Y$ as "relative sheaf cohomology".

13th lecture, March 16th 2012

Definition 5.1.21. We define $\text{Ext}^i(\mathcal{F}, \mathcal{G})$ to be the higher derived functor R^i of $\text{Hom}(\mathcal{F}, -)$ applied to $\mathcal{G}$.

Exercise 5.1.22. We have $\text{Ext}^i(\mathcal{O}_x, \mathcal{G}) \cong H^i(\mathcal{G})$.

Theorem 5.1.23 (Serre duality)**.** *Let X be a projective variety, possibly singular but Cohen–Macaulay and equidimensional (i.e. the dimensions of reduced components have the same dimension), then*

$$\text{Ext}^i(\mathcal{F}, \omega_X) \stackrel{\cong}{\to} H^{n-i}(X, \mathcal{F})^*.$$

For a variety X, define the K-group $K(X)$ of X to be the free abelian group generated by coherent sheaves modulo $\mathcal{F}_1 + \mathcal{F}_3 = \mathcal{F}_2$ if $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$. Now, coherent sheaves have Chern classes just like vector bundles do, since locally (globally even?) coherent sheaves fit into short exact sequences of vector bundles. Let $\text{ch} : K(X) \to H^*(X, \mathbb{Z})$ denote the Chern character, which is a particular combination of Chern classes. The K-group $K(X)$ is a ring with multiplication induced by $\otimes_{\mathcal{O}(X)}$ and $\text{ch} : K(X) \to H^*(X, \mathbb{Z})$ is a ring morphism.

Theorem 5.1.24 (Grothendieck–Riemann–Roch)**.** *Let $f : X \to Y$ be a projective morphism between $\mathbb{C}$-varieties. Then we can define a map $K(X) \to K(Y)$ which maps a single sheaf $\mathcal{F}$ to the (finite) sum $\sum_{i=0}^{\infty}(-1)R^i f_* \mathcal{F} \in K(Y)$. Then the square*

$$\begin{array}{ccc} K(X) & \xrightarrow{\text{ch}} & H^*(X, \mathbb{Q}) \\ \downarrow & & \downarrow f_* \\ K(Y) & \xrightarrow{\text{ch}} & H^*(Y, \mathbb{Q}) \end{array}$$

does not commute, but

$$\text{ch}(\sum_i (-1)^i R^i f_*(\mathcal{F})) = f_*(\text{ch}(\mathcal{F})\text{td}(\mathcal{T}_f)),$$

where $\text{td}(\mathcal{T}_f)$ is the Todd class of the "relative tangent sheaf". If X and Y are both smooth, then $\mathcal{T}_f = TX - f^*TY$.

The usual (Hirzebruch–)Riemann–Roch theorem is the special case where Y is a point.

5.2 Construction of the sheaf of conformal blocks

Consider a family $\mathcal{F} = (C \xrightarrow{\pi} B, s_1, \ldots, s_N, \eta_1, \ldots, \eta_N)$, such that s_i are sections of π, and such that $(\pi^{-1}(P), s_1(P), \ldots, s_N(P))$ is a stable curve for all $p \in B$ (see Figure 1.7 of [Uen08]) with formal neighbourhoods (i.e. for all $j = 1, \ldots, N$, take the completion of the sheaf $\mathcal{O}_C$ localized at the image s_j and choose an isomorphism of π_* of this with $\mathcal{O}_B((\xi_j))$). Here, B is some manifold, and C a $\mathbb{C}$-analytic space.

We want to have coherent locally free sheaf $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ on B such that $\tilde{B} \xrightarrow{f} B$ is a morphism, then $\mathcal{V}^\dagger_{\vec{\lambda}}(f^*\mathcal{F}) \cong f^*\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$. E.g. if $\tilde{B} = \{\text{pt}\} \hookrightarrow B$, we recover everything we have done so far.

Define a quasi-coherent sheaf of Lie algebras $\hat{\mathfrak{g}}_N$ on B by

$$\hat{\mathfrak{g}}_N(B) := \mathfrak{g} \otimes_{\mathbb{C}} \left(\bigoplus_{j=1}^{N} \mathcal{O}_B((\xi_j)) \right) \oplus \mathcal{O}_B \cdot c$$

with bracket

$$[(X_1 \otimes f_1, \dots, X_N \otimes f_N), (Y_1 \otimes g_1, \dots, Y_N \otimes g_N)]$$
$$= ([x_1, y_1] \otimes f_1 g_1, \dots) \otimes \bigoplus_{j=1}^{N} \sum (X_i, Y_i) \operatorname{Res}_{\xi_j = 0} g_j \, df_j.$$

Similarly define a sheaf on B by $\hat{\mathfrak{g}}(\mathcal{F}) \subseteq \hat{g}_N(B)$ by $\mathfrak{g} \otimes_{\mathbb{C}} \pi_*(\mathcal{O}_C(*S))$, where S is a divisor on C that is the sum of the images of the sections s_i, and $\mathcal{O}_C(*S) = \lim_{n \to \infty}(\mathcal{O}(nS))$.

Remark 5.2.1. This is a remark on the relationship between divisors and line bundles. Let $s : M \to \mathcal{L}$ be a meromorphic section of a holomorphic line bundle $\mathcal{L}$ over a holomorphic manifold M. Let s have zeroes and poles of order n_i at codimension 1 irreducible analytic subspaces D_i. From this we get the divisor $D = \sum n_i D_i$.

Proposition 5.2.2. *Given $D = \sum n_i D_i$, this determines a line bundle $\mathcal{O}(D) \to M$, unique up to isomorphism, with a meromorphic section $s_D : M \to \mathcal{O}(D)$, unique up to scalar multiplies, such that the zeroes and poles of s_D are given by D.*

In particular, let Σ be a Riemann surface, and let $p \in \Sigma$. Then $\mathcal{O}(p)$ is some line bundle. So far, we have been thinking of $H^0(\mathcal{O}(*\sum p_i))$ as meromorphic functions with poles at the p_i, but we can think of the space as the space of holomorphic sections of the line bundle. The explination for this is that if $D = \sum n_i D_i$ is effective (i.e. all n_i are positive), let us choose s_D a holomorphic section of $\mathcal{O}(D)$. Then we get a map of sheaves

$$\mathcal{O}_M \overset{\cdot s_D}{\to} \mathcal{O}(D)$$

where $\mathcal{O}$ is the structure sheaf of M.

Claim 5.2.3. *Multiplying by s_D gives an isomorphism between the sheaf of meromorphic functions with poles only on D_i of order at most n_i with $\mathcal{O}(D)$.*

Let D effective be given, choose s_D and consider

$$\mathcal{O} \overset{\cdot s_D}{\to} \mathcal{O}(D) \overset{\cdot s_D}{\to} \mathcal{O}(2 \cdot D) \to \cdots .$$

Then the direct limit $\mathcal{O}(*D) = \lim_{n \to \infty} \mathcal{O}(nD)$ is well-defined as a quasi-coherent sheaf and is isomorphic to the sheaf of meromorphic functions with poles (of arbitrary order) only on D_i.

14th lecture, March 19th 2012

Before we return to our discussion of the sheaf of conformal blocks, we add a few extra things to our discussion from last time. We stated last time that given a divisor D on a holomorphic manifold, there is a line bundle $\mathcal{O}(D)$ associated to it, unique up to isomorphism, and we get a meromorphic section of that bundle, which is in general unique up to nowhere zero holomorphic functions (i.e. up to a constant scalar if M is compact). The transitions functions for $\mathcal{O}(D)$ are described as follows: If $D = \sum n_i D_i$, where each D_i is an irreducible hypersurface, cover M by U_α such that on U_α, eaech D_i is the zero locus of $h_{\alpha i}$ and each $h_{\alpha i}$ has zero of first order along D_i. Put $f_\alpha = \prod_i h_{\alpha i}^{n_i}$ on U_α and $g_{\alpha\beta} = f_\alpha / f_\beta$ on $U_\alpha \cap U_\beta$. It is easy to check that $g_{\alpha\beta}$ are the transition functions of a line bundle; the one we denote $\mathcal{O}(D)$. Furthermore, f_α is the section s_D in the local trivialization with respect to these transition functions.

Consider as last time $\mathcal{F} = (C \xrightarrow{\pi} B, s_1, \ldots, s_N, \eta_1, \ldots, \eta_N)$.

Definition 5.2.4 (Sheaf of conformal blocks)**.** Define $\mathcal{H}_{\vec{\lambda}}(B) = \mathcal{O}_B \otimes_{\mathbb{C}} \mathcal{H}_{\vec{\lambda}}$ and

$$\mathcal{H}^\dagger_{\vec{\lambda}}(B) = \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\mathcal{H}_{\vec{\lambda}}(B), \mathcal{O}_B) \cong \mathcal{O}_B \otimes_{\mathbb{C}} \mathcal{H}^\dagger_{\vec{\lambda}},$$

where the isomorphism is an exercise. There is a pairing $\langle\, , \rangle$between $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}^\dagger_{\vec{\lambda}}$ (i.e. a map $\mathcal{H}_{\vec{\lambda}} \times \mathcal{H}^\dagger_{\vec{\lambda}} \xrightarrow{\langle , \rangle} \mathcal{O}_B$). Now $\hat{\mathfrak{g}}_N(B)$ acts on $\mathcal{H}_{\vec{\lambda}}$ and $\mathcal{H}^\dagger_{\vec{\lambda}}$ in a natural way, and we considered the Lie subalgebra $\mathfrak{g}(\mathcal{F})$ of $\hat{\mathfrak{g}}_N(B)$ (depending on $\mathcal{F}$ and not just on B as everything else has).

The *sheaf of covacua* is

$$\mathcal{V}_{\vec{\lambda}}(\mathcal{F}) = \mathcal{H}_{\vec{\lambda}}(B)/\hat{\mathfrak{g}}(\mathcal{F})\mathcal{H}_{\vec{\lambda}},$$

and the *sheaf of conformal blocks* or *sheaf of vacua* is

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}) = \underline{\mathrm{Hom}}_{\mathcal{O}_B}(\mathcal{V}_{\vec{\lambda}}, \mathcal{O}_B),$$

which is isomorphic to the subsheaf of $\mathcal{H}^\dagger_{\vec{\lambda}}(B)$ that is trivial for the action of $\hat{\mathfrak{g}}(\mathcal{F})$ (exercise).

The only statement that is not a direct translation is the following Proposition. Note that given the family $\mathcal{F}$ over B and given a morphism $f : Y \to B$, we obtain a pullback family $f^*\mathcal{F}$ (over the base Y).

Proposition 5.2.5. *There are canonical isomorphisms*

$$\begin{aligned} f^*\mathcal{H}_{\vec{\lambda}}(B) &\cong \mathcal{H}_{\vec{\lambda}}(Y), \\ f^*\hat{\mathfrak{g}}_N(B) &\cong \hat{\mathfrak{g}}_N(Y), \\ f^*\hat{\mathfrak{g}}(\mathcal{F}) &\cong \hat{g}(f^*\mathcal{F}), \\ f^*\mathcal{V}_{\vec{\lambda}}(\mathcal{F}) &\cong \mathcal{V}_{\vec{\lambda}}(f^*\mathcal{F}), \\ f^*\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}) &\cong \mathcal{V}^\dagger_{\vec{\lambda}}(f^*\mathcal{F}). \end{aligned}$$

The proof is a direct verification.

Our aim is now to prove that $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ is a locally free coherent sheaf. Before we do so, we need some more tools.

5.3 Deformations of stables pointed curves

We restrict ourselves to the language of holomorphic manifolds and later specify to the case of stable marked curves with formal neighbourhoods.

Let C_0 be a holomorphic manifold. A *deformation* of C_0 is a holomorphic submersion $C_0 \xrightarrow{\pi} W$ with a point $0 \in W$ together with an isomorphism $f : \pi^{-1}(0) \to C$ (topologically, $C \to W$ is locally trivial). Given a deformation of C_0, we can find a small open neighbourhood $0 \in U \subseteq W$ such that on $\pi^{-1}(U)$, we can take holomorphic coordinate charts, where each is holomorphic to $U \times U_\alpha$ for open subsets U_α of C_0.

Pick coordinates (z_α, w), where $\mathcal{Z}$ are coordinates on U_α and $w = (w_k)_k$ are coordinates on Y. Transition functions look like

$$(z_\alpha, w) = (g_{\alpha\beta}(z_\beta, w)z_\beta, w),$$

the transition function $g_{\alpha\beta}$ depending on w. Consider

$$\theta^{(k)}_{\alpha\beta} := \left(\frac{\partial g_{\alpha\beta}(z_\beta, w)}{\partial w_k} \frac{\partial}{\partial z_\beta} \right).$$

The following is an easy exercise.

Claim 5.3.1. *The $\theta^{(k)}_{\alpha\beta}$ are* 1*-cocycles for the tangent sheaf* Θ *(i.e. the sheaf of sections of the tangent bundle) of* C_0 *and thus determine elements in* $\check{H}^1(C_0, \Theta)$

Definition 5.3.2 (Kodaira–Spencer map)**.** The map

$$T_0W \to H^1(C_0, \Theta)$$
$$\sum a_k \frac{\partial}{\partial w_k} \mapsto [\sum a_k \theta^{(k)}_{\alpha\beta}]$$

is called the *Kodaira–Spencer map.*

The question of deformation theory is when this map is an isomorphism in which case we see that C_0 itself knows in how many ways it can be deformed.

There is also a global version of this: Given $C \overset{\pi}{\to} B$ as before, we have a short exact sequence of sheaves

$$0 \to \Theta_{C/W} \to \Theta_C \to \pi^*\Theta_W \to 0,$$

where $\Theta_{C/W}$ is the *relative tangent sheaf*, i.e. sheaf of sections of Θ_C that are tangent to the fiber. Pushing this forward, we get a long exact sequence

$$\pi * (\Theta_{C/W}) \to \pi_*(\Theta_C) \to \Theta_W \to R^1\pi_*(\Theta_{C/W}) \to R^1\pi_*(\Theta_C),$$

and the map $\Theta_W \to R^1\pi_*(\Theta_{C/W})$ is the *global Kodaira–Spencer map.*

Definition 5.3.3. A deformation $\pi : C \to W$ of C_0 is called a *complete family at* 0 if locally any other deformation is a pullback of it. That is if $\tilde{\pi} : \tilde{C} \to \tilde{W}$, $\tilde{0} \in W$ is another deformation, then there is a neighbourhood $\tilde{0} \in U \subseteq \tilde{W}$ map $g : U \to W$, such that $g(\tilde{0}) = 0$, $\tilde{C}|_U \cong g^*C$ and such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{\pi}^{-1}(\tilde{0}) & \xrightarrow{\quad g \quad} & \pi^{-1}(0) \\ & {}_{\tilde{f}}\searrow \quad \swarrow_{f} & \\ & C_0 & \end{array}$$

Definition 5.3.4. A complete family is called *versal at* 0 if $dg|_{\tilde{0}}$ is unique. If the germ of g is unique, then the family is called *universal at 0.* It is called a , if it is universal at every point (viewing it as a deformation of every fiber, choosing for both $\tilde{f}$ and f the identity).

Remark 5.3.5. The group $\mathrm{Aut}(C_0)$ will be an obstruction of a versal family at 0 to be universal at 0.

Theorem 5.3.6. *If a family of compact Riemann surfaces of genus at least* 2 *is complete at every point of a neighbourhood of* 0 *and is versal at* 0*, then it is universal at* 0*.*

Theorem 5.3.7. *If the Kodaira–Spencer map of a family is surjective, then the family is complete at* 0*.*

Theorem 5.3.8. *A deformation is versal at* 0 *if and only if it is complete near* 0 *and the Kodaira–Spencer map is an isomorphism at* 0*.*

15th lecture, March 21st 2012

The outline for today is the following:

- We talk a bit about versal families for stable curves.
- The sheaf of conformal blocks is coherent.

- We introduce a particular family of differential operators on the sheaf of conformal blocks.

Before going into this, we note the following theorem.

Theorem 5.3.9 (Kuranishi)**.** *If M is a compact holomorphic manifold, then there exists a versal family for M.*

The reason we are interested in versal families roughly is that a versal family of stable curves will give orbifold-charts $W/\mathrm{Aut}(C_0)$ for the Deligne–Mumford compactification $\overline{\mathcal{M}_{g,n}}$ around $[C_0]$ (see [**?**] for details).

Now, we want to allow families of stables pointed curves; Ueno argues that formal neighbourhoods play no role in the definition (which might not be clear).

Definition 5.3.10. A *family of stable N-pointed curves* is $(C \xrightarrow{\pi} B, s_1, \ldots, s_N)$ such that for all $p \in B$, $(\pi^{-1}(p), s_1(p), \ldots, s_N(p))$ is a stable curve for all p. Here C and B are complex analytic spaces, and π is flat, which in the smooth case means that it is topologically locally trivial. In the case as ours where we are dealing with families of projective varieties, flatness means the following: If $\pi : C \to B$ is a family of projective varieties, it is called *flat* if the Hilbert series of $\pi^{-1}(p)$ is constant on B. The Hilbert series can be interpreted, by the index theorem, as something completely topological; for families of curves, it translates to the arithmetic genus being constant.

We need to adapt the Kodaira–Spencer map to this case: If B is a manifold, we have a map $T_pB \to \mathrm{Ext}'(\Omega'_{\pi^{-1}(p)}, \mathcal{O}_{\pi^{-1}(p)}(-\sum_{j=1}^N s_j(p)))$. Again we have a global version $\rho : \Theta_B \to R^1\pi_*\underline{\mathrm{Hom}}(\Omega^1_{C/B}, \Theta_B(-\sum_{j=1}^N s_j(B)))$. The notions of completeness, versality and universality are the same. A non-trivial result from deformation theory is the following:

Theorem 5.3.11. *Any stable pointed curve has a versal deformation. If $\mathrm{Aut}(C, Q_1, \ldots, Q_N)$ is trivial, we have a universal deformation.*

Proposition 5.3.12. *For this versal deformaiton, C and B are smooth, and the dimension of the base is*

$$\dim B = 3g - 3 + N = \dim \overline{\mathcal{M}_{g,n}}.$$

The total space has dimension

$$\dim C = 3g - 2 + N.$$

Define the *critical locus*

$$\Sigma = \{P \in C \mid d\pi_p : T_pC \to T_{\pi(p)}B \text{ is not surjective}\}.$$

We write $D = \pi(\Sigma)$. In fact, Σ is a smooth submanifold of codimension 2 and D is an (effective reduced) divisor with *normal crossings* in B: That is, locally it looks like the union of coordinate hyperplanes.

Definition 5.3.13. Suppose that (X, D) is a smooth manifold with a divisor with normal crossings. Then we can define a the *sheaf of logarithmic vector fields* $\Theta_X(-\log D)$, which can be defined algebraically as the sheaf of derivations of $\mathcal{O}_X$ that preserve the ideal sheaf $\mathcal{I}_D$ of D, or as the sheaf of vector fields tangent to D. We have $\Theta_X(-\log D)|_{X\setminus D} \cong \Theta_{X\setminus D}$. Similarly, we have the *sheaf of logarithmic differentials* $\Omega_X(\log D)$, which is the sheaf of differential forms with poles of order 1 along D.

Proposition 5.3.14. *Let $(C \xrightarrow{\pi} B, s_1, \ldots, s_N)$ be a versal family of stable curves. Then there is an isomorphism of sheaves*

$$\rho : \Theta_B(-\log D) \to R^1\pi_*(\Theta_{C/B}(-s)),$$

where $s = \sum_{i=1}^N s_i(B)$, which is also (somewhat confusingly) referred to as the Kodaira–Spencer map.

We now want to show that the sheaf of conformal blocks is coherent. Ueno shows that we have a surjection, locally, $\mathcal{O}_B^M \twoheadrightarrow \mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}) \to 0$ for some $M < \infty$ (which is not enough to show that it is coherent). To prove it, we need the following algebraic lemma.

Lemma 5.3.15. *Let $\mathcal{A}$ be a Lie algebra over k, $\mathcal{H}$ an $\mathcal{A}$-module of finite type, i.e. $\mathcal{H} = U(\mathcal{A}) \cdot V$ for some finite-dimensional k-vector space V. Suppose there is a basis e_i of $\mathcal{A}$ such that the action of e_i on $\mathcal{H}$ is locally finite (i.e. one can find a basis of $\mathcal{H}$ such that the action of e_i with respect to this basis consists of finite blocks). Let $\mathcal{A}_+ = \{x \in \mathcal{A} \mid x \cdot V = 0\} = \operatorname{Ann}(V)$. If K is a Lie subalgebra of $\mathcal{A}$ such that $K + \mathcal{A}_+$ has finite codimension in $\mathcal{A}$, then $\mathcal{H}/K\mathcal{H}$ is a finitely generated module.*

Theorem 5.3.16. *$\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ and $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ are coherent $\mathcal{O}_B$-modules for all (versal?) families of stable pointed curves.*

Proof. For $\mathcal{F}$ some versal family of stable pointed curves, we apply Lemma 5.3.15 to $\mathcal{A} = \hat{\mathfrak{g}}_N(B)$, $K = \hat{\mathfrak{g}}(\mathcal{F})$, $\mathcal{H} = \mathcal{H}_{\vec{\lambda}}(B)$. We know from before that the action of $\mathcal{A}$ is locally finite, and that $(\hat{\mathfrak{g}}(F) + \hat{\mathfrak{g}}_N(B)_+)$ is of finite codimension in $\hat{\mathfrak{g}}_N(B)$. Then $\mathcal{A} = (\mathcal{A}_+ + K) \oplus \bigoplus_{j=1}^N (\mathcal{O}_B e_j)$, and we have

$$U(\mathcal{A}) = \sum_{m_1,\dots,m_N} U(K) e_1^{m_1} \cdots e_N^{m_N} U(\mathcal{A}_+).$$

Now $\mathcal{A}_+$ acts trivially on $\mathcal{H}$, so $U(\mathcal{A}_+)V = V$, and

$$\mathcal{H} = U(\mathcal{A}) \cdot V = U(K) \sum e_1^{m_1} \cdots e_N^{m_N} \cdot V,$$

and since each e_i acts locally finite, the sum is a finite dimensional space, $\tilde{L}$ say. In particular we have a surjection $\tilde{L} \to \mathcal{H}/K\mathcal{H} \to 0$, and $\mathcal{H}/K\mathcal{H} \cong \mathcal{V}_{\vec{\lambda}}(B)$. This says that $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ (and therefore also $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$) is coherent. □

As discussed before, we should also explain why the kernel of the map is finitely presented, but we leave it at this.

16th lecture, March 26th 2012

Some extra references that might be useful are the following:

- A Master's thesis called "Decomposition of conformal blocks" by Swarnava Mukhopadhyay (can be Googled).
- "Conformal blocks revisited" by Looijenga (is on arXiv).

5.4 Locally freeness of the sheaf of conformal blocks

Theorem 5.4.1. *The sheaf $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ is locally free.*

The outline is the following:

- We introduce a sheaf of differential operators, and
- we use this to show that $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F})$ is locally free if the family has only smooth curves,
- and finally show that $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ is locally free for any family of stable marked curves.

Before we can do that, we need to discuss "some exact sequences of sheaves". Ueno at this point assumes that the formal local coordinates in our versal family are actually holomorphic. Assume again that our family is written $\mathcal{F} = (C \xrightarrow{\pi} B, s_1, \dots, s_N, \eta_1, \dots, \eta_N)$. Recall that we have an exact sequence of sheaves

$$0 \to \pi_*(\Theta_{C/B}(*S)) \xrightarrow{b} \bigoplus_{j=1}^{N} \mathcal{O}_B[\xi_j^{-1}] \frac{d}{d\xi_j} \to R^1\pi_*\Theta_{C/B}(-S) \to 0,$$

coming from the short exact sequence of sheaves on C,

$$0 \to \Theta_{C/B}(-S) \to \Theta_{C/B}(*S) \to \bigoplus \pi^*\mathcal{O}_B[\xi_j^{-1}] \frac{d}{d\xi_j} \to 0.$$

We also have a short exact sequence

$$0 \to \Theta_{C/B} \to \Theta_C \xrightarrow{d\pi} \pi^*\Theta_B \to 0.$$

Define $\Theta'_{C,\pi}$ to be the sheaf of vector fields on C tangent along Σ (where Σ is the locus of singularities of the fibers of $C \xrightarrow{\pi} B$) whose horizontal component is constant along the fibers. Likewise $\Theta'_C(mS)_\pi$ is the sheaf of those having poles of order at most m along S (recall that $S \cap \Sigma = \emptyset$ and that we write $D = \pi_*\Sigma$). We have a short exact sequence of sheaves of Lie algebras

$$0 \to \Theta_{C/B}(mS) \to \Theta'_C(mS)_\pi \xrightarrow{d\pi} \pi^*\Theta_B(-\log D) \to 0.$$

We push this forward for every m and get sequences

$$0 \to \pi * \Theta_{C/B}(*S) \to \pi_*\Theta'_C(*S)_\pi \to \Theta_B(-\log D) \to 0,$$
$$0 \to \pi_*\Theta_{C/B}(*S) \to \bigoplus_{j=1}^{N} \mathcal{O}_B[\xi_j^{-1}] \frac{d}{d\xi_j} \to R^1\pi_*\Theta_{C/B}(-S) \to 0,$$

that fit into a commutative diagram. The first spaces are equal here, and bteween the third ones of each sequence, we have the Kodaira–Spencer map which is an isomorphism, since the family is versal. Then by the 5-lemma, we have an isomorphism $p : \pi_*\Theta'_C(*S)_\pi \cong \bigoplus_{j=1}^N \mathcal{O}_B[\xi_j^{-1}] \frac{d}{d\xi_j}$. Taking Laurent expansions, we also have sheaves of Lie algebras

$$\tilde{p} : \pi_*\Theta'_C(*S)_\pi \hookrightarrow \bigoplus_{j=1}^{N} \mathcal{O}_B(\!(\xi_j)\!) \frac{d}{d\xi_j} \left(\to \bigoplus \mathcal{O}_B[\xi_j^{-1}] \frac{d}{d\xi_j} \right).$$

We also have We write $\mathcal{L}(\mathcal{F}) := \tilde{p}(\pi_*\Theta'_C(*S)_\pi)$. This is a sheaf of Lie algebras on B, whose bracket can be written down explicitly: We have

$$0 \to \pi_*\Theta_{C/B}(*S) \to \mathcal{L}(\mathcal{F}) \to \xrightarrow{\theta} \Theta_B(-\log D) \to 0,$$

where θ is induced by the map $\pi_*\Theta'_C(*S)_\pi \to \Theta_B(-\log D)$.

For $\vec{l}, \vec{m} \in \mathcal{L}(\mathcal{F})(U)$, we have

$$[\vec{l}, \vec{m}]_d = [\vec{l}, \vec{m}]_0 + \theta(\vec{l})(\vec{m}) - \theta(\vec{m})(\vec{l}).$$

where $_0$ denotes the usual bracket of formal vector fields.

Definition 5.4.2. Let $\vec{l} \in \mathcal{L}(\mathcal{F})$, $\vec{l} = (l_1, \dots, l_N)$. We get an operator $D(\vec{l})$ acting on $\mathcal{H}_{\vec{\lambda}}(B)$, defined by

$$D(\vec{l})(f|\Phi\rangle) = (\theta(\vec{l})(f))|\Phi\rangle - f \cdot (\sum \rho_j(T(l_j)))|\Phi\rangle.$$

Proposition 5.4.3. *This operator satisfies*

- $D(f\vec{l}) = fD(\vec{l})$,
- $[D(\vec{l}), D(\vec{m})] = D([\vec{l}, \vec{m}]_d) + \frac{c_v}{12} \sum_{j=1}^{N} \operatorname{Res}_{\xi_j=0} \left(\frac{d^3 l_j}{d\xi_j^3} m_j d\xi_j \right) \operatorname{Id}$,
- $D(\vec{l})(f|\Phi\rangle) = (\vec{\theta}(\vec{l})(f))|\Phi\rangle + fD(\vec{l})|\Phi\rangle$, *and*
- $D(\vec{l})(\hat{\mathfrak{g}}(\mathcal{F}) \cdot \mathcal{H}_{\vec{\lambda}}(B)) \subseteq \hat{\mathfrak{g}}(\mathcal{F})\mathcal{H}_{\vec{\lambda}}(B)$ *(and as a result, $D(\vec{l})$ acts on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$).*

18th lecture, April 2nd 2012

What we want to do today is talk a little bit about local freeness and sketch the argument of Ueno's book. Recall that we had a sheaf $\mathcal{L}(\mathcal{F})$ and a short exact sequence of sheaves of Lie algebras (modulo the discussions from the 16th lecture)

$$0 \to \pi_* \Theta_{C/B}(*S) \to \mathcal{L}(\mathcal{F}) \xrightarrow{\theta} \Theta_B(-\log D) \to 0.$$

We get a differential operator $D(\vec{l})$, acting on $\mathcal{H}_{\vec{\lambda}}(B), \mathcal{H}^{\dagger}_{\vec{\lambda}}(B), \mathcal{V}_{\vec{\lambda}}(\mathcal{F})$, and $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F})$, for $\vec{l} = (l_1, \dots, l_n)$, a section of $\mathcal{L}(\mathcal{F})$.

Now, we want to show that $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F})$ is locally free.

Step 1. This is true if $D = \emptyset$, i.e. if $\mathcal{F} = (C \xrightarrow{\pi} B)$ is a family of smooth curves.

Proof. The idea is to choose a basis in some minimal way and use minimality to show that it satisfies no relationships.

Choose local holomorphic sections of $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ in a neighbourhood U near x, $v_1, \dots, v_m$, such that at $x \in B$, these form a basis of the conformal block; this is always possible since the sheaf is coherent. Here, we view $v_m(x)$ as $v_m \otimes 1$ in $\mathcal{V}_{\vec{\lambda}} \otimes_{\mathcal{O}(U)} \mathbb{C}_x$. Assume that (*) $a_1 v_1 + \cdots + a_m v_m = 0$ for a_i local holomorphic functions. Then $a_i(x) = 0$ for all i, so $a_i \in \mathcal{M}_x$, the maximal ideal of the local ring at x. Change order such that $a_1 \in \mathcal{M}_x^k \setminus \mathcal{M}_x^{k+1}$, $a_i \in \mathcal{M}_x^l$, $l \geq k$ when $i = 2, \dots$ and choose the basis such that k is minimal. Let τ be a nonvanishing local vector field $\tau \in \mathcal{M}_x^{k-1} \setminus \mathcal{M}_x^k$. Choose $\vec{l}$ such that $\theta(\vec{l}) = \tau$. Apply $D(\vec{l})$ to (*) and get a new relation

$$D(\vec{l}) \left(\sum_{i=1}^{m} a_i v_i \right) = \sum_{j=1}^{m} \left(\tau(a_i) + \sum_j a_j \alpha_{ji} \right) v_i = 0.$$

But the coefficient of v_1 is $\tau(a_1) + \sum_{j=1}^{m} a_j \alpha_{j1} \in \mathcal{M}_x^{k-1}$. This contradicts minimality, so the v_i generate $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})(U)$. □

Step 2. The sheaf is locally free for general versal families $\mathcal{F}$.

Proof. Recall that if $\mathcal{F} = (C \xrightarrow{\pi} B, \dots)$ is a versal family, then D is a normal crossing divisor – recall that a normal crossing divisor in a smooth variety/complex manifold is $D = \sum D_i$, where the D_i are smooth such that the D_i intersect (locally in the analytic topology) as coordinate hyperplanes. In particular $D = \sum D_i$, where the D_i are smooth.

We introduce a bit of notation: If $\mathcal{F}$ is a versal family of stable nodal curves and we have a morphism $Y \to B$, we denote the pull-back of $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ by $\mathcal{V}_{\vec{\lambda}}(\mathcal{F}_Y)$. We need the concept of *sewing*: Let $D = \sum_i D_i \subseteq B$ be the normal crossing divisor. Look at $\tilde{\pi}_{D_i} : C_{D_i} \to D_i$ be $\mathcal{F}$ restricted to D_i. This family has a "global node" – one particular node which is in every fiber. We "normalize" this node to get $\tilde{\pi} : \tilde{C}_{D_i} \to D_i$, a family $\tilde{\mathcal{F}}_{D_i}$ of stable marked curves with $N+2$ marked points (where N is the number of marked points of the original family over D_i); this makes sense since normalization is a local procedure. Note that the family is not necessarily connected.

Choose now $\mu, \mu^\dagger$ in the Weyl alcove. Then $\mathcal{V}_{\mu,\mu^\dagger,\vec{\lambda}}(\tilde{\mathcal{F}}_{D_i})$ is locally free over $D_i \setminus (\bigcup_{j \neq i} D_j)$.

A main technical tool is the following: Given $\langle\Psi|$, a section of $\mathcal{V}^\dagger_{\mu,\mu^\dagger,\vec{\lambda}}(\tilde{\mathcal{F}}_{D_i})$, define $\langle\Psi_d| \in \mathcal{H}^\dagger_{\vec{\lambda}}(D_i)[\![q]\!]$ by

$$\langle\Psi_d|\Phi\rangle = \sum_{i=1}^{m_d} \langle\Psi|v_i(d) \otimes v^i(d) \otimes \Phi\rangle,$$

where v_i is the dual basis of $\mathcal{H}_\mu(d)$ for "some" pairing.

Assume that D has a single component. Then $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ is locally free and

$$\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}) \otimes \mathcal{O}_B\mathcal{O}_Y \cong \mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}_Y).$$

We proceed by using the following results.

Lemma 5.4.4. *In general, if $\mathcal{V}^\dagger_{\vec{\lambda}}(\mathcal{F}D_i)$ is locally free, then $\mathcal{F}^\dagger_{\vec{\lambda}}(\mathcal{F})$ is locally free.*

Lemma 5.4.5. *This is the case.*

□

17th lecture, March 28th 2012

5.5 Projectively flat connections

Our aim today is to understand Chapter 5.1 of [Uen08]. References are J. D. Fay, "*Theta functions on Riemann surfaces*" and A. N. Tyurion "*On Periods of quadratic differentials.*

Assume that X is a compact Riemann surface. Consider the two projections $p_i : X \times X \to X$ and the diagonal $\Delta_X \subseteq X \times X$. Let ω_X be the canonical bundle. We have an invertible sheaf $p_1^*\omega_X \otimes p_2^*\omega_X$. Consider a symmetric bidifferential (of the second kind) $\omega \in H^0(X \times X, (p_1^*\omega_X \otimes p_2^*\omega_X)(2\Delta_X))^{\mathbb{Z}_2}$, where $\mathbb{Z}_2$ acts by $(x, y) \mapsto (y, x)$. Let x be a coordinate on $U \subseteq X$, and let y denote the corresponding coordinate on the second factor, so (x, y) is a coordinate on $U \times U \subseteq X \times X$. Then one can show that ω can be written as

$$\omega = \omega(x, y)\, dx\, dy = \left(\frac{\alpha}{(x-y)^2} + H(x, y)\right) dx\, dy,$$

where H is holomorphic. Here, α is called the *biresidue* of ω, $\operatorname{Res}^2 \omega = \alpha$.

Assume from now on that $\operatorname{Res}^2 \omega = 1$ and let

$$S_\omega(z)\, d^2z := 6 \lim_{w \to z} \left(\omega(w, z)\, dw\, dz - \frac{dw\, dz}{(w-z)^2}\right).$$

We refer to S_ω as a *projective connection.*

Theorem 5.5.1 (Tyurion)**.** *Under change of coordinates this transforms like*

$$S_\omega(z)(dz)^2 = S_\omega(w)(dw)^2 - \{z; w\}(dw^2).$$

As an aside, note that if $z = z(w)$, one can show that

$$\left\{\frac{\alpha z + \beta}{\gamma z + \delta}; z\right\} = \{w; z\},$$

and that $\{z; w\} = 0$ if and only if $z = (\alpha w + \beta)/(\gamma w + \delta)$. We also have

$$\{w; z\}(d^2z) = \{z; w\}(dw)^2,$$
$$\{z; t\}(dt)^2 = \{w; t\}(dt)^2 + \{z; w\}(dw)^2.$$

Theorem 5.5.2 (Fay). *There always exists a symmetric bidifferential ω with $\operatorname{Res}^2 \omega = 1$.*

Remark 5.5.3. This ω seems to be related to the Bergman kernel.

Actually, we need existence of ω in versal families. Let $\mathcal{F} = (\mathcal{C} \xrightarrow{\pi} \mathcal{B}, s_1, \ldots, s_N, \eta_1, \ldots, \eta_N)$ be a versal family. Let S be the degeneracy locus, and $D = \pi(S)$.

Remark 5.5.4. There exists a symmetric bidifferential

$$\omega \in H^0(\mathcal{C} \times_{\mathcal{B}} \mathcal{C}, \omega_{\mathcal{C} \times_{\mathcal{B}} \mathcal{C}/\mathcal{B}}(2\Delta))$$

with $\operatorname{Res}^2 \omega = 1$ (if necessary we have to make $\mathcal{B}$ a little bit smaller).

Recall that we have an exact sequence of Lie algebras over $\mathcal{O}_{\mathcal{B}}$,

$$0 \to \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \xrightarrow{t} \mathcal{L}(\mathcal{F}) \xrightarrow{\theta} \Theta_{\mathcal{B}}(-\log D) \to 0,$$

with a bracket $[\,,\,]_d$ on $\mathcal{L}(\mathcal{F})$. Now $\mathcal{L}(\mathcal{F})$ acts on $\mathcal{H}_{\vec{\lambda}}$. For $\vec{l} = \sum l_j(\xi_j) \frac{d}{d\xi_j}$, we have

$$D(\vec{l})(F \otimes |\varphi\rangle) = \theta(\vec{l})(F) - F \sum_{j=1}^{N} \rho_j(T(l_j))|\varphi\rangle,$$

and as mentioned last time, the action descends to give an action of $\mathcal{L}(F)$ on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ (and since they will turn out to just be vector bundles, also on $\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F})$).

Lemma 5.5.5 (Lemma 5.1 of [Uen08]). *Choose a symmetric bidifferential ω with $\operatorname{Res}^2 \omega = 1$. There exists a unique $\mathcal{O}_{\mathcal{B}}$-module morphism $a : \pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S)) \to \mathcal{O}_{\mathcal{B}}$, independent of ω such that for any $\vec{l} \in t(\pi_*(\Theta_{\mathcal{C}/\mathcal{B}}(*S))) = \ker(\theta)$, we have $D(\vec{l}) = a(\vec{l}) \operatorname{Id}_{\mathcal{V}_{\vec{\lambda}}(\mathcal{F})}$ (resp. $D(\vec{l}) = -a(\vec{l}) \operatorname{Id}_{\mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F})}$).*

Proof. Take $\langle\psi| \in \mathcal{V}^{\dagger}_{\vec{\lambda}}(\mathcal{F})$ and $|\varphi\rangle \in \mathcal{V}_{\vec{\lambda}}(\mathcal{F})$. Recall that $T[\underline{l}] = \operatorname{Res}_{z=0}(l(z)T(z)\,dz)$, where $\underline{\vec{l}} = \sum \underline{l}_j(\xi_j) \frac{d}{d\xi_j}$. Since $\theta(\vec{l}) = 0$, we have

$$\langle\psi|D(\vec{l})|\varphi\rangle = -\sum_{j=1}^{N} \langle\psi|\rho_j(T[\underline{l_j}])|\varphi\rangle = -\sum_{j=1} \operatorname{Res}_{\xi_j=0}(\underline{l}_j(\xi_j)\langle\psi|T(\xi_j)|\varphi\rangle\,d\xi_j).$$

By Theorem 3.26 of [Uen08] (which we have not proved),

$$\langle\psi|T(z)|\varphi\rangle d^2z = \frac{1}{2(g^*+l)} \lim_{w\to z} \left(\sum_a \langle\psi|J^a(z)J^a(w)|\varphi\rangle - \frac{c_V}{(z-w)^2}\langle\psi|\varphi\rangle\,dz\,dw \right).$$

Define $\tilde{T}(z)$ by

$$\langle\psi|\tilde{T}(z)|\varphi\rangle d^2z = \frac{1}{2(g^*+l)} \lim_{w\to z} \left(\sum_a \langle\psi|J^a(z)J^a(w)|\varphi\rangle\,dz\,dw - \frac{c_V}{2}\omega(z,w)\langle\psi|\varphi\rangle\,dz\,dw \right).$$

It follows that

$$\langle\psi|T(z)|\varphi\rangle(dz)^2 = \langle\psi|\tilde{T}(z)|\varphi\rangle(dz)^2 + \frac{c_V}{12}\langle\psi|\varphi\rangle S_\omega(z)(dz)^2,$$

and from this one (apparently) gets

$$\langle\psi|D(\vec{l})|\varphi\rangle = -\sum \operatorname{Res}_{\xi_j=0} \left(\underline{l}_j(\xi_j)\langle\psi|\tilde{T}(z)|\varphi\rangle\,d\xi_j\right) - \frac{c_v}{12}\langle\psi|\varphi\rangle \sum_{j=1}^{N} \operatorname{Res}_{\xi_j}(\underline{l}_j(\xi_j)S_\omega(\xi_j)\,d\xi_j).$$

The first term vanishes since $\underline{l}_j(\xi_j)\langle\psi|\tilde{T}(z)|\varphi\rangle$ is a globally defined meromorphic 1-form. We put

$$a_\omega(\vec{l})\langle\psi|\varphi\rangle = -\frac{c_v}{12}\langle\psi|\varphi\rangle\sum_{j=1}^{N}\operatorname{Res}_{\xi_j}(\underline{l}_j(\xi_j)S_\omega(\xi_j)\,d\xi_j).$$

Then

$$a_\omega(\vec{l}) = -\frac{c_v}{12}\sum_{j=1}^{N}\operatorname{Res}_{\xi_j}(\underline{l}_j(\xi_j)S_\omega(\xi_j)\,d\xi_j). \tag{4}$$

If ω' is another symmetric bidifferential with $\operatorname{Res}^2\omega' = 1$, then $\omega - \omega'$ is another symmetric bidifferential, hence $S_\omega - S_{\omega'}$ is a holomorphic section in $H^0(\mathcal{C}\times_{\mathcal{B}}\mathcal{C}, \omega_{\mathcal{C}\times_{\mathcal{B}}\mathcal{C}/\mathcal{B}})$. Note that if $\tilde{\vec{l}} \in \operatorname{im}(t)$, then

$$l_j(\xi_j)(S_\omega(\xi_j - S_\omega(\xi_j)))\,d\xi_j$$

is a meromorphic 1-form. It follows that the residues vanish, which proves the Lemma. □

Corollary 5.5.6 (Corollary 5.2 in [Uen08])**.** *By (4), we can extend the action to an (ω-dependent) morphism of $\mathcal{O}_{\mathcal{B}}$-modules, $a_\omega : \mathcal{L}(\mathcal{F}) \to \mathcal{O}_{\mathcal{B}}$.*

For each $X \in \Theta_B(-\log D)$, there exists an $\vec{l} \in \mathcal{L}(\mathcal{F})$ with $\theta(\vec{l}) = x$. Define $\nabla_X^{(\omega)} : \mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{F}) \to \mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{F})$ by

$$\nabla_X^{(\omega)}\langle\psi| = D(\vec{l})\langle\psi| + a_\omega(\vec{l})\langle\psi|.$$

The proof of the following is an exercise.

Proposition 5.5.7. $\nabla_X^{(\omega)}$ *is well-defined and satisfies the Leibniz rule*

$$\nabla_X^{(\omega)}(f\langle\psi|) = X(f)\langle\psi| + f\nabla_X^{(\omega)}\langle\psi|.$$

We refer to $\nabla_X^{(\omega)}$ as a connection with regular singularities on $\mathcal{V}_{\vec{\lambda}}^\dagger(\mathcal{F})$.

Theorem 5.5.8 (Theorem 5.5 in [Uen08])**.** *The connection $\nabla_X^{(\omega)}$ on $\mathcal{B}\setminus D$ is projectively flat (i.e. a connection whose curvature 2-form is scalar).*

Proof. We can in fact do everything without removing D, so without loss of generality, assume $D = \emptyset$. Write $X = \theta(\vec{l}), Y = \theta(\vec{m})$. The curvature tensor is $R(X,Y) = [\nabla_X^{(\omega)}, \nabla_Y^{(\omega)}] - \nabla_{[X,Y]}^{(\omega)}$. An easy calculation shows that

$$[\nabla_X^{(\omega)}, \nabla_Y^{(\omega)}] = [D(\vec{l}), D(\vec{m})] + X(a_\omega(\vec{m})) - Y(a_\omega(\vec{l})).$$

Define

$$\vec{n} := [\vec{l}, \vec{m}]_d = [\vec{l}, \vec{m}]_0 + X(\vec{m}) - Y(\vec{l}).$$

Then (perhaps with a sign issue – see Proposition 4.6 in [Uen08])

$$D(\vec{n}) = [D(\vec{l}), D(\vec{m})] - \frac{c_v}{12}\sum_{j=1}^{N}\operatorname{Res}_{\xi_j=0}\left(\frac{d^3 l_j}{d\xi_j^3}m_j\,d\xi_j\right).$$

From this it follows that

$$
\begin{aligned}
R(X,Y) =& X(a_\omega(\vec{m})) - Y(a_\omega(\vec{l})) - a_\omega(\vec{n}) + \frac{c_v}{12}\sum_{j=1}^{N} \mathrm{Res}_{\xi_j=0}\left(\frac{d^3 l_j}{d\xi_j^3} m_j \, d\xi_j\right) \\
=& \frac{c_v}{12}(\sum_{j=1}^{N} \mathrm{Res}_{\xi_j} = 0(m_j(\xi_j)l_j'(\xi_j)S_\omega(\xi_j) + m_j(\xi_j)X(S_\omega(\xi_j))d\xi_j \\
& - l_j(\xi_j)m_j'(\xi_j S_\omega(\xi_j) + l_j(\xi_j)X(S_\omega(\xi_j))d\xi_j)) + \sum_{j=1}^{N} \mathrm{Res}_{\xi_j=0}\left(\frac{d^3 l_j}{d\xi_j^3} m_j d\xi_j\right)).
\end{aligned}
$$

(Maybe this contradicts the similar expression in [Uen08, Thm. 5.6]) □

The dependence of $\nabla_X^{(\omega)}$ on the choice of ω is given by

$$\nabla_X^{(\omega)} - \nabla_X^{(\omega')} = -\frac{c_v}{12}\langle \varphi_{\omega,\omega'}, X\rangle$$

for some holomorphic 1-form $\varphi_{\omega,\omega'}$ (exercise).

19th lecture, April 16th 2012

Previously, we introduced certain integrable connections with regular singularities along the discriminant locus on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ and $\mathcal{V}_{\vec{\lambda}}^{\dagger}(\mathcal{F})$. For that, we needed a bidifferential ω satisfying

$$\langle\psi|\tilde{T}(z)|\varphi\rangle(dz)^2 = \lim_{w\to z}\left(\frac{1}{2(l+g^*)}\sum_{d=1}^{\dim\mathfrak{g}}\langle\psi|J^a(w)J^a(z)|\varphi\rangle\, dw\, z - \frac{c_v}{2}\omega(w,z)\langle\psi|\varphi\rangle\, dw\, dz\right).$$

Today we discuss the dependence of $\nabla^{(\omega)}$ on ω and define the sheaf of twisted differential operators with first order part equal to $\nabla^{(\omega)}$.

Let M is a complex manifold and $D = \bigcup_{i=1}^k D_i$ a divisor with normal crossings.

Definition 5.5.9. The *sheaf of twisted differential operators* $\mathcal{D}_\xi(-\log D)$ *with regular singularities along* D is anything fitting into a short exact sequence of abelian algebras

$$0 \to \mathcal{O}_B \to \mathcal{D}_\xi(-\log D) \to \Theta_M(-\log D) \to 0.$$

The ξ will parametrize such objects. Locally, $\mathcal{D}_\xi(-\log D) \cong \mathcal{D}(-\log D)$, the sheaf of first order holomorphic differential operators with regular singularities along D. Then for a fine enough open covering $\{U_\alpha\}_{\alpha\in A}$ of M, we have local trivializations

$$\varphi_\alpha : \mathcal{D}_\xi(-\log D)|_{U_\alpha} \cong \Theta_{U_\alpha}(-\log D) \oplus \mathcal{O}_{U_\alpha}$$

with transition functions given by

$$\varphi_\alpha \circ \varphi_\beta^{-1}((X,f)) = (X, f + \varphi_{\alpha\beta}(X))$$

for $U_\alpha \cap U_\beta = U_{\alpha\beta} \neq \emptyset$, and

$$\varphi_{\alpha\beta} \in \mathrm{Hom}_{\mathcal{O}_{U_{\alpha\beta}}}(\Theta_{U_{\alpha\beta}}(-\log D), \mathcal{O}_{U_{\alpha\beta}}).$$

Since

$$H^0(U_{\alpha\beta}, \underline{\mathrm{Hom}}_{\mathcal{O}_\mathcal{B}}(\Theta_M(-\log D), \mathcal{O}_\mathcal{B})) \cong H^0(U_{\alpha\beta}, \Omega_M^1(\log D)),$$

we see that $\varphi_{\alpha\beta}$ is a meromorphic 1-form on M with regular singularities along $D \cap U_{\alpha\beta}$. We claim that $\varphi_{\alpha\beta}$ is d-closed: We have

$$\begin{aligned}\varphi_\alpha \circ \varphi_\beta^{-1}([(X,f),(Y,g)]) &= \varphi_\alpha \circ \varphi_\beta^{-1}([X,Y], X(g) - Y(f)) \\ &= ([X,Y], X(g) - Y(f) + \varphi_{\alpha\beta}([X,Y])).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi_\alpha \circ \varphi_\beta^{-1}([(X,f),(Y,g)]) &= [\varphi_\alpha \circ \varphi_\beta^{-1}(X,f), \varphi_\alpha \circ \varphi_\beta^{-1}(Y,g)] \\ &= [(X, f + \varphi_{\alpha\beta}(X)), (Y, g + \varphi_{\alpha\beta}(Y))] \\ &= ([X,Y], X(g + \varphi_{\alpha\beta}(Y)) - Y(f + \varphi_{\alpha\beta}(X))),\end{aligned}$$

and it follows that

$$d\varphi_{\alpha\beta}(X,Y) = X(\varphi_{\alpha\beta}(Y)) - Y(\varphi_{\alpha\beta}(X)) - \varphi_{\alpha\beta}([X,Y]) = 0.$$

Let $\Omega_M^{1,\mathrm{cl}}(\log D)$ denote the sheaf of d-closed meromorphic 1-forms with singularities along D, then $\{\varphi_{\alpha\beta}\}$ is a cocycle and in particular defines a cohomology class $\xi \in H^1(M, \Omega_M^{1,\mathrm{cl}}(\log D))$. This proves the following

Proposition 5.5.10. *The set of sheaves of twisted differential operators with regular singularities along D is in 1-1-correspondence with $H^1(M, \Omega_M^{1,\mathrm{cl}}(\log D))$.*

Example 5.5.11. We can associate to these sheaves certain characteristic classes such that if this class vanishes, we can construct the sheaf as the sheaf of sections of some line bundle.

Assume that $D = \emptyset$. If $\mathcal{L}$ is the sheaf of sections of a line bundle over M and $\{U_\alpha\}$ is an open covering of M such that we have trivializations $f_\alpha : \mathcal{L}|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}$. Let $f_{\alpha\beta} := f_\alpha \circ f_\beta^{-1}$. Let $\mathcal{D}_{\mathcal{L}}|_{U_\alpha} := f_\alpha^{-1} \circ \mathcal{D}_{U_\alpha} \circ f_\alpha$ which is a Lie algebra homomorphism. Then $d \log f_{\alpha\beta} \in H^1(M, \Omega_M^{1,\mathrm{cl}})$ is the cohomology class associated to $\mathcal{D}_{\mathcal{L}}$.

We can also define a characteristic class of $D_\xi(-\log D)$ for $D = \emptyset$. We have a short exact sequence

$$0 \to \mathbb{C} \to \mathcal{O}_M \overset{d}{\to} d\mathcal{O}_M = \Omega_M^{1,\mathrm{cl}} \to 0,$$

from which we obtain a long exact sequence

$$\cdots \to H^1(m, \Omega_M^{1,\mathrm{cl}}) \overset{c}{\to} H^2(M, \mathbb{C}) \to \cdots .$$

Then c applied to a sheaf of twisted differential operators is a characteristic class. Similarly, by exponentiating, we have

$$0 \to \mathcal{O}_M^* \overset{d\log}{\to} \Omega_M^{1,\mathrm{cl}} \to 0,$$

giving a long exact sequence and another characteristic class

$$\cdots \to H^1(M, \Omega_M^{1,\mathrm{cl}}) \overset{\bar{c}}{\to} H^2(M, \mathbb{C}^*) \to \cdots ,$$

with $\bar{c} = \exp \circ c$.

Lemma 5.5.12. *The sheaf D_ξ is defined by a line bundle $\mathcal{L}$ if and only if $\bar{c}(\xi) = 0$.*

Recall from previously that any stable poitned curve has a versal deformation. Concretely, if $\mathcal{F}^{(\emptyset)} = (\mu : \mathcal{C}^{(\emptyset)} \to \mathcal{B}^{(\emptyset)})$ is a versal family, then we set (with N copies)

$$\begin{aligned}\mathcal{B} &= \mathcal{C}^{(\emptyset)} \times_{\mathcal{B}^{(\emptyset)}} \cdots \times_{\mathcal{B}^{(\emptyset)}} \mathcal{C}^{(\emptyset)} \setminus \bigcup_{i<j} \Delta_{ij}, \\ \mathcal{C} &= \mathcal{C}^{(\emptyset)} \times_{\mathcal{B}^{(\emptyset)}} \mathcal{B}.\end{aligned}$$

We have projections $\tilde{p} : \mathcal{C} \to \mathcal{C}^{(\emptyset)}$, $p : \mathcal{B} \to \mathcal{B}^{(\emptyset)}$. We can always choose a bidiferential $\omega^{(\emptyset)}$ on $\mathcal{C}^{(\emptyset)} \times_{\mathcal{B}^{(\emptyset)}} \mathcal{C}^{(\emptyset)}$. Then

$$\omega_\alpha^{(\emptyset)} \in H^0(\mu^{-1}(U_\alpha) \times_{U_\alpha} \mu^{-1}(U_\alpha), \omega_{\mathcal{C}^{(\emptyset)} \times_{\mathcal{B}^{(\emptyset)}} \mathcal{C}^{(\emptyset)}/\mathcal{B}^{(\emptyset)}}(2\Delta))$$

with $\operatorname{Res}^2 \omega_\alpha^{(\emptyset)} = 1$. Let $U_\alpha = p^{-1}(U_\alpha)$ and $\omega_\alpha = \tilde{p}^*(\omega_\alpha^{(\emptyset)})$. We can define an action of $\Theta_{U_\alpha}(-\log D) \oplus \mathcal{O}_{U_\alpha}$ on $\mathcal{V}_{\vec{\lambda}}(\mathcal{F})$ by

$$(X, f) \cdot [|\varphi\rangle] = \nabla_X^{(\omega_\alpha)}[|\varphi\rangle] + f \cdot [|\varphi\rangle].$$

We can show that

$$\varphi_{\alpha\beta} = -\frac{c_v}{2} \Pi^* d \log \det(C\tau + D),$$

where $\Pi : U_\alpha \to \mathcal{S}_g$ is the period map (where T is Teichmüller space and $\mathcal{S}_g$ is the Siegel upper half space), and

$$\tau_{ij} = (\int_{\beta_i} \omega_j)(\int_{\alpha_i} \omega_j)^{-1}$$

is the period matrix, where the ω_j is a basis of holomorphic 1-forms, and α_i, β_i form a symplectic basis of $H_1(C, \mathbb{Z})$, and

$$\begin{pmatrix} \tilde{\beta}_1 \\ \vdots \\ \tilde{\beta}_g \\ \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_g \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \\ \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix},$$

for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(g, \mathbb{Z})$, and where the $\tilde{\alpha}_i, \tilde{\beta}_i$ is some other basis depending on $\varphi_{\alpha\beta}$? As a reminder

$$\Pi(t) = \left(\int_{\hat{\beta}_i(t)} \hat{\omega}_j(t) \right) \left(\int_{\hat{\alpha}_i(t)} \hat{\omega}_j(t) \right),$$

where $\hat{\alpha}, \hat{\beta}$ are global sections of $R^1\pi_*\mathbb{Z} \cong H^1(\pi^{-1}(t), \mathbb{Z})$. Again $\varphi_{\alpha\beta}$ is d-closed on $U_{\alpha\beta}$ and we get a cohomology class

$$\xi(\mathcal{F}) = [\{\varphi_{\alpha\beta}\}] \in H^1(\mathcal{B}, \Omega_{\mathcal{B}}^{1,\mathrm{cl}}(\log D)).$$

This has a nice description via the cohomology of the mapping class group ([Uen08, Theorem 5.13] and [Uen08, Corollary 5.14]).

In the end of Lecture 18, we considered

$$\langle \psi_d | \varphi \rangle = \sum_{i=1}^{m_d} \langle \psi | v_i(d) \otimes v^i(d) \otimes \varphi \rangle,$$

where $\psi_d \in \mathcal{H}_{\vec{\lambda}}^\dagger(D_i)[\![q]\!]$ and the $v_i(d)$ form a basis of $\mathcal{H}_{\vec{\lambda}}(d)$, and

$$\langle \hat{\psi} | \varphi \rangle := \sum_{d=0}^{\infty} \langle \psi_d | \varphi \rangle q^d.$$

The goal is to see that $\langle \hat{\psi} |$ converges. The proof of Ueno uses the locally freeness property (even if that's what it's used to prove).

20th lecture, April 17th

5.6 The Verlinde formula

One reference for this is "La formule de Verlinde", by Sorger, in Astérisque 1994–1995.

Fix as always a Lie algebra $\mathfrak{g}$ and a level l. The Weyl alcove (intersected with the characters) we denoted P_l. We introduce what is called the Verlinde algebra or the fusion ring. Look at $\mathbb{Z}^{P_l}$, i.e. the free abelian group generated by $\lambda \in P_l$. This can be made into a ring with the following multiplication:

$$\alpha \cdot \beta = \sum_{\lambda \in P_l} (N^{\lambda}_{\alpha,\beta})\lambda,$$

where $N^{\lambda}_{\alpha,\beta} \in \mathbb{N}$ is given by

$$N^{\lambda}_{\alpha,\beta} = \dim(\mathcal{V}_{(\alpha,\beta,\lambda^{\dagger})}(\mathbb{P}^1, a, b, c, \eta_a, \eta_b, \eta_c)).$$

Remark 5.6.1. The element $0 \in P_l$ is not the 0 in $\mathbb{Z}^{P_l}$.

Theorem 5.6.2. *Extending the product $\mathbb{Z}$-linearly, $(\mathbb{Z}^{P_l}, \cdot)$ becomes a commutative ring.*

Proof. Commutativity is easy. We check associativity:

$$\alpha \cdot (\beta \cdot \gamma) = \sum_{\lambda \in P_l} N^{\lambda}_{\beta\gamma} \alpha \cdot \gamma = \sum_{\lambda,\mu \in P_l} N^{\lambda}_{\beta\gamma} N^{\mu}_{\alpha\lambda} \mu,$$
$$(\alpha \cdot \beta) \cdot \gamma = \sum_{\lambda \in P_l} N^{\lambda}_{\alpha\beta} \lambda \cdot \gamma = \sum_{\lambda,\mu \in P_l} N^{\lambda}_{\alpha\beta} N^{\mu}_{\lambda\gamma} \mu,$$

and the two right hand sides are equal because of factorization. □

For convenience, choose a labelling λ_i, $i = 1, \ldots, m$, for the elements in P_l, and recall that we have an involution $\dagger : \{1, \ldots, m\} \to \{1, \ldots, m\}$. Define the matrix N_i by

$$(N_i)^k_j = N^{\lambda_k}_{\lambda_i \lambda_j}.$$

Theorem 5.6.3. *If the genus is $g = 0$, then*

$$\dim(\mathcal{V}_{\lambda_{i_1} \ldots \lambda_{i_N}}(\mathbb{P}^1, a_1, \ldots, a_N, \eta_1, \ldots, \eta_N)) = (N_{i_1} \cdots N_{i_N})^0_0.$$

If $g > 0$, then

$$\dim(\mathcal{V}_{\lambda_{i_1},\ldots,\lambda_{i_N}}(C, a_1, \ldots, a_N, \eta_1, \ldots, \eta_N)) = (N_{i_1} \cdots N_{i_N} W^g)^0_0,$$

where

$$W = \sum_{i=1}^{m} (\operatorname{Tr} N_{k^{\dagger}}) N_k.$$

Sketch of proof. For $g = 0$: Add 2 points with weight 0, call them a_0, a_{N+1}. By propagation of vacuum, this does not change the dimension. Degenerate the $\mathbb{P}^1$ such that each of the $a_1, \ldots, a_N$ is on its own irreducible component (a drawing would be nice here). By factorization, we get

$$\sum_{k_1,\ldots,k_{N-1}} N^{k_1}_{i_1 0} N^{k_2}_{k_1 i_2} N^{k_3}_{k_2 i_3} \cdots N^{k_{N-1}}_{k_{N-2} i_{N-1}} N^{0}_{k_{N-1} i_N} = \sum_{k_1,\ldots,k_{N-1}} N^{k_1}_{i_1 0} N^{k_2}_{i_2, k_1} \cdots N^{k_{N-1}}_{i_{N-1} k_{N-2}} N^{0}_{i_n k_{N-1}},$$

which proves the claim. For $g > 0$ one playes the same game using induction on g. □

The question now is how to calculate this matrix product. The miracle is that it is possible to simultaneously diagonalize all of the N_i (and hence W).

We look now at elliptic curves; i.e. a 1-pointed genus 1 curves. Say p is the marked point. Any such is isomorphic to $\mathbb{C}/\langle 1,\tau\rangle$ for some $\tau \in H = \{z \mid \mathrm{Im} z > 0\}$, unique up to the action of $\mathrm{SL}(2,\mathbb{Z})$ on H, given as follows: We know that

$$\mathrm{SL}(2,\mathbb{Z}) = \left\langle T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

and these act by $T(z,\tau) = (z, 1+\tau)$, $S(z,\tau) = (z/\tau, -1/\tau)$. The action has fundamental domain that can be visualized by (insert pretty drawing). In any case, we have a family of elliptic curves over H. Take now an elliptic and degenerate it to pick up a single node. Normalizing, we get a $\mathbb{P}^1$ with three marked points. By factorization, if $\tilde{C}$ is the degenerate surface, then

$$\mathcal{V}_0(\tilde{C}, p) = \bigoplus_{\lambda \in P_l} \mathcal{V}_{0,\lambda,\lambda^\dagger}(\mathbb{P}^1, p, a_1, a_2).$$

Now, each summand on the right hand side is 1-dimensional.

Fact 5.6.4. *In fact, the bundle $\mathcal{V}_0(C \to H, p)$ over H splits as a sum of line bundles, enumerated by $\lambda \in P_l$.*

A basis is given by the characters of the $\hat{\mathfrak{g}}$-modules $\mathcal{H}_\lambda$, as functions of $(z,\tau) \in H$, given by a Weyl character formula,

$$\chi(z,\tau) = \frac{\sum_{w \in W} (-1)^{|w|} \Theta_{w(\lambda+\rho), g^*+l}}{\sum_{w \in W} (-1)^{|w|} \Theta_{w(\rho), g^*}},$$

where the $\Theta_{\lambda,l}(z,\tau)$ are the so-called theta-functions.

The action of $\mathrm{SL}(2,\mathbb{Z})$ lifts to the vector bundle, and are given by

$$\begin{aligned} \chi_\lambda(z, \tau+1) &= q^{\delta_\lambda - c_v/24} \chi_\lambda(z,\tau), \\ \chi_\lambda(z/\tau, -1/\tau) &= \sum_{\mu \in P_l} S_{\lambda,\mu} \chi_\mu(z,\tau), \end{aligned}$$

where here

$$\begin{aligned} S_{\lambda,\mu} &= S_{\lambda,0} \mathrm{ch}_\mu \left(e^{-\frac{-2\pi i(\lambda+\rho)}{l+g^*}} \right), \\ S_{\lambda,0} &= |P/((l+g^*)Q^V)|^{-1/2} \prod_{\alpha>0} 2 \sin\left(\frac{\pi(\lambda+\rho,\alpha)}{l+g^*} \right). \end{aligned}$$

Theorem 5.6.5. *Conjugation by S diagonalizes the N_i.*

E.g. for $\mathfrak{sl}(2,\mathbb{C})$, level l,

$$S_i^j = \sqrt{\frac{2}{l+2}} \sin\left(\frac{(i+1)(j+1)\pi}{l+2} \right),$$

and from this we get the Verlinde formula

$$\begin{aligned} \dim \mathcal{V}_{\lambda_{i_1} \cdots \lambda_{i_n}}(\mathcal{X}) &= \sum_{j=0}^{l} (S_0^j)^{2-2g} \prod_{\alpha=1}^{N} \frac{S_{i_\alpha}}{S_0^j} \\ &= \left(\frac{l+2}{2} \right)^{g-1} \sum_{j=0}^{l} \frac{\prod_{a=1}^N \sin\left(\frac{(i_a+1)(j+1)\pi}{l+2} \right)}{\left(\sin\left(\frac{(j+1)\pi}{l+2} \right) \right)^{2g-2+N}}. \end{aligned}$$

We end up by relating conformal blocks and non-abelian theta functions. Recall that for G a simple (1-connected) group G, B a Borel, let $\lambda \in \mathrm{Hom}(T_G, \mathbb{C}^*)$. We have a bundle $L_\lambda = G \times \mathbb{C}/B \to G/B$ over the flag variety.

Theorem 5.6.6 (Borel–Weil(–Bott)). *The irreducible weight of G determined by the highest weight λ is isomorphic to $H^0(G/B, L_\lambda)$.*

Suppose now we have a curve C and a holomorphic G-bundle $P \to C$. Choose a point $x \in C$.

Theorem 5.6.7 (Harder). *The bundle $P|_{C\setminus\{x\}}$ is trivial.*

Choose a small disk D around x. Then $P|_D$ is also trivial, and we now have a trivializing open cover of C, and we can determine the bundle by the transition functions, defined on $D^\times = D \setminus \{x\} = D \cap (C \setminus \{x\})$. Morally speaking, morphisms from $D^\times$ to G are given by elements of the formal loop group $G(\!(z)\!)$. Likewise, morphisms from D to G are given by elements of $G[\![z]\!]$.

Proposition 5.6.8. *The set of isomorphism classes of holomorphic G-principal bundles* Bun_G *on C is determined (as a set but also as a stack) by transition functions, $G(C \setminus \{x\})\backslash G(\!(z)\!)/G[\![z]\!]$, where elements of $G(C \setminus \{x\})$ are morphisms $C \setminus \{x\} \to G$.*

Here $G(\!(z)\!)/G[\![z]\!]$ is the affine Grassmannian. It is also isomorphic to $\hat{G}(\!(z)\!)/\hat{G}[\![z]\!]$. Here, $G[\![z]\!]$ is a parabolic subgroup of $G(\!(z)\!)$, $\hat{G}(\!(z)\!)$ is the universal central extension, and in particular $\mathrm{Lie}(\hat{G}(\!(z)\!))$ is the affine Lie algebra $\hat{\mathfrak{g}}$.

Irreducible positive energy representations of $\hat{\mathfrak{g}}$ are realized again as $H^0(\hat{G}(\!(z)\!)/\hat{G}[\![z]\!], L_\lambda)$.

We are interested in $H^0(\mathcal{M}_G, L)$, where $\mathcal{M}_G$ is the moduli space of G-bundles, and L is some line bundle, which is isomorphic to $H^0(\mathrm{Bun}_G, L)$ for some L. We think of Bun_G as the quotient of the affine Grassmannian by something else, so we can take the line bundle L and pull it back to the affine Grassmannian, and by the above we get a representation of $\hat{\mathfrak{g}}$. Pulling back, we get more sections, and we are only interested in those that are pullbacks of sections from Bun_G. I.e. those that are invariant under $G(C \setminus \{x\})$, so in terms of Lie algebras, those invariant for $\hat{\mathfrak{g}}(C, x)$. What we have written down is literally the definition of the space of covacua.

This is the theorem of Beauville and Laszlo for $G = \mathrm{SL}(n, \mathbb{C})$.

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