Quantum field theory

Lectures by Richard Borcherds, notes by Søren Fuglede Jørgensen

Math 275, UC Berkeley, Fall 2010

Contents

Disclaimer

These are notes from the first 4 weeks of a course given by Richard Borcherds in 2010.[1] They are discontinued from the 7th lecture due to time constraints. They have been written and TeX'ed during the lecture and some parts have not been completely proofread, so there's bound to be a number of typos and mistakes that should be attributed to me rather than the lecturer. Also, I've made these notes primarily to be able to look back on what happened with more ease, and to get experience with TeX'ing live. That being said, feel very free to send any comments and or corrections to `fuglede@imf.au.dk`.

1st lecture, August 26th 2010

1 Introduction

1.1 Defining a QFT

The aim of the course is to define a quantum field theory, to find out what a quantum field theory is, and to define Feynman measures, renormalization, anomalies, and regularization. Many mathematical definitions exist, including the following:

[1]The course homepage is located at `http://math.berkeley.edu/~reb/courses/275/index.html` – that probably won't be true forever though.

(1) Wightman axioms: Roughly, these include a Hilbert space H of physical states with operators satisfying various axioms.

(2) Operator algebras: A quantum field theory is a net of operator algebras A with a state ω. For example the set of all bounded operators on H with $\omega : A \to \mathbb{C}$ such that $\omega(aa^*) \geq 0$.

(3) Osterwalder–Scharder axioms: A quantum field theory is given by a measure on the space of distributions on $\mathbb{R}^n$.

(4) Topological quantum field theory: A quantum field theory is a functor from a cobordism category to the Hilbert spaces.

One common property of these is that they don't work; no physical theory (such as ϕ^4-theory, QED, QCD, ...) is modelled by any of them. One reason QED/QCD etc. do not satisfy the axioms is that the theories are perturbative: The "answers" are not real numbers but formal power series in coupling constants (these power series do not converge, but the first couple of terms give good results). We can modify the Wightman axioms, for example, to allow formal power series. We will also allow composite operators, time ordered operators and curved space-time. The general problem will be to guess the physicists' secret definitions.

The first naive attempt is the following: Let H be a Hilbert space. For every point x of space-time, suppose we are given a so-called field operator $\phi(x) : H \to H$ satisfying Lorentz invariance and locality; that is, $\phi(x)\phi(y) = \phi(y)\phi(x)$, if x and y are space-like separated. Here, we assume that space-time has an ordering $x \leq y$ (which physically means that x can influence y), and we call x and y space-like separated, if neither $x \leq y$ or $y \leq x$. The problem is that no such $\phi(x)$ exist; the expression $\phi(x)\phi(y)$ will become singular, as x approaches y – a phenomenon known as ultraviolet divergence.

We should consider $\phi(x)$ as a *distribution*: While $\phi(x)$ does not make sense, the expression $\int \phi(x)f(x)\,dx$ will when f is compactly supported and smooth. So, instead of demanding $\phi(x)$ be linear, we want a linear operator $\phi(f) : H \to H$ for test functions f. This also fails: $\phi(f)$ is only defined on a dense open subset of H. However, all the $\phi(f)$ will keep invariant a suitable dense subset $D \subset H$, so that composite operators make sense.

This D can also be a module over a ring $\mathbb{C}[\![\lambda]\!]$ of formal power series. Consider the algebra A generated by the $\phi(f)$. We will consider the structure of A. D has a vacuum vector v, and we can define a linear form $A \to \mathbb{C}$ by letting $\omega(a) = \langle v, av\rangle$. It is then possible to reconstruct D and $\langle,\rangle$ from A and ω as follows: Define $\langle,\rangle$ on A by letting $\langle a, b\rangle = \omega(a^*b)$, where * denotes the adjoint in A; that is, $\langle ax, y\rangle = \langle x, a^*y\rangle$. Let D be $(A, \langle,\rangle)/\ker(\langle,\rangle)$. We conclude that our aim should be to find such an algebra A, an involution * on A as well as a linear form $\omega : A \to \mathbb{C}$ (or $\mathbb{C}[\![\lambda]\!]$). It turns out that finding the algebra and the involution is going to be the easy part, while finding ω is going to be somewhat harder.

To summarize, we want to solve the following problem: Assume that we are given

- space-time
- a space of classical field (essentially a bundle over space-time)
- a Lagrangian (such as $\phi^4 + m^2\phi^2 + \sum(\partial_i\phi)^2$).

Construct from this $(A, ^*, \omega)$.

1.2 Constructing a QFT

In order to construct A, we begin by constructing a dictionary explaining the physicists' terminology in terms of mathematical constructions.

- The space-time of physics should be a manifold M (such as $\mathbb{R}^{1,3}$).
- A classical field ϕ should correspond to a section of a vector bundle Φ (for example, the set of scalar fields is $C^\infty(M)$).

- Differentiation $\partial_i \phi$ corresponds to a section of the jet space $J\Phi$.
- Multiplying fields and taking composite fields (as in the expression $\phi^4 + m^2\phi^2 + \sum(\partial_i\phi)^2$) corresponds to taking the symmetric algebra $SJ\Phi$.
- To get a Lagrangian density $L\,dx^4$, we take the product with the bundle ω of densities (that is, n-forms tensored with orientations), so that we end up with $\omega SJ\Phi$.
- To make sense of $\int L\,dx^4$, we consider compactly supported sections $\Gamma_c\omega SJ\Phi$.
- To make time-ordered products, we add another symmetrical algebra and obtain $S\Gamma_c\omega SJ\Phi$.
- Finally, to be able to take tensor products, we consider $T_0 S\Gamma_c\omega SJ\Phi$ (where T_0 denotes the even part of the tensor algebra).

Now, this monstrosity is going to be our A, and we will construct ω on this. It should be noted that all of these steps are necessary in order to make sense of the relevant physical objects and operations.

2nd lecture, August 31st 2010

The construction of ω will involve Feynman integrals. We consider expressions like

$$\int_{\infty\text{-dim. space}} \phi(x_1)\phi(x_2)\cdots\cdot e^{i\int L(\phi)d^4x} D\phi.$$

The first factor of the integrand is called a Green's function $G(x_1, x_2, \dots)$. The problems we would like to address are how to make sense of $D\phi$ and how to reconstruct the QFT from G. This turns out to be a huge problem, as we have no Haar measure on our infinite-dimensional space. The solution to this problem will be to cheat and change the definition of measure. Normally, we think of a measure as a real-valued map μ on a σ-algebra as measurable sets from which we can reconstruct an integral $\int f(x)\,d\mu(x)$. More suitable for our problem is the following:

Definition 1. A *measure* is a linear space of functions on X together with a linear map $S \to \mathbb{R}$.

This generalizes the concept of a Radon measure, which is what we get when X is locally compact and $S = C_c(X)$. There is no reason to stick to $C_c(X)$ though; instead, we take S to be a the functions on the space of fields that we want to be able to integrate; that is, functions of ϕ given by integrating over space-time, taking polynomials in ϕ, derivates etc.. That is, expressions like

$$\int \int \phi^4(x) f(x)\,dx \int \phi(x) g(x)\,dx e^{i\int L(\phi)\,dx} D\phi,$$

where f, g are test functions as before. The exponential part of the integrand is harder to deal with. In general, L consists of a quadratic part, like $m^2\phi^2 \sum(\partial_i\phi)^2$ plus some higher order part. Integrating the quadratic part roughly corresponds to doing some Gaussian integral, and the problem lies in the higher order part. The solution this time is to multiply the higher order terms with an infinitesimal *coupling constant* λ;

$$e^{i\lambda\int \phi^4\,dx} = \sum(\lambda \int \phi^4\,dx)^n/n!$$

Summarizing, we want to integrate polynomials in $\int \phi^k(x) f(x)$ as well as Gaussians $e^{i\cdot\text{quad. part of } L}$, and so, our space S is the space $e^{iL_0} S\Gamma_c\omega SJ\Phi$.

We begin our analysis by constructing such a measure on $\mathbb{R}$. The usual measure in this case would be $f \mapsto \int f(x)\,dx$. Instead, take S to be the set of polynomials times $e^{-\pi x^2}$. Comparing to what we usually get by doing integration by parts, we should map $e^{-\pi x^2} \mapsto 1$, $xe^{-1\pi x^2} \mapsto 0$,

$x^2e^{-\pi x^2} \mapsto \frac{1}{2\pi}$ and so forth; in general, $\int x^n e^{-\pi x^2}\,dx$ can be expressed in terms of $\int x^{n-2}e^{-\pi x^2}\,dx$, using integration by parts.

Integration by parts rougly corresponds to translation invariance, $\int f(x+dx) = \int f(x)$, so by the above, we have formally defined a translation invarant map $\{\text{polynomials}\cdot e^{-\pi x^2}\} \to \mathbb{R}$. The idea is to copy this approach in the infinite-dimensional case. However, the resulting measure is not going to be unique. In fact, neither is the Lebesgue measure on $\mathbb{R}$ – however, in this case we know that measures differ only by scaling, and modding out by the action of $\mathbb{R}_{>0}$ given by scaling, we obtain a unique orbit. In the infinite dimensional case, something rather similar will happen; while we have no canonical choice of measure, any two measures will be related by a unique renormalization. The renormalizations are going to constitute a infinite-dimensional group. This group will be a projective limit of nilpotent groups. As before, we thus get a unique orbit of Feynman measures. The old picture was to use the Lagrangian to construct a quantum field theory. The above suggests that we should instead construct the QFT using a Lagrangian and a Feynman measure. The approach using both and modding out by renormalization is isomorphic to just using a Lagrangian. This isomorphism is not canonical, though, and will give what is known as *anomalies*.

1.3 Feynman diagrams

Roughly, a Feynman diagram is what you get, if you expand the expression

$$\int \prod \text{polynomials}\cdot e^{-\pi x^2}\,dx.$$

Consider as an example the 1-dimensional case and the integral

$$\int x^i x^j x^k e^{-ax^2}\,dx$$

forgetting for a second that $x^i x^j = x^{i+j}$. Such an integral is computed using integration by parts, and since $x^i = x^{i-1}x$, we get

$$\int \frac{d}{dx}(x^{i-1}x^j x^k)e^{-ax^2}/(2a)\,dx = \int ((i-1)x^{i-2}x^j x^k + x^{i-1}jx^{j-1}x^k + \cdots)\cdot e^{-ax^2}/(2a)\,dx.$$

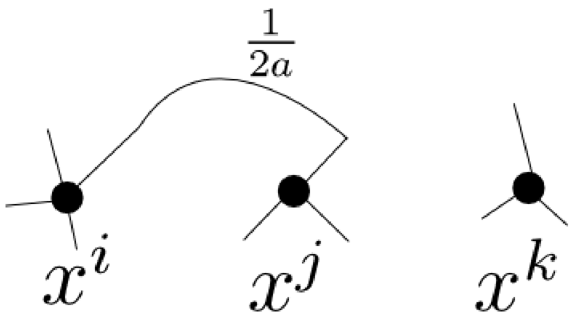

Figure 1: Feynman diagram illustrating the one-dimensional case.

That is, we reduce i and one of j and k by one. This can be pictured diagrammatically as in Figure 1. Here, we have i lines at the vertex x^i and a line between x^i and x^j corresponds to reducing i and j by 1. Now, a *Feynman diagram* is the result of connecting all lines (notice that this is possible, exactly when the number of lines is even; if the number is odd, we want the integral to be 0). That is, we can calculate

$$\int x^i x^j x^k e^{-ax^2}, dx = \sum_{\text{Feynman diagrams}} \prod_{\text{Edges}} 1/(2a).$$

In general, the factors in the last products will be whatever we have on our edges, which happens to be $1/(2a)$ in our case.

In the infinite dimensional case, something similar happens. However, the constants on the edges are replaced by propagators $\Delta(x,y)$, and we would want to multiply these. The singularities in Δ are too ill-behaved for this to be possible. A typical singularity looks like $\Delta(x,y) = 1/(x-y)^2$. The solution to this problem will be to regularize the expression: Replace the propagator $1/(x-y)^2$ by $1/(x-y)^s$ for some complex s. For $\mathrm{Re}(s) \ll 0$, this is well-behaved. Roughly speaking, the Feynman measures will be defined for some complex s, and we will take the analytical continuation to $s = 2$. This will always be possible by the Bernstein theorem, but the continuation will have poles for $s = 2$.

3rd lecture, September 2nd 2010

1.4 Renormalization

The story so far: We found that ω has something to do with a sum over Feynman diagrams. "Feynman diagram" is shorthand for product of *propagators* $\Delta(x,y)$ (on each edge). The problem is that propagators have (ultraviolet) singularities a la $1/(x-y)^2$ at $x = y$. We introduced a complex parameter s, such that $1/(x-y)^s$ is nice for $\mathrm{Re}(s) \ll 0$. This was then analytically continued using the Bernstein theorem (this process is known as regularization). The problem is that we get poles at $s = 2$; the case, we are interested in. It does however give a perfectly fine QFT defined over the ring of meromorphic functions, in case we don't care about the poles. We do, though. Naive attempts on using this do not work; for example, we could try taking the constant term of the Laurent expansion of meromorphic functions. This doesn't work as it destroys subtle relations between Feynman diagrams, that are needed for proving locality, positive definiteness, and so on ... Instead, we have to find a suitable transformation, preserving all necessary structure. These transformations are known as renormalizations; the same group as we used when considering the action on Feynman measures – now, these are taken over the field of meromorphics functions (we could call them *infinite renormalizations*). We will show that we can find an infinite renormalization that removes all poles. (Note that this is far from unique: we can also act by any *finite* renormalization – this reflects that there is no canonical choice of a Feynman measure).

Definition 2 (Renormalization)**.** A *renormalization* is an automorphism of $SSJ\Phi$.

As before, we have a few problems with a statement like this: It is incomprehensible and it seems to have nothing to do with the renormalizations of physics. These renormalizations do, however, have all the properties we want them to have:

(1) We will show that renormalizations act on $S\Gamma_c\omega SJ\Phi$. The dual of this is the set of Feynman measures, so renormalizations act on these too. This action is simply transitive (giving us the unique orbit, we were looking for).

(2) $\Gamma_c\omega SJ\Phi$ is more or less the space of compactly supported Lagrangians. We want renormalizations to act on these. The key idea is to look at the exponential map $\Gamma_c\omega SJ\Phi \to \hat{S}(\Gamma_c\omega SJ\Phi)$, where $\hat{S}$ denotes completion of symmetric algebra. Renormalizations preserve the image of exp, o we get a nonlinear action on the Lagrangians. The action is given by something like $\rho(L) = \log\rho\exp(L)$. The renormalizations preserve this image since they preserve a *coproduct* on $S\Gamma_c\omega SJ\Phi$: Recall that if V is a vector space (an abelian Lie algebra), SV is a ring (which we can think of as the universal enveloping algebra) as well as a Hopf algebra; there is a map $\Delta : SV \to SV \otimes SV$, which is an algebra homomorphism mapping v to $v \otimes 1 + 1 \otimes v$ for $v \in V$. This extends to a completion $\Delta : \widehat{SV} \to \widehat{SV \otimes SV}$. We have an exponential map $\exp : V \to \widehat{SV} : v \mapsto 1 + v + v^2/2 + \cdots$. Elements of $\widehat{SV}$ satisfying $\Delta g = g \otimes g$ with constant term 1 form a group. Now, exp is an isomorphism of sets from V to these "group-like" elements. As an example, take G a group with group ring $\mathbb{Z}G$ and Hopf algebra product $\Delta g = g \otimes g$. G is exactly the group of group-like elements in ZG.

The next question is what $SSJ\Phi$ really is. As previously, $J\Phi$ is the jet bundle/sheaf of Φ, and S is simply the symmetric algebra of sheaves. We want to now discuss the structure of $SSJ\Phi$. Recall that if V is a vector space/sheaf, then SV is a *coalgebra* (as above with $v \mapsto 1 \otimes v + v \otimes 1$). So $SSJ\Phi$ is a coalgebra, and renormalizations have to preserve the coalgebra structure. Furthermore, $SJ\Phi$ is a coalgebra, acting on $SSJ\Phi$, and renormalizations must also preserve this coaction. (Note that we throw away the two algebra structures here.) This construction uses four iterations of "function space" construction; this compresses a lot of complexity into 4 symbols.

The result of all this regularization/renormalization is that we get a (non-unique) map $\omega : e^{iL_0} S\Gamma_c \omega SJ\Phi \to \mathbb{C}$. This map encodes Greens functions of a free field theory with *composite operators*. Also, the map is sort of a Feynman measure. Finally, it gives expectation values of time-ordered operators.

To get a free field theory, we extend ω to $T_0 S\Gamma_c SJ\Phi$ (where T_0 means the even part of the tensor algebra). Fortunately, this can be done canonically. We will check that ω satisfies analogues of the Wightman axioms.

Notice that all the complications of renormalization/regularization are needed for a free field theory with composite operators. The advantage of this is that it is now really easy to construct interacting theories. This separates problems of regularization and renormalization from problems of pertubation theory and Lagrangians.

1.5 Constructing an interacting theory

We will now look at the construction of the interacting theory from a Lagrangian L and free field theory ω: This will be nothing but $e^L\omega$. As before, ω is a linear map $T_0 S\Gamma_c \omega SJ\Phi \to \mathbb{C}$, and Lagrangians L turn out to act as *derivations* on this algebra. So, $\exp L$ is an automorphism of this algebra and acts on the dual (that is, on ω). The problem here being that we don't really get an automorphism, as the exponential map gives us an infinite series that doesn't converge. As before, the solution is to use formal power series. Apart from this rather unfortunate glitch, we finally obtain a quantum field theory. Another apparent problem is that interacting quantum field theories are isomorphic to free ones; e^L gives an isomorphism. The isomorphism changes the simple operators $\phi(x)$ (as opposed to composite operators such as $\phi(x)^4$), and so if we restrict to simple operators (as is usual), there is no longer an isomorphism.

1.6 Anomalies

Now, we consider anomalies. Suppose we have a group G of symmetries of the bundle Φ over spacetime M. For example, the Lorentz group, if M is flat spacetime, spacetime translations or a gauge group fixing points of spacetime but acting on fibers Φ. We get an action of G on $T_0 S\Gamma_c \omega SJ\Phi$. Suppose that G fixes a Lagrangian in $\Gamma_0 \omega SJ\Phi$. Does G act on the quantum field theory $e^L\omega$? In other words, is $e^L\omega$ fixed by G? In fact, ω need not be fixed by G, as ω was not canonical. In some cases, we can find an ω fixed by G; for example, if G is the Lorentz group, all constrctions can be made invariant under G. Sometimes it's hard, though; for example if G is a gauge group. In general, we want to find obstructions to finding G-invariant ω – these we will call *anomalies*. Suppose we have a group U acting simply transitively on a set S. Suppose another group G acts on both U and S preserving the action of U on S. Is it then possible to find a fixed point of G on S? The relevant example in our case is G a gauge group, U the group of renormalizations and S the set of Feynman measures. Another example would be G an absolute Galois group of $\mathbb{Q}$, U a Jacobian and S a genus 1 curve. In both cases, the obstruction to finding a G-invariant point of S is an element of the first cohomology $H^1(G, U)$. Pick any point $s \in S$. Define a_σ for $\sigma \in G$ by $\sigma(s) = a_\sigma(s)s$ (which is possible as U acts simply transitively on S). Then $a_{\sigma\tau} = \sigma(a_\tau)a_\sigma$ (this is an exercise; $\sigma \mapsto a_\sigma$ is a 1-cocycle of G with values in U). Also, us is fixed for some $u \in S$ if and only if $u\sigma(u)^{-1}$ for all σ. In other words, a is a coboundary.

4th lecture, September 7th 2010

Recall that we are in the following general setup: A group U acts simply transitively on a set S. A group G acts on U and S preserving the action of U. In our application, S is the set of Feynman measures, U the group of renormalizations and G a gauge group. Last time we saw that the obstruction to finding a G-invariant element of S is given by an element of $H^1(G, U)$, so one can think of this set as a set anomalies.

We will modify this slightly. The group U is non-abelian. This means that $H^1(G, U)$ is non-abelian cohomology, so it's not a group. Secondly, we will not really need a G-invariant measure ω but only a G-invariant QFT $e^{iL}\omega$. We can sometimes "absorb" the non-invariance of ω into the Lagrangian L. The result of this is that the anomalies in fact lie in $H^1(G, S\Gamma_c\omega SJ\Phi)$. In particular, this is an abelian vector space and easier to deal with than $H^1(G, U)$. (Fujikawa (around 1970) discovered that anomalies arise because Feynman measure is not invariant under the gauge group.)

1.7 Infrared divergences

There are two reasons why $\int_{-\infty}^{\infty} f(x)\,dx$ might not exist. One is that f behaves badly locally (say $f(x) = 1/x^10)$. This is called ultraviolet divergences (ultraviolet meaning short distance). Another one is that f is bad globally (say $f(x) = x^2$), in which case we talked about infra red divergences. The former we have dealth with by renormalization, regularization and so on. The latter we have more or less defined out of existence by using comapactly supported global sections, Γ_c. This works fine if the interaction terms of the Lagrangian have compact support. We don't have reason to think so then; we would like our physical laws to be translation invariant.

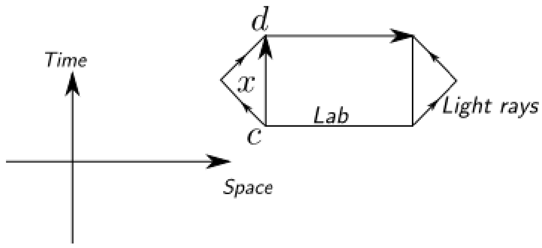

Figure 2: The experiment in laboratory together with the convex hull in spacetime.

The key point is that the result of any physics experiment only depends on the interaction in a compact region of spacetime; instead of simply considering what goes on in laboratory, we take the laboratory × the time of experiment, which will be a compact set C, and take the "convex hull": All points x of spacetime such that $c \le x \le d$ for some $c, d \in C$ (see the figure). We will show – roughly – that the experiment in the lab is not affected by changing the interaction outside the convex hull of C. Suppose that the convex hull of any compact set is compact; then we are fine, as we can just cut the interaction down to something of compact support without changing it on the convex hull. If this holds, we can (almost) ignore infrared divergences (it's not quite true for a reason, we will see in a moment). The spacetimes with this property are called *globally hyperbolic*. Reasonable spacetimes are going to be globally hyperbolic, more or less by the definition of the word "reasonable". For example, Minkowski $\mathbb{R}^{1,3}$ is globally hyperbolic. Hyperbolic space with a point missing is not. One can consider the property as saying, that we have no holes in spacetime, which is a reasonable thing to demand.

1.8 Summary of problems and solutions

So far, we have come across the following problems and solutions.

- It is unclear what a QFT is. The solution is to write down what it is: $e^{iL}\omega : T_0 S\Gamma_c\omega SJ\Phi \to \mathbb{C}$.

- Feynman measure is not a well-defined concept. The solution is to change the definition of Feynman measure.
- A product of distributions is not well-defined. We then "regularize" by introducing a complex parameter s.
- We need to analytically continue. To do this, we use Bernstein's theorem.
- We then get poles. To get rid of them, we "renormalize".
- The measure is not defined on e^L. Here, the solution is to introduce infinitesimal interactions $e^{\lambda L}$.
- Symmetries do not fix the measure. Our solution is to calculate anomalies and hope they vanish.

2 Review of Wightman axioms for a QFT

The goal is to state the Wightman axioms, to construct free field theories as examples. We will the explain why the axioms are inadequate and finally extend them to fix the problems.

The basic data we are given are as follows: A Hilbert space H (representing the space of physical states), a "vacuum vector" in H (representing vacuum), an (unbounded) operator $\phi(f)$ on H for each compactly supported smooth function on spacetime, which we take to be flat $\mathbb{R}^{1,3}$. (Take ϕ to be a Hermitian scalar field.)

These should satisfy the following properties:

(1) The inner product of H is positive definite (which is really implied by the above).

(2) Lorentz invariance: The group of all Lorentz rotations and translations acts on the space of compactly supported functions. This action should extend to an action on H preserving the action of the operator on a vector.

(3) Positive energy: If E is operator of infenitesimal translations forward in time. Then $(Ev, v) \geq 0$. (We can think of E as giving the energy of a state. That is, any state has energy at least the energy of the vacuum.)

(4) Locality: $[\phi(f), \phi(g)] = 0$, if f and g are spacelike separated, i.e. the supports of f and g are spacelike separated. (This is related to the fact that signals cannot propagate faster than light.)

These are the "big 4" conditions. The remaining are minor and more technical conditions:

(1) The vacuum vector is unique up to scalars; that is, it is the unique vector which is invariant under translations of spacetime.

(2) There is a dense subspace D of H, such that all operators $\phi(f)$ map D to itself.

(3) Wightman distributions – things on the form $\langle \text{vacuum} \mid \phi(f_1) \cdots \phi(f_n) \mid \text{vacuum} \rangle$ – are *tempered*; i.e., they don't grow too fast at ∞.

2.1 Free field theories

We consider now consider the simplest example of something satisfying the Wightman axioms.

Definition 3. A *free field theory* is a representation of a *Heisenberg algebra*.

Definition 4 (Heisenberg algebras)**.** Consider first a toy example: Look at the algebra generated by $\frac{d}{dx}$ and x acting on $\mathbb{C}[x]$. Notice that $[\frac{d}{dx}, \frac{d}{dx}] = [x,x] = 0$, but $[\frac{x}{dx}, x] = 1$. What we get is a representation of the 3-dimensional Lie algebra generated by elements X, Y and H such that $[X,Y] = H$ and $[H,X] = [H,Y] = 0$, as this Lie algebra acts on $\mathbb{C}[X]$. Exercise: This representation is irreducible.

This representation is really the symmetric algebra generated by Y.

Now consider the case of several variables, $x_1, \dots, x_n$, $\frac{d}{dx_1}, \dots, \frac{d}{dx_n}$. The commutator relations are slightly less trivial, $[\frac{d}{dx_i}, \frac{d}{dx_j}] = [x_i, x_j] = 0$ while $[\frac{d}{dx_i}, x_j] = \delta_{ij}$. So we get a $2n+1$ dimensional Lie algebra spanned by $\frac{d}{dx_i}$, x_j and 1. The structure of the Heisenberg algebra is rather simple: It has 1-dimensional center (spanned by 1 or H). The quotient by the center is abelian.

Suppose now that V is a vector space with a skew-symmetric space $\langle,\rangle$. Then we can define a Lie algebra on $V \oplus \mathbb{C}$ by letting $\mathbb{C}$ be the center and defining $[w,v] = \langle w,v\rangle$

Suppose $V = V_1 \oplus V_2$, $\langle,\rangle = 0$ on V_1 and 0 on V_2. Then this algebra has a representation of $S(V_2)$ as follows: Elements of V_2 just act as multiplication by themselves on $S(V_2)$. Elements of V_1 act as order 1 differential operator on $S(V_2)$: Suppose V_2 is spanned by $x_1, x_2, \dots$, $S(V_2) = \mathbb{C}[x_1, x_2, \dots]$, and suppose $d \in V_1$ and that $[d, x_i] = a_i$. So d acts as $a_1 \frac{d}{dx_1} + a_2 \frac{d}{dx_2} + \cdots$. This also works if V_1 and V_2 are infinite-dimensional; we get an infinite-dimensional Heisenberg algebra $V_1 \oplus V_2 \oplus \mathbb{C}$ acting on $S(V_2)$.

We will construct an infinite-dimensional Heisenberg algebra spanned by 1, creation operators $\phi^+(f)$, annihilation operators $\phi^-(f)$, where f are compactly supported test functions. We define the relations $[\phi^-(f), \phi^-(g)] = [\phi^+(f), \phi^+(g)] = 0$. The element $[\phi^-(f), \phi^+(g)]$ should be translation invariant. We can think of it as a "distribution" on $M \times M$ (where M is spacetime). The Fourier transform of this will be a distribution of two momentum variables p_1, p_2 which vanishes unless $p_1 + p_2 = 0$. We have

$$[\phi^-(f), \phi^+(g)] = \int_{p^2=m^2, p>0} \tilde{f}(p)\tilde{g}(-p)\, d^3p,$$

where $\tilde{f}(p) = \int_{\mathbb{R}^4} e^{ipx} f(x)\, dx$ is the Fourier transform of f. We consider momentum space, which is the dual of $\mathbb{R}^{1,3}$. The formula $p^2 = m^2$ gives a hyperboloid of 2 sheets. We throw one away, and integrate $\tilde{f}(p)\tilde{g}(-p)$ over the remaining sheet with repsect to a Lorentz-invariant measure on the sheet. This gives some complex number which we use as the definition: That is, we put $\langle\phi^-(f), \phi^+(g)\rangle = \int \tilde{f}(p)\tilde{g}(-p)d^3p$ and use this as our $[\phi^-(f), \phi^+(g)]$.

So as above, we get a space H as $S(\phi^+(f))$ acted on by the operators $\phi^-(f)$, $\phi^+(f)$. This will essentially be our Hilbert space H (after completion). We now define $\phi(f) = \phi^-(f) + \phi^+(f)$, and we can go on to checking the Wightman axioms. Lorentz invariance is going to be a routine check. A non-trivial axiom to check is locality: Why is $[\phi(f), \phi(g)] = 0$ for f, g space-like separated?

5th lecture, September 9th 2010

Last lecture, we were constructing the free field theory. The free field operators were given by $\phi(f) = \phi^+(f) + \phi^-(f)$ with f some test function. These satisfy the relations $[\phi^+(f), \phi^+(f)] = [\phi^-(f), \phi^-(f)] = 0$ and $[\phi^-(f), \phi^+(f)] = \int_{p^2=m^2,p>0} \tilde{f}(p)\tilde{g}(-p)d^3p$ where the integral is over 1 sheet of a hyperboloid (see Fig. 3).

The main point was to check locality; that $[\phi(f), \phi(g)] = 0$ whenever f and g are spacelike separated. Writing it out, we obtain

$$\begin{aligned}[\phi(f), \phi(g)] &= [\phi^+(f) + \phi^-(f), \phi^+(g) + \phi^-(g)] \\ &= [\phi^-(g), \phi^+(f)] - [\phi^-(f), \phi^+(g)] \\ &= \pm \int \tilde{g}(p) - \tilde{f}(-p) \mp \int \tilde{f}(p)\tilde{g}(-p).\end{aligned}$$

Comparing the two integrals, we have essentially replaced p by $-p$, corresponding to the Fourier transform of the measure m as in Fig. 4. That $\phi(f), \phi(g)$ commute for f, g spacelike separating

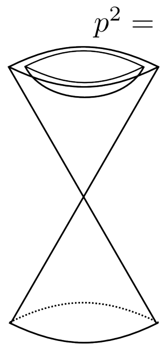

Figure 3: The hyperboloid in momentum space.

reduces to the fact that the Fourier transform of m, $\tilde{m}$, vanishes on space-like vectors. This is an exercise; there are two methods of doing it. One can calculate the Fourier transform $\int e^{ixp}\, dm$ explicitly, using Bessel functions. An easier way to do it is to consider symmetries of m. There are two obvious ones: m changes sign under $p \to -p$ and m is invariant under rotations preserving the time-direction. So the Fourier transform $\tilde{m}$ has the same properties, as the Fourier transform commutes with rotations of spacetime. Any function f with these symmetry properties vanishes outside the double cone: If a is outside the double cone, we can find a reflection σ fixing a, reversing the direction of time; $\sigma(a) = a$. This means that $f(\sigma x) = -f(x)$ for any x, and $f(a) = f(\sigma a) = -f(a)$ so $f(a) = 0$. Strictly speaking, $\tilde{m}$ is a distribution rather than a function, but a slight variation of the argument goes through.

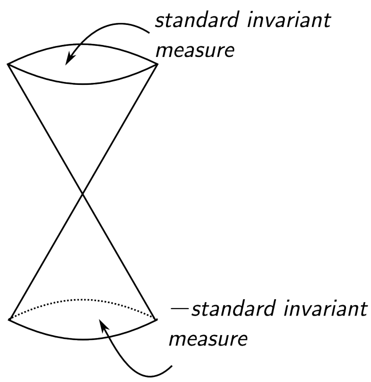

Figure 4: The measures on the hyperboloid.

What is the reason for the expression $[\phi^-(f), \phi^+(g)] = \int \tilde{f}(p)\tilde{g}(-p)$?

(1) We want translation invariance. Any translation invariant distribution can be written as $\int \tilde{f}(p)\tilde{g}(-p)d(\text{some measure})$.

(2) The positive energy condition turns out to mean that this measure should have support in the positive cone.

(3) We use Lorentz invariance to see that the measure should be Lorentz invariant.

Note that we do not have to restrict to a hypersurface $p^2 = m^2$. In fact any Lorentz invariant measure on the cone will do. This give rise to what is called "generalized free theories" (which nobody has ever found any use of).

2.2 Fixing the Wightman axioms

The problems with these axioms are, that they do not allow pertubation theory, curved spacetime, composite operators or time-ordered operators. We will try to stretch the axioms to allow these extra features. Note that we will extend them rather than changing them. First off, we will get rid of the Hilbert space to do perturbation theory. To do this we reformulate the Wightman axioms in terms of Wightman distributions (n-point functions).

2.2.1 Reformulation in terms of distributions and states

Wightman distributions are given by

$$W(f_1, f_2, \dots, f_n) = \langle \mathrm{vac} \mid \phi(f_1) \cdots \phi(f_n) \mid \mathrm{vac} \rangle.$$

The f_i are test functions on $\mathbb{R}^{1,3}$, and W is a distribution on $(\mathbb{R}^{1,3})^n$. A key point is that we can (more or less) reconstruct the QFT if we just know the Wightman distribution; if we lose the Hilbert space, this doesn't matter too much, if we just know the Wightman distributions, as we have the following theorem which is a rather sketchy version of the Wightman reconstruction theorem; for a full version see [?].

Theorem 5. *We can reconstruct the Hilbert space H from $W(f_1, \dots, f_n)$.*

Proof. Let H_1 be the space of test functions f (something like 1-particle states). Let $H_n = T^n(H_1)$ be the n'th tensor power of H_1 (which we think of as n-particle states. We then put $H = \mathbb{C} \oplus H_1 \oplus H_2 \oplus \cdots$. This will be pretty close to our Hilbert space. We will use the wightman distributions to define an inner product $\langle , \rangle$ on H. The Hilbert space of physical states will be the completion of H in the norm of this inner product. The essential part is going to be the construction of the inner product: H is spanned by elements of the form $f_1 \cdot f_2 \cdots$, $f_i \in H_1$, so we need to define $\langle f_1 \cdot \cdots \cdot f_m, g_1 \cdot \cdots \cdot g_n \rangle$. Here, m need not be n and H_i, H_j need not be orthogonal. We put

$$\langle f_1 \cdot \cdots \cdot f_m, g_1 \cdot \cdots \cdot g_n \rangle = W(f_m^*, \dots, f_1^*, g_1, \dots, g_n).$$

We have

$$\langle \phi(f_1) \cdots \phi(f_m) \mathrm{vac}, \phi(g_1) \cdot \phi(g_n) \mathrm{vac} \rangle = \langle \mathrm{vac}, \phi(f_m)^* \dots \phi(f_1)^* \phi(g_1) \cdot \phi(g_n) \mathrm{vac} \rangle.$$

□

We will now convert the Wightman axioms for H into properties of the Wightman distributions W.

- Locality becomes the requirement that

  $$W(f_1, \dots, f, g, f_{n+1}, \dots) = W(f_1, \dots, g, f, f_{n+1}, \dots),$$

  whenever f, g are spacelike separated.

- Lorentz invariance becomes requiring that W is Lorentz invariant: It should be unchanged if we act on $f_1, \dots, f_n$ by the Lorentz group.
- Positive energy becomes something about the supports of Fourier transforms of W lying on a cone (see [?]).
- Positivity becomes $\langle a, a \rangle \geq 0$ for $a \in H$. This is really important and really tiresome to write it down. We have to write $a = a_0 + a_1 + \cdots$, where $a_i \in H_i$, where a_0 is a constant, $a_1 = f_1$, $a_2 = f_{21} \cdot f_{22} \cdots$, and

  $$\langle a, a \rangle = a_0 \cdot a_0 + a_0 \cdot W(f_1) + W(f_1 \cdot f_1) + W(f_{21} \cdot f_{22} \cdot f_1) + \cdots \geq 0$$

  for all a.

- The Hermitian condition becomes $W(f_1,\dots,f_n) = W(f_n^*,\dots,f_1^*)^*$.

The main problem here is that the positivity problem is a mess, and we will reformulate the axioms a second time in order to make that specific expression a lot simpler, this time formulating the distributions in terms of *states* on Borchers algebras (note the missing d; except from having a beard, he seems to have nothing to do with Borcherds).

Let $H_1 = \Gamma_c\Phi$ be the space of all compactly supported test functions and let $H = T(H_1)$ be the tensor algebra of H_1. Furthermore, H has an involution * with the property that $(ab)^* = b^*a^*$, and $a^* = a$ for $a \in H_1$. In other words, $(a_1\cdots a_n)^* = a_n^*\cdots a_1^*$, so it's actually a *-algebra. Now the Wightman distribution $W(f_1,\dots,f_n)$ is really just a linear map $H_n \to \mathbb{C}$. So we can combine all the Wightman distributions into a linear map $\omega : H \to \mathbb{C}$. Now we translate the Wightman axioms into properties of ω.

(1) Locality: ω vanishes on the ideal generated by $f\cdot g - g\cdot f \in H_2$ for f,g spacelike separated, called the locality ideal.

(2) Lorentz invariance is obvious: The Lorentz group acts on H, so on the dual of H, invariance means that the group fixes ω.

(3) Hermitian: We define ω^* by $\omega^*(a^*) = (\omega(a))^*$. The condition that the field theory is Hermitian is just $\omega = \omega^*$; that is, $\omega(a^*) = \omega(a)^*$.

(4) Positivity of $\langle,\rangle$: This becomes $\omega(a^*a) \geq 0$ for all $a \in H$. We normalize by $\omega(1) = 1$. This condition is just that ω is a *state* in the language of operator algebras.

(5) Positive energy (omitted).

As a summary, we have reduced a quantum field theory to a (real) *-algebra H with a state ω. If we want, we can take $H/(\text{locality ideal})$. Therefore, a quantum field theory is more or less just a state on a *-algebra.

In conclusion, we now have three definitions of a QFT; in therms of the Wightman axioms of operators on Hilbert space, in terms of Wightman distributions (equivalent to the first one via the Wightman reconstruction theorem) and finally in terms of states on *-algebras. From now on we will only consider the last definition.

2.2.2 Extension of the Wightman axioms

Now, we extend to perturbation theory, curved spacetimes and so on.

Perturbation theory is easy. We extend ω to a $\omega : {}^*\text{-algebra} \to \mathbb{C}[\![\lambda]\!]$. Denote the *-algebra by H as before. (In other cases, we use other rings than $\mathbb{C}[\![\lambda]\!]$. In statistical field theory we consider as target of ω the ring $\mathbb{R}$ or $\mathbb{R}[\![\lambda]\!]$.)

We now consider the Hermitian condition and the reconstruction of physical states. As before, we require $\omega^* = \omega$. We can define $\langle,\rangle$ on H by $\langle a,b\rangle = \omega(a^*b)$. Note that H is not an algebra over $\mathbb{C}$, so we define $\langle,\rangle$ on $H\otimes_{\mathbb{R}}\mathbb{C}$ or $H\otimes_{\mathbb{R}}\mathbb{C}[\![\lambda]\!]$ – we can extend ω by linearity to these larger spaces. After doing this, the requirement that ω be Hermitian translates to the inner product being Hermitian; $\langle a,b\rangle = \langle b,a\rangle^*$.

Concerning positivity, we have $\omega(a^*a) \geq 0$ if and only if $\langle a,a\rangle \geq 0$. The problem is what $\omega(a^*a) \geq 0$ should mean in $\mathbb{C}[\![\lambda]\!]$. we can define positive elements of $\mathbb{C}[\![\lambda]\!]$ to be finite sums of elements of the form f^*f. Notice that this gives an order $\leq$ on $\mathbb{C}[\![\lambda]\!]$ but it is only a partial and not a total order; in particular $0 \nleq \lambda$ and $\lambda \nleq 0$. We have one problem: We can define the positive definite inner product $\langle,\rangle$ on H/kernel, but we cannot take Hilbert space completion! The ring $\mathbb{R}[\![\lambda]\!]$ is not Archimedean, so the theory of Hilbert spaces breaks down.

6th lecture, September 14th 2010

Last lecture we saw how to translate the Wightman axioms into properties of a map $\omega : {}^*\text{-algebra} \to \mathbb{C}$.

Locality means that ω vanishes on a certain locality ideal in I (so replace A by A/I).

To allow for curved spacetimes, take A to be $T(\Gamma_c\omega\Phi)$, the tensor algebra of compactly supported sections, so we add in the sheaf of densities, so we can take integrals.

Lorentz invariance is easy to deal with, as it is meaningless for curved spacetimes. More generally, we can ask for invariance under any subgroup G of automorphisms of the sheaf Φ: G acts on A and invariance means that G fixes ω.

Finally, we extend to time-ordered and composite operators by enlarging A to $TS\Gamma_c\omega SJ\Phi$, where the SJ part refers to composite operators and S to time-ordered operators.

So, there is a change of philosophy here – instead of writing down a list of axioms we want a quantum field theory to satisfy, we simply say that a quantum field theory is the map ω, and Lorentz-invariance, locality etc. are optional extra properties.

2.3 Free field theory revisited

We will see what this ω looks like for a free field theory. We have a 2 point function $\Delta(f,g) \in \mathbb{C}$, f, g test functions. From this we can reconstruct the free field theory. We consider $\langle \text{vac} \mid \phi(f_1)\cdots\phi(f_n) \mid \text{vac}\rangle$, where $\phi(f_i) = \phi^+(f_i) + \phi^-(f_i)$, and $\Delta(f,g) = [\phi^-(f), \phi^+(g)]$. In the above inner product, we push all ϕ^- to the right, and the expression becomes a sum of lots of products of commutators. Join all the $\phi(f_i)$ in pairs. There are $(n-1)\cdot(n-3)\cdot\dots\cdot 3\cdot 1$ ways to do this. For example, pairing $\phi(f_1)$ and $\phi(f_3)$ corresponds to a $[\phi^-(f_1), \phi^+(f_3)] = \Delta(f_1 f_3)$. We have

$$\begin{aligned}
\langle \text{vac} \mid \text{vac}\rangle &= 1 = \omega(1)\\
\langle \text{vac} \mid \phi(f_1) \mid \text{vac}\rangle &= 0 = \omega(f_1)\\
\langle \text{vac} \mid \phi(f_1)\phi(f_2) \mid \text{vac}\rangle &= \Delta(f_1 f_2) = \omega(f_1 f_2)\\
\langle \text{vac} \mid \phi(f_1)\phi(f_2)\phi(f_3) \mid \text{vac}\rangle &= 0\\
\langle \text{vac} \mid \phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4) \mid \text{vac}\rangle &= \Delta(f_1 f_2)\Delta(f_3 f_4) + \Delta(f_1 f_3)\Delta(f_2 f_4) + \Delta(f_1 f_4)\Delta(f_2 f_3)\\
&= \omega(f_1 f_2 f_3 f_4).
\end{aligned}$$

So, this gives ω: For free field theories, ω is determined by ω on $T^2(\Gamma_c\Phi)$, where it is the same as the propagator Δ: Free field theories are in correspondence with propagators Δ.

2.4 Propagators

In physics, a propagator is an amplitude for a particle to go from one point to another. In math, informally, a scalar propagator is a function $\Delta(x,y)$, x, y in spacetime. Slightly more formally, it is a distribution $\Delta(f,g)$ depending on 2 test functions, where

$$\Delta(f,g) = \iint_{M\times M} f(x)g(y)\Delta(x,y)\,dx\,dy.$$

Again this integral is meaningless, when Δ is a function, why we should consider it as a distribution.

The following is a catalogue of propagators.

(1) We can choose the dimension of spacetime $(1,2,3,\dots)$. Propagators have exceptional behavior in low dimensions $1,2$. They also behave differently in even and odd dimensions.

(2) Each propagator on flat spacetime can be viewed in either position space or, taking its Fourier transform space, in momentum space.

(3) Propagators can be massive $(m > 0)$ or massless $(m = 0)$.

(4) Propagators can exist in either Euclidean or Lorentzian space.

(5) In Lorentzian space, there are 6 important types: Advanced, retarded, two Feynman type and two cut propagators.

(6) Propagators have different spins. Important cases are $0, \frac{1}{2}, 1$ (and $\frac{3}{2}, 2$ in other applications).

The total number of cases or important propagators is $4\cdot 2\cdot 2\cdot 7\cdot 3 = 336$.

2.5 Review of properties of distributions

Take a space of test functions on $\mathbb{R}^n$. This can be smooth functions of compact support, smooth functions with all derivatives (of all orderes) rapidly decreasing (so that the product with any polynomial is bounded) such as e^{-x^2}.

A *distribution* g is a continuous linear function from the test functions to either $\mathbb{R}$ or $\mathbb{C}$ for some reasonable topology on the space of test functions. The value $g(f)$ represents $\int f(x)g(x)\,dx$, if we pretend g is a function.

Example 6. Any locally integrable function g is a distribution, for define $g(f) = \int f(x)g(x)\,dx$. Another example is the Dirac delta function δ given by $\delta(f) = f(0)$, and again this can be written as $f(0) = \int \delta(x)f(x)\,dx$, although no such function exists.

We can differentiate any distribution. We want to make sense of $\frac{dg}{dx}(f)$ where g is a distribution. If g is just a function, we should have

$$\int \frac{dg}{dx}(x)f(x)\,dx = -\int g(x)\frac{df}{dx}(x)\,dx = -g(\frac{df}{dx}),$$

so we simply define $\frac{dg}{dx}(f) = -g(\frac{df}{dx})$.

Example 7. If $g(x) = 0$ for $x \le 0$ and $g(x) = 1$ for $x > 0$, we have

$$\frac{dg}{dx}(f) = -g\left(\frac{df}{dx}\right) = -\int_0^\infty \frac{df}{dx}\,dx - [f(\infty) - f(0)] = f(0),$$

so $\frac{dg}{dx}$ is the delta distribution.

Locally any distribution looks like some high derivative of a continuous function.

It is possible to take Fourier transforms of distributions. Let S be the set of smooth functions with all derivatives rapidly decreasing. The Fourier transform $f \mapsto \tilde{f}$, $\tilde{f}(p) = \int e^{ixp}f(x)\,dx$ takes S to S. As the Fourier transform acts on S, it also acts on the dual of S, which we will call the space of tempered distribution. Informally, these increase at most polynomially at infinity.

Example 8. We will work out the Fourier transform $\tilde{\delta}$ of δ. We have

$$\tilde{\delta}(f) = \delta(\tilde{f}) = \int 1 \cdot f(x)\,dx,$$

so $\tilde{\delta} = 1$.

Fourier transforms exchange x with $\frac{d}{dx}$ as

$$\int e^{ipx}\frac{d}{dx}f(x)\,dx = -\int \frac{d}{dx}e^{ipx}f(x)\,dx = -ip\int e^{ipx}f(x)\,dx,$$

so acting on f by $\frac{d}{dx}$ acts on $\tilde{f}$ by ip (note that we might have to include some signs here, depending on convention).

2.6 Construction of a massive propagator

We will now consider the construction of a massive propagator Δ for Hermitean scalar field on flat spacetime. We will try to figure out what $\Delta(x,y)$ should be by considering the properties it should have:

(1) It should be translation invariant: $\Delta(x,y) = \Delta(x+z, y+z)$. We can use this to eliminate on of the variables, putting $\Delta(z) = \Delta(x+z, y)$ for any x. Note that this only works on flat spacetime.

(2) We want Δ to be a solution of $(\sum \pm \frac{\partial}{\partial x_i^2} - m^2)\Delta = \delta(x)$. That is, Δ is a Greens function for this operator. (The operator in question is the Klein–Gordon operator, which comes from the Euler–Lagrange equations.)

Let us solve $(\sum \frac{\partial}{\partial x_i^2} - m^2)\Delta = \delta$ in Euclidean space first. We will do this using Fourier transform as these change differentiation $\frac{dd^2}{\partial x_j^2}$ into $(ip_j)^2 = -p_j^2$, and the equation becomes

$$(\sum p_j^2 + m^2)\tilde{\Delta} = 1,$$

which has the solution $\tilde{\Delta} = 1/(\sum p_j^2 + m^2)$. The key here is that the denominator is non-zero.

$$\Delta = \widetilde{\frac{1}{\sum p_j^2 + m^2}}.$$

In fact, $\Delta(x)$ can be written as some Bessel function of (x, x).

We try the same thing in Lorentzian space. That is, we consider

$$\left(\sum \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial t^2} - m^2\right)\Delta = \pm\delta.$$

We can try to take Fourier transforms as before. We obtain

$$\left(\sum p_i^2 + m^2 - p_t^2\right)\tilde{\Delta} = 1,$$

but now we can't simply take the inverse, as the function $\sum p_i^2 + m^2 - p_t^2$ might be zero and is in

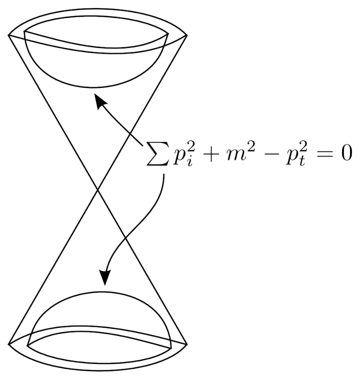

Figure 5: The sheets in question.

fact not locally integrable. We will have to make sense of this singularity, and a problem is that there are several different ways to do this. The solution of $(\sum p_i^2 + m^2 - p_t^2)\tilde{\Delta} = 1$ is not unique, as we can add to $\tilde{\Delta}$ any function on one of the sheets (with the product measure) in Fig. 5. (Another problem is the following: Inverses in a ring are unique but distributions do not form a ring under multiplication – consider the product $\frac{1}{x}x\delta$, where $\frac{1}{x} = \frac{d}{dx}\log(x)$. Now $\frac{1}{x}x = 1$, so $(\frac{1}{x}x)\delta = \delta$, and $x\delta = 0$, so $\frac{1}{x}(x\delta) = 0$, so a product is not always defined, and if it is, it is not associative.)

To make sense of the inverse, we should make sense of $f \mapsto \int f(p)/(p^2 + m^2)\,dp$, where $p^2 = \sum p_i^2 - p_t^2$. The trick is to deform the space we are integrating over into complex space $\mathbb{C}^4$ to avoid the poles as in Fig. 6. Look at the integral over p_t (for some fixed choice of p_1, p_2, p_3). Note that there are in total four reasonable ways to go around the poles. Similarly, we could come up with a lot of unreasonable ways to do so. The four different ways of doing the integral corresponds to particular propagators – the one in Fig. 6 to an advanced propagator; similarly, going under both gives a retarded propagator and going over one and other the over gives a Feynman propagator and its complex conjugate.. We can also get the cut propagators by going once around one of the poles.

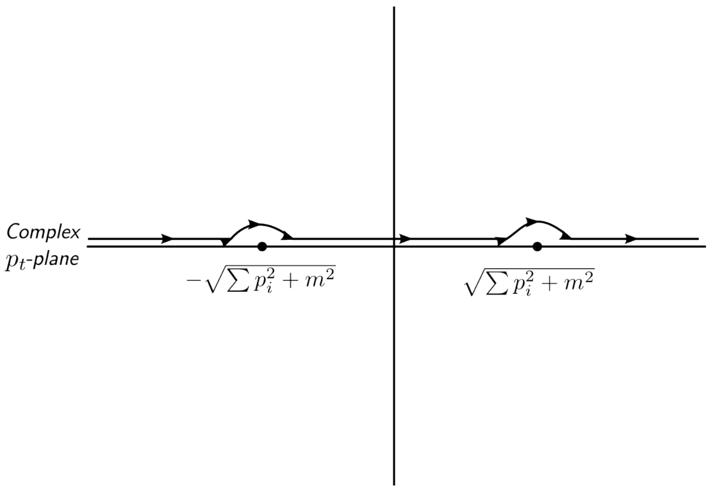

Figure 6: Deforming the integral to the complex plane.

Note that there are lots of linear relations between them (going over both corresponds to going under the first one, over the last one, and adding the curve obtained by going once around the first one). In fact the distributions obtained by the various contour integrals give a 3-dimensional space of functions; taking any four of the six propagators, they satisfy some linear relation. Note also that advanced and retarded propagators have supports inside the cones in position space.

7th lecture, September 16 2010

Last lecture we were looking at 6 propagators given by $\int \frac{1}{p^2+m^2}$ over 6 contuours to miss poles. In the complex p_t-plane, there are poles at $p_t = \pm\sqrt{m^2 - p_1^2 - p_2^2 - p_3^2}$ (see Fig. 6). We will see that advanced and retarded propagators really are advanced and retarded: That is, these have support in certain closed cones in position space; this will imply, using the linear relations, that Feynman and cut propagators coincide on certain regions.

Consider the advanced propagators. Taking the Fourier transform of the above expression, we have (up to possible sign issues)

$$\int e^{ip_t t} p_t^2 - (m^2 + p_1^2 + p_2^2 + p_3^2)\, dp_t.$$

If $t > 0$, this expression is equal to 0, as $e^{ip_t t} \to 0$ rapidly, as $\mathrm{Im}(p_t) \to \infty$ (so we can push the

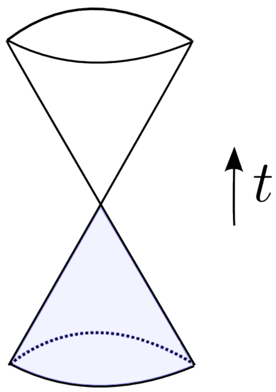

Figure 7: The retarded propagator has support in the negative cone.

contour up in the picture). If $t < 0$, it is not zero, as we can push the contour down and pick up residues at the 2 poles. In other words, the retarded (so it seems, we swap retarded and advanced from now on) propagator vanishes in position space for $t > 0$. It is also rotationally invariant under the Lorentz group, so it vanishes except in the negative cone (See Fig. 7). A useful consequence is the following: Each Feynman propagator is equal to a cut propagator outside some cone, so all Feynman/cut propagators are equal on spaceline vectors.

The following is a summary of the use of propagators:

(1) The advanced/retarded propagators are not used in QFT. They are mainly used in classical mechanics to solve the Cauchy problem etc.

(2) Feynman propagators appear in Feynman diagrams: Each line in a Feynman diagram represents a Feynman propagator, and a diagram corresponds to their product.

(3) Cut propagators are 2-point functions (i.e. Wightman distributions) of quantum field theories.

We have two problems:

(1) What is special about these 6 propagators/choices of contours? The answer is that the propagators have small wave front sets.

(2) When can we multiply operators? This is possible when wave front sets are compatible.

2.7 Wave front sets

For example, there is no way to make sense of the product of distributions $\delta(y)\cdot\delta(y)$. On the other hand, we have $\delta(x)\cdot\delta(x-1) = 0$; in general, it is easy to multiply distributions, provided their singular supports (that is, the closure of the set where the distribution is not a smooth function) are disjoint.

If a is a distribution, b a smooth function, then ab is defined by $ab(f) = a(bf)$, so there is no problem multiplying one distribution by a smooth function (as we saw last time, one has to be careful about expressions like $\frac{1}{x}\cdot x\cdot\delta$). Let γ be a curve running once around the origin and γ_1, γ_2

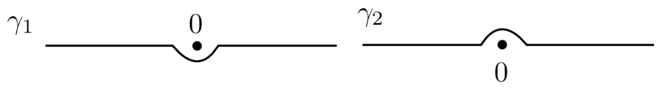

Figure 8: The curves γ_1 and γ_2.

as in Fig. 8. By definition,

$$\delta(f) = f(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z}\,dz = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(z)}{z}\,dz - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(z)}{z}\,dz.$$

The two last expressions can be considered as distributions $\frac{1}{2\pi i z}$ in the lower resp. upper half plane. Call them $(\frac{1}{z})^{mp}$. Now, $(1/z)^-$ can be multiplied by itself as often as we like – $(1/z)^{-n}$ should be $(1/z)^n$ in the lower half plane. Similarly for $(1/z)^+$, but we cant multiply $(1/z)^-$ and $(1/z)^+$, as it is not clear where to perform the integration.

We will now find a better definition of the distribution $(1/z)^+$. This should be $(1/z)^+ = \frac{d}{dz}\log(z)$, where $\log(z)$ is defined in the upper half plane (with $\log(-1) = \pi i$). Similarly we can define $(1/z)^-$ by using the lower half plane (and $\log(-1) = -\pi i$). These are well defined distributions. Note that we can sometimes multiply distributions with singularities at the same place (as we did with $(1/z)^-$ above) – so in general, when is this possible? This is possible when wave front sets go in rougly same directions.

Informally, a wave front set is something that tells you not just where the singularity is but also in what directions it is traveling. It consists (very) roughly of the hyperplanes at a point that are tangent to singularities (this is actually incorrect). It will turn out to be a conical subset of the cotangent space at each point. Here, conical means that it is closed by multiplication by reals, and cotangent spaces can be thought of as hyperplanes at the given point (we can put a Riemannian metric on the space so that contangent vectors correspond to tangent vectors).

Our first attempt to define the "direction" of singularities of a distribution f on $\mathbb{R}^n$ is the following: Recall that f is smooth, if and only if the Fourier transform $\tilde{f}$ is rapidly decreasing. This means, that if f is not smooth, then $\tilde{f}$ is not rapidly decreasing in some direction; we will use these directions to define the wave front set. Note however that we can't detect singularities at a individual point using only the Fourier transform. The solution will be to first multiply f by some cutoff function u, u a smooth function with compact support equal to 1 in a neighborhood of a point x. Then f and fu has the same singularities at the point x, but we get rid of all other singularities, and we consider instead the directions in which fu is not rapidly decreasing – for each u we get a closed subset of $(\mathbb{R}^n)^*$ of directions such that the Fourier transform $\widetilde{fu}$ is not rapidly decreasing. This depends on u but we simply take the intersection over all u with the above properties. This will be the *wave front set* of f. That is, a closed subset of the cotangent space of $\mathbb{R}^n$ (or, rather, it is i times a cotangent space).

One problem is whether or not 0 is in a wave front set. There are two conventions: We can include it, and in this case the waev front set is closed, and the wave front set of a product will be the sum of wave front sets. We can also exclude it in which case a function is non-singular at a point if and only if the wave front set is empty (and this convention seems to be standard).

Example 9. Consider the wave front set of δ. In this case we have $\delta u = \delta$ for any cutoff function, and $\widetilde{\delta u} = \tilde{\delta} = 1$, which is not rapidly decreasing in any direction, so the wave front set of δ at 0 is all vectors of cotangent space.

Example 10. Consider the distributions $(1/2\pi i x)^{\pm}$ from before. Thus we should consider the upper halfplane integral

$$\int \frac{1}{2\pi i x^+} e^{ixp} dx.$$

If $p > 0$ we get 0 (as before we can push the contour up), and if $p < 0$, we get 1. So, the Fourier transform of $(1/2\pi i x)^+$ is just a step function, and the wave fromt set is half the cotangent space. Similarly $(1/x)^-$ has the opposite wave front set.

The reason we can multiply the $(1/x)^{\pm}$ and not the delta distribution is exactly that their wave front sets are a bit smaller.

Theorem 11. *We can multiply distributions, if their wave front sets at any point are contained in the same proper cone.*

Here, proper means that it doesn't contain a $x, -x$ (for $x \neq 0$). Another way of saying this is that we cannot find $v_1, \dots, v_n$ in the wavefront sets of $f_1, \dots, f_n$ with $v_1 + \dots + v_n = 0$.)

For example, the wave front set of δ at 0 is not contained in a proper cone, but the wave front set of $(1/z)^+$ is, and we can multiply it with itself as often as we like. Similarly, we the wave front sets of $(1/z)^+$ and $(1/z)^-$ are not compatible.

"Proof". Multiplication of functions f, g corresponds under Fourier transform to taking convolutions; we have $\widetilde{fg}(p) = \int \tilde{f}(p) q \tilde{-} p \, dp$, and we can take convolutions of two functions if $\tilde{f}, \tilde{g}$ both have supports in a proper closed cone C. The reason for this is that the integral $\int \tilde{f}(p) q \tilde{-} p \, dp$ in this case is a integral of a function with compact support. Our functions don't have supports in a proper closed cone, but it is also true, if $\tilde{f}, \tilde{g}$ are rapidly decreasing outside a proper cone C and polynomial growth inside. The argument is similar, except now $\tilde{f}(p)\tilde{g}(q-p)$ is rapidly decreasing rather than 0 outside the a compact set.

We will now define fg at a point x, supposing their wave front sets are both contained in a proper closed cone. We can find a cutoff function u so that $\widetilde{fu}$ and $\widetilde{gu}$ are both smooth outside C, and the proof is to be continued next week. □

References

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