# Quantum representations for dummies

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## Contents



# 1 Introduction

As a PhD student of mathematics, I face what appears to be a common problem in our world: My friends outside of the mathematics community have no clue, what I am doing or why their tax money is being put to good use by my doing what from the outside seems to be completely detached from anything sensible. In fact, most of my friends inside the mathematics community probably don't know either. In fact, I'm not completely conviced I do myself. Now, I don't want to begin a discussion of how to reasonably spend your money – you'll have to believe me on that one. I can, however, try to give you some idea about what is going on, when I day in and day out linger in my office, dwelling upon the same piece of paper as the day before.

Even though this note looks all kinds of formal, being written in English and typeset like an ordinary mathematics article, I would like to emphasize that this is nothing close to a mathematics article. A precise formulation of my project would require an introduction to most parts of a standard education in mathematics, and my target group consists of people from outside mathematics. So rather than being precise I will try to explain the meaning behind the mathematical objects at hand. In particular, I am going to tell lies throughout the note, and I will have to ask the mathematicians to bear with me as I violate the precision that underlies the magnificense of our field. I will however also try to add some mathematical rigour when appropriate (that is, when I feel that those with just a little bit of mathematical background will benefit from it). However, anyone having trouble understanding the precise mathematical statements (which should include anyone who has never done math before) should simply skip these passages, and the note should be readable in its entirety anyway.

It is thus my hope that anyone, who survives the lengths of this note, will get some idea about what kind of machinery goes into doing modern mathematics. At the very least, I hope that non-mathematicians understand the kind of trouble they get us mathematicians into, every time they ask us what we're doing.

Feel very free to write me any comments or corrections at s@fuglede.dk. The most recent version of this note should be available at http://fuglede.dk/en/maths/notes/other/.

## 2 A short overview of this note

The title of my project is *Quantum representations of mapping class groups*. The title consists of 6 words; each of which (including the *of*), I will try to explain more or less individually. Actually, the Danish title is *Kvanterepræsentationer af afbildningsklassegrupper*, consisting of only half the number of words, so arguably it would have been easier for me to do this in Danish. Maybe I should consider just changing the title to *Kvanteafbildningsklassegrupperepræsentationer*.

The first part is devoted to the very abstract concept of a *group*. Then to accommodate the level of abstraction, I will discuss the concept of a *representation of a group*. I will then consider the *mapping class group* as an important example of a group, and finally I will explain a particular example of a representation, called the *quantum representation*, of this particular example of a group. The reader should then be able to stitch together the various concepts and to make sense of the title.

## 3 Groups and their representations

### 3.1 The concept of a group

Mathematics is all about abstraction. When faced with a problem, we strip away any unnecessary noise cluttering the problem, leaving only the central properties the problem exhibits, allowing for a much more streamlined approach to a solution. The concept of a *group* is a particularly good example of this happening. Very often in mathematics (as well as anywhere else), one comes across the problem of composing things. Here, “things” can be more or less anything, illustrating how general the concept is. As a warm-up example remember how in gymnasium (secondary school/high school), one goes to great lengths trying to understand how to compose functions into new functions – given two functions $f$ and $g$ of some variable $x$, what is $f(g(x))$? As an even more fundamental example, the first thing one learns in math is how to compose numbers – given the numbers 5 and 60 we compose these as $5 + 60$ and get a new number.

The concept of a group encompasses in full generality the idea of composition. To more precisely describe what constitutes a group, I will consider first a very abstract example. This will seem very odd at first reading, and the reader will have to bear with me here – I promise to explain what is actually going on somewhat more thoroughly in section 3.2.

In order to make sense of composition we should have something to compose. In my abstract example, what I would like to compose in various ways are 6 elements[1], that I call $R_0$, $R_1$, $R_2$, $S_0$, $S_1$, and $S_2$. Composing any two of them I would like to get another one of the elements. For example, if I compose $S_2$ and $S_1$, I would like to get $R_1$. I write this as $S_2S_1 = R_1$. If for example I compose $R_1$ and $S_0$, I would like to get $S_1$; that is, $R_1S_0 = S_1$. In general, to explain how to compose two elements, I could make a table like the following one:

| | $R_0$ | $R_1$ | $R_2$ | $S_0$ | $S_1$ | $S_2$ |
|---|---|---|---|---|---|---|
| $R_0$ | $R_0$ | $R_1$ | $R_2$ | $S_0$ | $S_1$ | $S_2$ |
| $R_1$ | $R_1$ | $R_2$ | $R_0$ | $S_1$ | $S_2$ | $S_0$ |
| $R_2$ | $R_2$ | $R_0$ | $R_1$ | $S_2$ | $S_0$ | $S_1$ |
| $S_0$ | $S_0$ | $S_2$ | $S_1$ | $R_0$ | $R_2$ | $R_1$ |
| $S_1$ | $S_1$ | $S_0$ | $S_2$ | $R_1$ | $R_0$ | $R_2$ |
| $S_2$ | $S_2$ | $S_1$ | $S_0$ | $R_2$ | $R_1$ | $R_0$ |

I should probably explain how to read this: Let’s say we want to compose $S_2$ and $S_1$. We then find the element of the table, which is in the same row as the $S_2$ on the left and in the same column as the $S_1$ in the top. This is an $R_1$, in accordance with what I wrote before: $S_2S_1 = R_0$. Similarly we find that $R_1S_0 = S_1$.

[1]Here, the term “elements” has a very precise meaning in terms of set theory. Unfortunately, set theory (which is really the fundament of all modern mathematics) turns out to be so insanely complicated that noone really bothers to think too much about it. Consequently, neither will I.

Let's take a second and see what kind of properties this table has: It seems that the order in which we compose really matters here: We have $R_1S_0 = S_1$, but $S_0R_1 = S_2$. In other words, $R_1S_0 \neq S_0R_1$. However, when we compose multiple objects, it doesn't matter if we do it "from the left" or "from the right": Say we want to compose $S_0$, $S_2$ and $R_1$. If we start with $S_0$ and $S_2$ we get $R_1$, and if we compose this $R_1$ with $R_1$, we get $R_2$. Writing this as an equality, we have $(S_0S_2)R_1 = R_1R_1 = R_2$. Instead, we could have started by composing $S_2$ and $R_1$. We see that $S_2R_1 = S_1$, and $S_0(S_2R_1) = S_0S_1 = R_2$. In other words, $(S_0S_2)R_1 = S_0(S_2R_1)$; both of them give $R_2$. This is true in general: It doesn't matter how we place the brackets, when we compute the composition of 3 objects – try checking a few yourself.

It also seems that the element $R_0$ is kind of special. No matter what we compose $R_0$ with, nothing really happens, and it doesn't even matter what order we do it in. For example, $S_1R_0 = S_1$ and $R_0S_1 = S_1$ – we just get $S_1$ back, when we compose with $R_0$. Finally, note that starting with any element, we can get to $R_0$ by composing with something: For example, if we start with $S_1$, and we compose with $S_1$ itself, we get $R_0$: $S_1S_1 = R_0$. Or if we start with $R_2$ we can get to $R_0$, since $R_1R_2 = R_0$. In this last case, the order didn't matter, and $R_2R_1 = R_0$ as well.

Let's sum up this abstract nonsense: We started out with a set of elements, and we defined a way of composing two elements to get another element. It turned out that there was a special element which corresponded to doing nothing at all: No matter how we composed it with other things, nothing happened. Finally, there is a way to invert elements: Every time I started with an element, I could compose it with another one to get the element, that didn't do anything.

Before I go on to explaining why I chose the above 6 elements, which admittedly probably seems a bit random at this moment, let me discuss another example with properties similar to the above: Consider the set of integers (*heltal* in Danish), i.e. $\dots, -2, -1, 0, 1, 2, \dots$. As in my first example I can compose two integers to get another one; for example $-2$ and 3 can be composed to make a new number $-2 + 3 = 1$. Notice that, unlike the example above, the order of composition doesn't matter here, since $a + b = b + a$ for all integers $a$ and $b$. Like before, though, when composing three elements, I can do this in two different ways, which give the same result: $(a + b) + c = a + (b + c)$. Also as before there is an element, which does nothing at all: 0. No matter how we compose 0 with another integer, we get the same integer: $a + 0 = a$ and $0 + a = a$ for all integers $a$. This corresponds to what happened with our $R_0$ before. Finally, every time we have an integer, we can find another integer such that if we compose the two, we get 0. For example, we can compose 2 with $-2$ or, in general, $a$ with $-a$.

The above two examples are very different in spirit. For example, in my first one I considered a set of 6 elements, while on the other hand there are infinitely many elements. However, both examples have compositions which exhibit the same three properties listed above: Compositions of three elements can be done in two ways giving the same result, there is an element doing nothing at all, and every element has a particular "inverse" element with composition the special element. Any set with a composition with these three properties is called a *group* (*en gruppe*, in Danish).

This corresponds well with my motivation in the introduction: If I am given a problem involving the composition of some set, I am led to consider groups. Also, considering these abstract groups, I am able to consider tons of concrete examples at once. Before going on to resolving a bit of the abstractness, let me give the precise mathematical of a group.

**Definition 1.** A *group* is a set $G$ with a composition taking two elements $x$ and $y$ in $G$ and giving a third element in $G$ denoted $xy$ satisfying the following three conditions:

1) Associativity: For any $x$, $y$ and $z$ in $G$, we have $(xy)z = x(yz)$.

2) Neutral element: There exists an element $e$ in $G$, such that $ex = xe = x$ for all $x$ in $G$.

3) Inverse elements: For every element $x$ in $G$ there exists an element $y$ in $G$ such that $xy = yx = e$.

## 3.2 Another way of representing groups

As motivated above, it should be worthwhile considering general abstract groups to see what kind of properties they might have. Often in practical cases, it turns out to be a lot easier to translate the language of groups to the language of what we call linear algebra. Let me illustrate this by a concrete example.

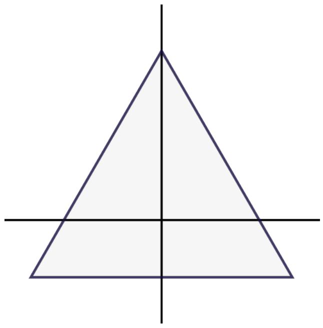

Figure 1: A triangle in the plane.

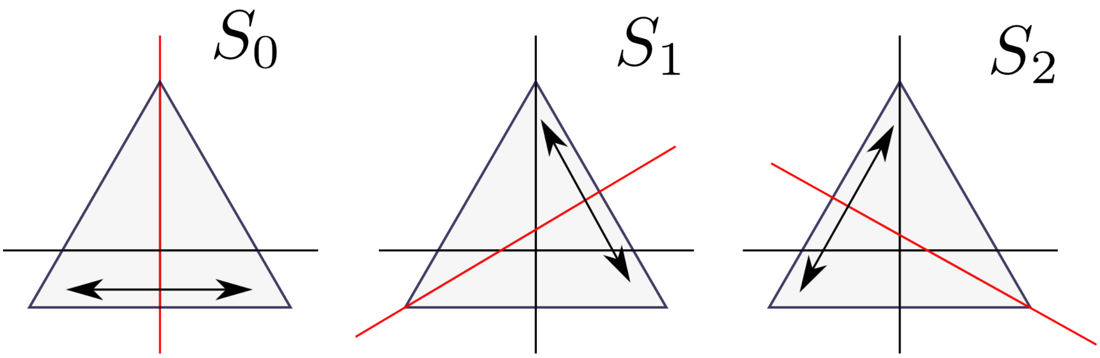


Figure 2: Three reflections of the triangle.

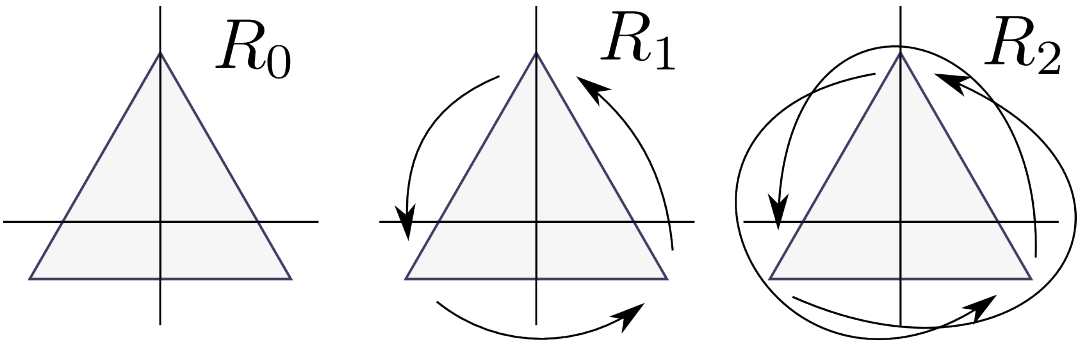


Figure 3: Three rotations of the triangle.

Consider again the set consisting of the elements $R_0, R_1, R_2, S_0, S_1, S_2$ as in section 3.1. We saw that the elements constitute a group, but the composition seemed somewhat arbitrary at first. Let me try to remedy this. Consider a triangle in the plane as in Fig. 1. The triangle has a bunch of rotational and mirror symmetries: If for example I reflect the triangle in the vertical axis, I get exactly the same triangle back; see the left figure in Fig. 2. Two other possible axes of reflections are also shown in Fig. 2 – reflecting the triangle in either gives me back my triangle.

I could also rotate the triangle by either 0 degrees, 120 degrees or 240 degrees as in Fig. 3. These rotations all give me back my triangle. Denote these six symmetries by $R_0, R_1, R_2, S_0, S_1$, and $S_2$ as in the figures. Now I could start composing these various operations. Note that $R_0$, the rotation by 0 degrees, does nothing at all. One possible composition of operations would be composing $S_0$ with itself. That is, I reflect the triangle in the vertical axis, and afterwards I reflect it back again. This corresponds to doing nothing at all; in other words, $S_0S_0 = R_0$. Similarly $S_1S_1 = R_0$ and $S_2S_2 = R_0$. Another possibility would be to rotate the triangle by 120 degrees twice. This is the same as rotating 240 degrees. In other words, $R_1R_1 = R_2$. At this point you should start comparing the compositions with the table in section 3.1. It turns out to be exactly what I get, when I compose the symmetries of the triangle. In particular, we could consider compositions of three operations, and the way we put the brackets will not make any difference. Similarly, it is completely clear that every operation has an inverse: The reflections can be undone by reflecting back, and the rotations can be undone by further rotation.

So, let's review what we did here. To every element of the group, we associated a particular operation on the plane (in this case reflections and rotations preserving a triangle) such that the associated operations had the same composition as in the group. Doing this, we realize the group as what we call *linear* operations on the plane, which might in some cases be easier to understand. We call this procedure a *representation of the group*.[2]

This representation is a bit special in some sense: It completely preserves all information about the composition in the group. To illustrate what I mean by this, consider instead this representation: To every element of the group, I associate not the reflections/rotations as above, but the operation which doesn't do anything at all. This is a representation (see the precise definition below), but this time around we don't really learn anything from it, as all compositions in the picture with the plane become boring: All operations are trivial, and so are all compositions. In this case we say that the representation has a non-trivial *kernel*: An element of the group is said to be in the kernel of the representation if it ends up being represented by a trivial operation on the plane, even though it is not trivial itself (like $R_0$ was). In other words, if some element of the group is in the kernel, we can't see it in the representation. Kernels play a central role in the theory of representations (and in mathematics in general), as they more or less determine how much information is lost when translating from the group picture to the linear picture.

In all of the above, I chose to represent my group as operations on the plane. There is really nothing special about the plane, and I might as well have used linear operations on 3-dimensional space or even spaces of higher dimensions. In mathematics, the relevant concept turns out to be that of a *vector space*, which is a natural generalization of the concept of $n$-dimensional space (just as the group was a natural generalization of the concept of having a composition). This is explained more precisely in the next section, which might seem daunting to the non-mathematican.

## 3.3 Precise statements

As hinted above, a vector space is the natural way to consider vectors. Recall that the gymnasium intuition about vectors is something like "vectors are some things we can add together or scale". Trying to put that into a mathematical framework, we get the following.

**Definition 2.** A real vector space[3] is a set $V$ together with two operators $+$ and $\cdot$. Elements of $V$ are called vectors. The operator $+$ takes two vectors and gives another vector. The operator $\cdot$ takes a real number $a$, a vector $v$ and gives a vector $a \cdot v$. The operators satisfy the following axioms, where $u, v, w$ are vectors and $a, b$ are real numbers:

- Associativity of addition: $u + (v + w) = (u + v) + w$.
- Commutativity of addition: $u + v = v + u$.

[2] To be honest, this is a bit backwards. The group consisting of the elements $R_0, R_1, R_2, S_0, S_1$, and $S_2$ is of interest *because* it is the group of symmetries of the triangle – I sort of presented it the other way around. This group, by the way, is called the dihedral group (diedergruppen in Danish) $D_3$ – see http://en.wikipedia.org/wiki/Dihedral_group for more nice drawings and explanations.

[3] See also http://en.wikipedia.org/wiki/Vector_space for a thorough exposition of the subject.

- Identity element of addition: There exists an element 0 of $V$, such that $v + 0 = v$ for all vectors $v$.
- Inverse element of addition: For every vector $v$, there exists a vector $w$ such that $v + w = 0$.
- Distributivity: $a \cdot (v + w) = a \cdot v + a \cdot w$.
- Distributivity: $(a + b) \cdot v = a \cdot v + b \cdot v$.
- Compatibility: $(ab) \cdot v = a \cdot (b \cdot v)$.
- Identity element of multiplication: $1 \cdot v = v$.

Usually, we just write $av$ for $a \cdot v$.

Notice that all of these make perfect sense, if $v, w, u$ are simply vectors of the plane. In general the set $\mathbb{R}^n$ consisting of elements of the form $(a_1, a_2, \dots, a_n)$, where all the $a_i$ are real numbers, form a vector space. For our purposes, it will suffice to think of this particular vector space, and as students of linear algebra know, in general any finite-dimensional vector space can be thought of as $\mathbb{R}^n$ (by choosing a basis).

We will now do the translation from groups to vector spaces, which is given by the notion of a representation. Before doing so, we will discuss what it means for a map or a function to "preserve a structure". Remember, that if $f$ is a function taking elements of a set $A$ and giving elements of a set $B$, we will usually just write $f : A \to B$. We will consider maps from groups to other groups.

**Definition 3.** A *group homomorphism* between two groups $G$ and $H$ is a map $f : G \to H$ such that $f(gh) = f(g)f(h)$ for all $g$ and $h$ in $G$.

So how should one think of such a map as "preserving structures"? Note first that in the equality $f(gh) = f(g)f(h)$ we have two group compositions involved. On the left hand side, $g$ and $h$ are elements of $G$, so $gh$ means taking the composition in $G$. On the right hand side, the elements $f(g)$ and $f(h)$ are elements of $H$, so $f(g)f(h)$ means taking the composition in $H$. That $f$ is a group homomorphism then means that $f$ takes the composition in $G$ to the composition in $H$. Now, the composition is really what makes a group, so it is be a natural thing to consider maps preserving this composition. I will give an explicit example of a group homomorphism in the end of this section.

In a similar way, we could consider the maps preserving the structure carried by vector spaces. This gives rise to the following definition:

**Definition 4.** A *linear map* between two vector spaces $V$ and $W$ is a map $f : V \to W$ such that $f(a \cdot v + b \cdot w) = a \cdot f(v) + b \cdot f(w)$.

Exactly as above, linear maps are the ones that take the vector space structure – that is, the addition and multiplication operators – from $V$ to the vector space structure on $W$.[4] For example, linear maps on the vector space $\mathbb{R}$ of real numbers to itself are exactly the maps $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax$ for some real number $a$.

Next up is trying to translate the world of vector spaces and linear maps into the world of groups. To do that, we need one final notion about linear maps: Invertibility. This reflects the invertibility in the group language. More precisely, we say that a linear map $f : V \to W$ between two vectors spaces $V$ and $W$ is invertible, if there exists a linear map $g : W \to V$, such that $f \circ g$ is the identity on $V$ and $g \circ f$ is the identity on $W$ – that is, $g(f(v)) = v$ for all $v$ in $V$ and $f(g(w)) = w$ for all $w$ in $W$. In other words, we are able to invert whatever we do with the functions $f$ and $g$. For example the map $f : \mathbb{R} \to \mathbb{R}$ considered above, $f(x) = ax$, is invertible if $a \neq 0$, but if $a = 0$, it is not.

Let $GL(V)$ denote the set of all invertible maps $V \to V$. The following proposition is left as an exercise to whoever got this far.

[4] Of course, all this structure business is also generalized in mathematics, and so-called category theory makes more precise what it means to preserve structure etc. – see http://en.wikipedia.org/wiki/Category_theory.

**Proposition 5.** *The set $GL(V)$ is a group, where the composition is given by composition of linear maps, the neutral element is given by the identity $V \to V$ and the inverse of a map $f$ is the map $g$ from the definition of $f$ being invertible.*

Finally, we are now able to say what a representation is.

**Definition 6.** A representation of a group $G$ on a vector space $V$ is a homomorphism $G \to GL(V)$.

Note that this definition kind of makes sense: From the beginning I wanted to associate to every element of $G$ a linear map. This is exactly what I'm doing here. Also, this map should somehow preserve the composition from $G$, which is why we require it to be a homomorphism.

In the case where $V$ is $\mathbb{R}^n$ for some $n$, we can consider this a bit more concretely. Remember that any linear map on $\mathbb{R}^n$ (or on a vector space with a given basis) can be realized as a $n \times n$-matrix[5]. That is, in this case a representation is a map from a group $G$ to the set of invertible $n \times n$-matrices. This set, by the way, is often denoted $GL(n)$.

Let's again consider the example from before. Let $D_3$ be the group with the six elements $R_0, R_1, R_2, S_0, S_1, S_2$ from before. We will describe a representation of this group on $\mathbb{R}^2$ corresponding exactly to my pictures above. By my above remark, it should be enough to associate to every element of the group a $2 \times 2$-matrix in a way that preserves the group structure. We do this as follows: Define a map $f : D_3 \to GL(2)$ by

$$\begin{aligned}
f(R_0) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
f(R_1) &= \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, \\
f(R_2) &= \begin{pmatrix} \cos(4\pi/3) & -\sin(4\pi/3) \\ \sin(4\pi/3) & \cos(4\pi/3) \end{pmatrix}, \\
f(S_0) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
f(S_1) &= \begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ \sin(2\pi/3) & -\cos(2\pi/3) \end{pmatrix}, \\
f(S_2) &= \begin{pmatrix} \cos(4\pi/3) & \sin(4\pi/3) \\ \sin(4\pi/3) & -\cos(4\pi/3) \end{pmatrix},
\end{aligned}$$

This might look a bit complicated, but really these matrices are nothing but the matrix representations of the rotations and reflections considered above. One can use these formulas directly to check that $f$ is indeed a group homomorphism. For example

$$f(S_0S_0) = f(R_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = f(S_0)f(S_0).$$

Again I leave it to the reader to check the other 34 equalities.

So, to conclude, I have described precisely how one can describe groups by instead describing considering the elements of the group as maps doing something to a vector space. Let me end up by describing precisely the notion of a kernel.

**Definition 7.** The kernel of a group homomorphism $f : G \to H$ are the elements in $G$ being mapped to the neutral element in $H$.

If $H$ is the group of invertible linear maps on $\mathbb{R}^n$ (that is, $GL(n)$), then the kernel is the set of elements in $G$ being mapped to the identity matrix. In our example above, the kernel of

[5] Up to this point, I didn't really step outside gymnasium territory, but I guess I am now, so if you don't know what a matrix is, just nod and smile.

$f : D_3 \to GL(2)$ is the element $R_0$ and nothing else. In general, however, the kernel might be more complicated. Here is an example using the same group:

Consider the map $g : D_3 \to GL(1)$ given by $g(R_0) = [1]$, $g(R_1) = [1]$, $g(R_2) = [1]$, and $g(S_0) = [-1]$, $g(S_1) = [-1]$, $g(S_2) = [-1]$. So the $R_i$ are represented by [1] and the $S_i$ by $[-1]$. Notice again that this is a representation: If we compose an $R$ with an $R$, we get an $R$. If we compose an $S$ with an $R$, we get an $S$, and so on. Here, we lose some information though: Using only the representation, we can't distinguish the individual $R$ or $S$, but we can say whether we have an $R$ or an $S$. In this particular case, the kernel of the representation $g$ consists of the elements $R_0$, $R_1$ and $R_2$ since these are the ones being mapped to the neutral element [1] in $GL(1)$.

We will not really use the kernels of linear maps in the following, but in fact one part of my project is trying to find the kernels of particular representations, as I will describe below.

## 4 The mapping class group

## 5 Topological quantum field theory and quantum representations

## 6 Conclusion