Quantum representations of mapping class groups and TQFTs for newcomers

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Abstract

Roughly, the mapping class group of a given surface can be seen as the symmetry group of the surface in question. In this talk, I will give several examples of mapping classes and see how so-called quantum representations of mapping class groups might enlighten us on the structure of the groups. Furthermore, I will discuss how they relate to and arise from topological quantum field theories.

Contents

1 Introduction

These are notes I used for a talk with the same title at University of California, Berkeley, during the fall of 2010. The talk is aimed towards a very general audience, and anyone with what corresponds to a typical bachelor's degree in mathematics should be able to follow the talk. Readers who want a more precise idea about what is going on might want to consider the excellent exposition [Mas03], which has inspired large parts of this note, while non-mathematicians who have no clue about what is going on might want to look up my short note named *Quantum representations for dummies* – all of my notes should be readily available at `http://home.imf.au.dk/pred`. Please feel very free to send me corrections and comments on the content.

2 Mapping class groups

Some of the first group one encounters in mathematics are symmetry groups of various geometrical objects; that is, groups of automorphisms of the given objects. We can make a similar construction for surfaces. Let $\Sigma_{g,d}$ denote an oriented surface of genus g with d marked points (or, equivalently, punctures). Denote by $\text{Homeo}(\Sigma_{g,d})$ the set of orientation preserving homeomorphisms from $\Sigma_{g,d}$ to itself, preserving the marked points setwise. This group is way too large for our purposes, and we want to remove the homeomorphisms that don't contribute with any knowledge from a topological point of view: Recall that two homeomorphisms $f_0, f_1 : \Sigma_{g,d} \to \Sigma_{g,d}$ are called isotopic, if there is a homotopy f_t, $0 \le t \le 1$ between them such that each f_t is a homeomorphism. Let $\text{Homeo}_0(\Sigma_{g,d})$ denote the normal subgroup of $\text{Homeo}(\Sigma_{g,d})$ consisting of diffeomorphisms of $\Sigma_{g,d}$ isotopic to the identity and define the *mapping class group*

$$M_{g,d} = \text{Homeo}(\Sigma_{g,d})/\text{Homeo}_0(\Sigma_{g,d})$$

with group structure given by composition of maps. In other words, $M_{g,d}$ is the set of orientation preserving homeomorphisms of $\Sigma_{g,d}$ considered up to isotopy. If the surface has boundary, one typically requires the homeomorphisms to fix the boundary pointwise, and the isotopy to be relative to the boundary. Let's not worry too much about that; we will mostly be dealing with closed surfaces.

2.1 Examples of mapping classes

We describe now the so-called *Dehn twists* of a surface. Let γ denote a simple closed curve on the surface Σ. Then the Dehn twist acts on Σ as the identity away from an annulus neighbourhood of γ, and on this neighbourhood it acts as in Fig. 1.

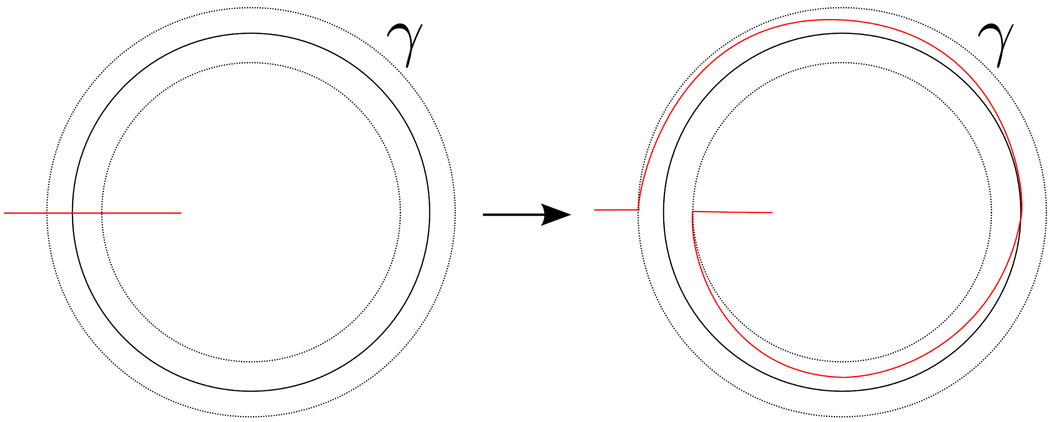

Figure 1: The action of a Dehn twist on a small line piece intersecting the curve γ.

Let us consider the torus $\Sigma_{1,0}$ with its two homology generators α the longitude and β the meridian as in Fig. 2. Considering β as a simple closed curve, we will describe the action of the Dehn twist t_β on α and β. This is illustrated in Fig. 3: The Dehn twist t_β leaves β invariant and twists α once around the meridian. Similarly a Dehn twist along the longitude would leave invariant the longitude and twist the meridian. The importance of the Dehn twists is due to the following theorems.

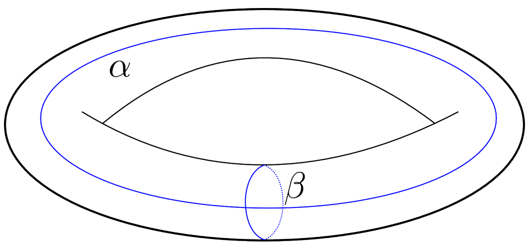

Figure 2: A torus with the two generators of homology, α and β.

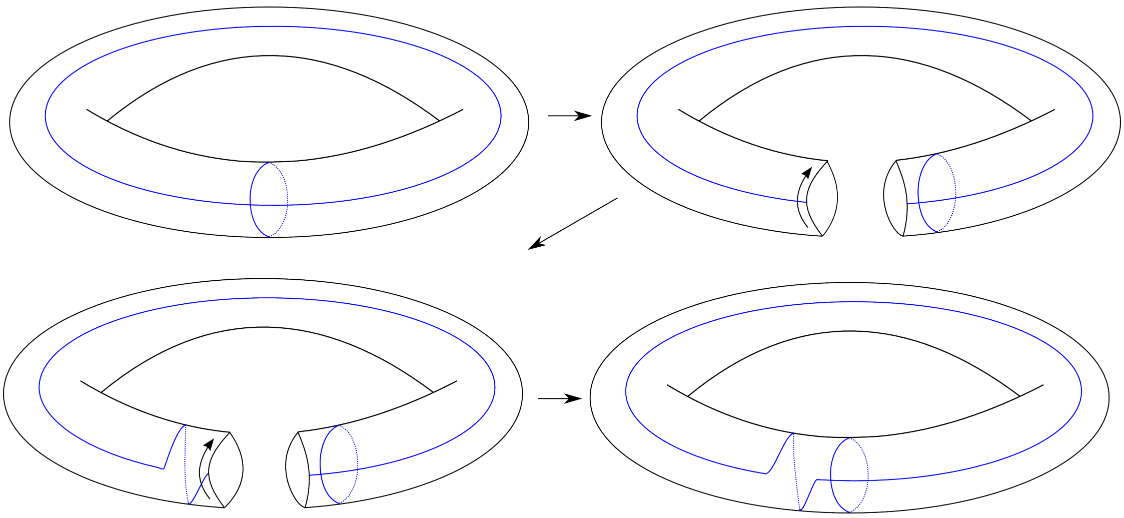

Figure 3: The Dehn twist along a curve can be seen as cutting up the surface along that curve, making a full twist, a putting the surface back together.

Theorem 1 (Dehn)**.** *The Dehn twists of a closed oriented genus g surface $\Sigma_{g,0}$ generate $M_{g,0}$.*

Even better, the mapping class group is finitely generated.

Theorem 2 (The Lickorish twist theorem, (Lickorish))**.** *The Dehn twists of $3g-1$ curves generate $\Sigma_{g,0}$ (see Fig. 4).*

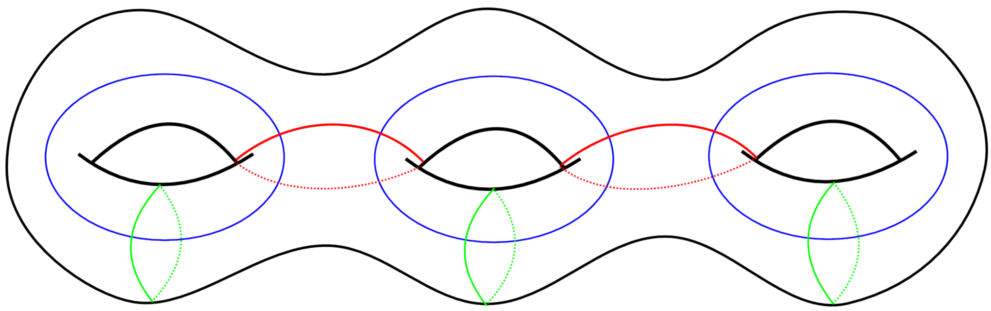

Figure 4: The $3g-1$ curves in the Lickorish twist theorem shown for $g=3$.

In fact, the minimal number of curves necessary has turned out to be $2g+1$ (for $g>1$), as was shown in [Hum79].

It is tempting to assume that every mapping class is determined by its action on isotopy classes of simple closed curves. The following example shows that this isn't completely true though: Consider again the torus; now as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. The map $x \mapsto -x$ on $\mathbb{R}^2$ preserves the lattice $\mathbb{Z}^2$ and induces a map on the torus. This induced map is the so-called hyperelliptic involution preserving four points on the torus with the property that the quotient of the torus by the map is a sphere; even though it preserves the curves, it is *not* isotopic to the identity. For closed surfaces, this is the only thing that goes wrong though: The action on closed curves determines a mapping class up to an element of the center of the mapping class group. The center is trivial for $g>2$ and is generated by hyperelliptic involution for $g=1,2$ (here, hyperelliptic involution on $\Sigma_{2,0}$ is a map with 6 fixed points, such that the quotient of $\Sigma_{2,0}$ with the map is a sphere). Here is a sketch of the proof: For $f \in M_{g,0}$ and γ a simple closed curve, we have $ft_\gamma f^{-1} = t_{\gamma'}$, where $\gamma' = f(\gamma)$. Thus f acts trivially on isotopy classes of all simple closed curves if and only if f is in the center of $M_{g,0}$, since the Dehn twists generate $M_{g,0}$.

Considering again the torus, every element of $SL(2,\mathbb{Z})$ determines a lattice-preserving map on $\mathbb{R}^2$, and it turns out that this map $SL(2,\mathbb{Z}) \to M_{1,0}$ is in fact an isomorphism.

2.2 The Nielsen–Thurston classification

In the above we saw some examples of mapping classes. The following classification of mapping classes was realized by William P. Thurston and completes work done by Jakob Nielsen in the 1930's. A typically cited reference is [FLP79]. Consider in the following a closed oriented surface of genus $g \geq 2$.

A mapping class is called *pseudo-Anosov*, if there is a pair F_s, F_u of transverse "measured" foliations and a representative f of the mapping class acting as $f_*(F_u) = \lambda F_u$, $f_*(F_s) = \lambda^{-1} F_s$ with $\lambda > 1$. Here, $_s$ is for stable and $_u$ for unstable. See Fig. 5.

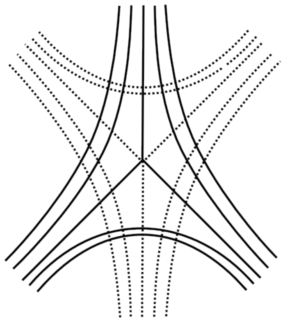

Figure 5: A pair of (singular) transverse foliations. Take for example the dotted line to be F_s and the other one to be F_u.

A mapping class is called *reducible* if some power of it preserves the non-trivial homotopy class of a simple closed curve on the surface.

We then have the following theorem classifying mapping classes.

Theorem 3. *A mapping class ϕ has exactly one of the following properties.*

- *The mapping class ϕ has finite order in the mapping class group.*
- *The mapping class ϕ is not finite order, but is reducible.*
- *The mapping class ϕ is pseudo-Anosov.*

In the pseudo-Anosov case, the stretching factor λ does not depend on the representative of ϕ. In the reducible case, one cuts the surface along a family of simple closed curves being preserved by a representative of some power of ϕ, and one continues the analysis on each component.

3 Representations of mapping class groups

We have above considered examples of the behaviour of mapping classes of surfaces and turn now to the representation theory of mapping class groups. In particular we will see how well the so-called quantum representations preserve the above mentioned structures.

3.1 The homology representation

Before going on to topological quantum field theories, let us consider the classical *homology representation* of the mapping class group. Recall that the homology $H_1(\Sigma_{g,0};\mathbb{Z})$ is generated by elements a_i, b_i, $i = 1,\ldots,g$. On $H_1(\Sigma_{g,0},\mathbb{Z})$ we have the intersection form given by $\langle a_i, b_j\rangle = -\langle b_j, a_i\rangle = \delta_{ij}$ and $\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$. It turns out that this skew-symmetric form is preserved by the mapping class group, which determines a representation $M_{g,0} \to \mathrm{Sp}(H_1(\Sigma_{g,0},\mathbb{Z})) =$

$\mathrm{Sp}(2g, \mathbb{Z})$. Furthermore, this map is surjective. We also have $\mathrm{Sp}(2, \mathbb{Z}) = SL(2, \mathbb{Z})$, and in the case $g = 1$, the homology representation is simply the isomorphism $M_{1,0} \cong SL(2, \mathbb{Z})$ discussed above.

3.2 Topological quantum field theories and quantum representations

In general the term "quantum representation" will be used for various constructions of countable families of representations of mapping class groups, each of which have in a sense for a "classical limit". Some of these can be to fit into the language of topological quantum field theories.

According to Wikipedia, a *topological quantum field theory* (TQFT) is a quantum field theory computing topological invariants. This is so vague that it's precise, and there seems to be no one widely used definition. The following discussion is based on the treatment in [Tur94] – a brilliant book outlining the main components in an abstract setting as well as containing a few constructions of TQFTs. The term was first introduced by Witten (see [Wit88]) and later axiomatized and constructed by a number of people – see for example [Ati88]. Time permitting, we will consider concrete constructions in this talk.

For our purposes, a TQFT will consist of the following data:

- To every closed oriented surface Σ we associate a vector space $V(\Sigma)$ over a field K, and to every homeomorphism $f : \Sigma_1 \to \Sigma_2$ we associate a vector space isomorphism $V(f) : V(\Sigma_1) \to V(\Sigma_2)$.
- To every cobordism M between two surfaces $\partial_-(M)$ and $\partial_+(M)$, we associate a linear map $Z_M : Z(\partial_-(M)) \to Z(\partial_+(M))$.

These associations should satisfy the following axioms:

- The vector spaces $V(\Sigma)$ and linear maps $V(f)$ should behave nicely with respect to disjoint union and composition; in particular $Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$, and we put $Z(\emptyset) = K$. See [Tur94] for details.
- Naturality: If $f : M_1 \to M_2$ is a homeomorphism of cobordisms preserving the boundaries, we have a commutative diagram

$$\begin{array}{ccc} Z(\partial_-(M_1)) & \xrightarrow{Z_{M_1}} & Z(\partial_+(M_1)) \\ {\scriptstyle V(f|_{\partial_-(M_1)})}\downarrow & & \downarrow{\scriptstyle V(f|_{\partial_+(M_1)})} \\ Z(\partial_-(M_2)) & \xrightarrow{Z_{M_2}} & Z(\partial_+(M_2)). \end{array}$$

- Multiplicativity: If $M = M_1 \sqcup M_2$, then $Z_M = Z_{M_1} \otimes Z_{M_2}$.
- The gluing axiom (see Fig. 6): If M is obtained from M_1 and M_2 by gluing together M_1 and M_2 along boundary components by a homeomorphism $f : \partial_+(M_1) \to \partial_-(M_2)$, then

$$Z_M = k Z_{M_2} \circ V(f) \circ Z_{M_1}$$

  for some invertible $k \in K$ called the *gluing anomaly* (depending on f, M_1, M_2).
- Normalization: Considering $\Sigma \times [0, 1]$ as a cobordism from Σ to itself, we have $Z_{\Sigma \times [0,1]} = \mathrm{id}_{V(\Sigma)}$.

Thus a TQFT can be viewed as a functor from a cobordism category to the category of vector spaces over K. Far-reaching generalizations of the above definition exist: For example, in [Tur94] the functor can be any modular functor associating projective K-modules to surfaces for some unital commutative ring K.

Two special cases are of importance. If $\partial M = \Sigma$, then Z_M can be viewed as a vector in $V(\Sigma)$, and if $\partial M = \emptyset$, we define $Z(M) = Z_M \in K$. Thus, a TQFT gives rise to an invariant of closed 3-manifolds. Imposing a few extra conditions on the TQFT – that it be "non-degenerate"

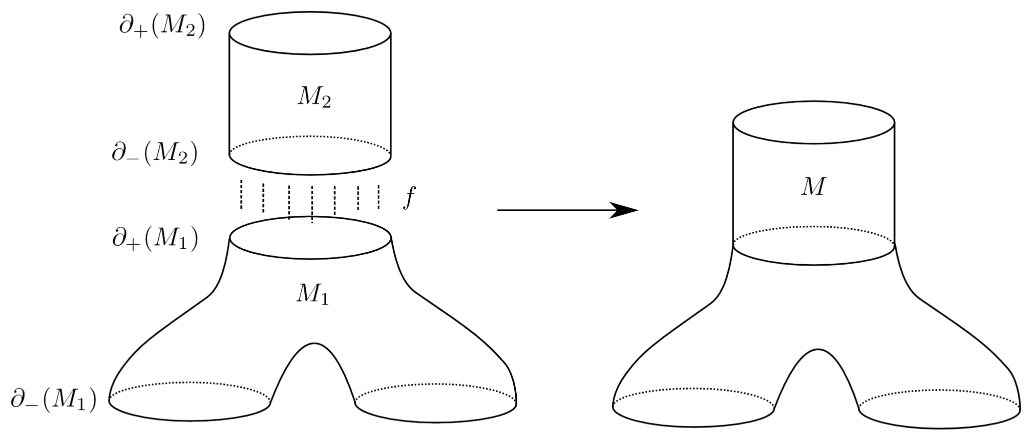

Figure 6: The gluing of two manifolds gives rise to a third.

(essentially meaning that for every surface Σ, the vector space $V(\Sigma)$ is spanned by elements $\tau(M)$ with $\partial M = \Sigma$) and without anomalies – then this invariant is a so-called *quantum invariant* (meaning again that it satisfies some extra axioms). It turns out that quantum invariants are in bijective correspondence with isomorphism classes (which I haven't introduced) of non-degenerate anomaly-free TQFTs.

We are now in a position to describe how TQFTs give rise to representations of the mapping class group of a given surface. When working it out, it seems that one always has to fix certain extra auxiliary structures on the surfaces involved, but the idea is rather simple: For a given surface Σ and a homeomorphism $g : \Sigma \to \Sigma$, consider the cylinder $\Sigma \times [0,1]$, where $\Sigma \times \{0\}$ is parametrized by g, and $\Sigma \times \{1\}$ is parametrized by id_Σ. Doing this, we get a cobordism $M(g)$ between the two copies of Σ and therefore a linear map $Z_{M(g)} : V(\Sigma) \to V(\Sigma)$. It turns out that this linear map depends only on the isotopy class of g. Thus it gives rise to a map ρ associating to every mapping class a linear map $V(\Sigma) \to V(\Sigma)$. However, due to the anomalies this is not quite a representation; rather, we only obtain a projective representation meaning that $\rho(gh) = C\rho(g)\rho(h)$ for some $C \in K$.

3.3 Construction of representations

So far, we have only discussed in generality how one might expect to obtain representations from topological quantum field theories. The construction of interesting representations is a non-trivial matter. In the following we consider three logically independent constructions. Each of these constructions give rise to an infinite number of representations ρ_k on spaces V_k, where k is called the *level* of the representation. It should be noted that many more constructions exist. The constructions are rather technical and are not necessary for understanding the last subsection of this note.

3.3.1 Quantum group approach

A streamlined introduction of the following, containing all basics, is available in [Tur94]. In the quantum group approach, a TQFT is constructed by first considering the graphical calculus of ribbons coloured with elements of a certain ribbon category (meaning a *monoidal* category with a certain *braiding*, *twist*, and *duality*). Using this, one might create invariants of links (or rather, ribbons), which may be further refined to give rise to invariants of 3-manifolds containing links. It turns out that this construction works whenever we are given a modular category. In general, a modular category is a ribbon Ab-category (see below) containing a collection of distinct so-called *simple* objects, and the TQFT functor associates to every surface a module of homomorphisms of certain tensor products of these simple objects. The TQFT invariant will be described in terms of the 3-manifold invariant coming from the graphical calculus. I will not here describe in detail the generalities going into the various definitions above. Instead, let us consider how the quantum groups enter the picture.

For any given Hopf algebra A over a commutative unital ring K, one might consider the category of representations of A, denoted $\mathrm{Rep}(A)$, with objects being finite rank A-modules. This category is *monoidal*, meaning that tensor products of objects make sense, and it is an Ab-category in the sense that any set of homomorphisms $\mathrm{Hom}(V,W)$, for V and W objects in $\mathrm{Rep}(A)$, is an abelian group with bilinear composition. Furthermore, the category has a natural duality pairing. To provide this algebra with a braiding, it should further have the structure of a *quasitriangular* Hopf algebra; this means that it has a distinct element $R \in A^{\otimes 2}$ (often refered to as an R-matrix) satisfying several equalities. Finally, to get a twist in $\mathrm{Rep}(A)$, one fixes an element v in the center of A satisfying again particular equalities. With all of these in place, $\mathrm{Rep}(A)$ aquires the structure of a ribbon Ab-category. For the representation category to be a *modular* category, there should exist a finite collection of finite rank A-modules $\{V_i\}_{i\in I}$ that are *simple*, in the sense that their only endomorphisms are multiplications by scalars and that they satisfy the following:

- For some element $0 \in I$, we have $V_0 = K$ (where A acts by the Hopf algebra counit).
- For every $i \in I$, there exists $i^* \in I$ so that V_{i^*} and $(V_i)^*$ are isomorphic.
- For every $k, l \in I$ the tensor product $V_k \otimes V_l$ splits as a finite direct sum of V_i, $i \in I$ and a module V satisfying $\mathrm{tr}_q(f) = 0$ for any $f \in \mathrm{End}(V)$. Here, tr_q denotes the so-called quantum trace of an endomorphism.
- Denoting by $S_{i,j}$ the quantum trace of $x \mapsto \mathrm{flip}(R)Rx$ on $V_i \otimes V_j$, we obtain an invertible matrix $[S_{i,j}]_{i,j\in I}$.

With all of these in place, we are almost good to go – if A-modules as above exist, then $\mathrm{Rep}(A)$ has a quasimodular subcategory. Here, quasi- denotes yet a technical difficulty, we won't consider here. It is known that any quasimodular category gives rise to a modular category in a canonical way.

It thus remains to construct Hopf algebras satisfying all of the above conditions. These arise in the language of quantum groups. While quantum groups can be defined for general simple Lie algebras, we consider only $\mathfrak{sl}_2$ for which the construction is known to work. The quantum group $U_q\mathfrak{sl}_2$ is defined to be the algebra over $\mathbb{C}$ generated by elements K, K^{-1}, E, F with relations

$$K^{-1}K = KK^{-1} = 1$$
$$KE = q^{-1}EK, \quad KF = qFK$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

Assume for simplicity that q is a primitive l'th root of unity; this turns out to be the most simple setting, and in general one cannot hope for the construction to work. The quantum group $U_q\mathfrak{sl}_2$ is not quite good enough for our purposes, and we consider instead the quotient $\tilde{U}_q\mathfrak{sl}_2$ of $U_q\mathfrak{sl}_2$ by the two-sided ideal generated by $E^{l'}, F^{l'}, K^l - 1$, where $l' = l$, if l is odd, and $l' = l/2$, if l is even. Then $\tilde{U}_q\mathfrak{sl}_2$ can be endowed with an R-matrix and a twist turning it into a ribbon Hopf algebra. Now the role of the simple modules described above will be played by certain irreducible $\tilde{U}_q\mathfrak{sl}_2$-modules. The construction for $\mathfrak{sl}_2$ was first done in [RT90], [RT91], and has since be generalized to other Lie algebras.

3.3.2 Skein theory approach

While the above constructions have a strong algebraic flavour, the following is considerably more combinatorial. It was first put forward in [BHMV95]. Rather than describing the quantum representations in terms of a TQFT (which is possible), we describe the vector spaces and corresponding maps more or less explicitly. The description comes from [Mas03], and is due in part to Roberts (see [Rob94]).

Given a 3-manifold M, we define the skein module $K(M)$ to be the module over the Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ generated by isotopy classes of banded links in M modulo the so-called *skein relations* – see Fig. 7.

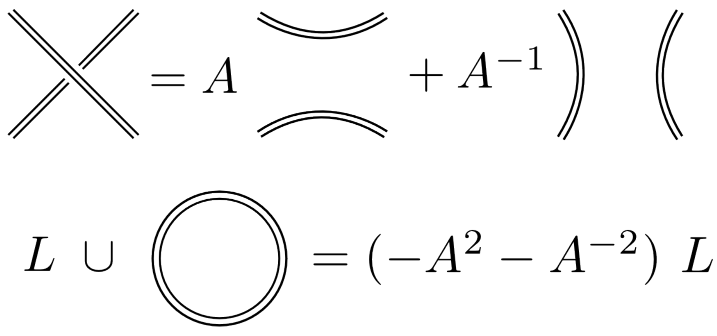

Figure 7: The skein relations.

For example, for a link in S^3, its corresponding element of $K(S^3)$ is nothing but the Kauffman bracket of the link. Denote by $K_\xi(M)$ the vector space obtained from $K(M)$ by the homomorphism $A \mapsto \xi$ for a non-zero complex number ξ. Assume in the following that ξ is a primitive root of unity of order $4k+8$. In this case, $K_\xi(S^3)$ is isomorphic to $\mathbb{C}$. Let Σ be a surface embedded into S^3, such that its complement is a union of two handle-bodies H and H'. We define a symmetric bilinear form

$$\langle \, , \, \rangle : K_\xi(H) \times K_\xi(H') \to K(S^3) = \mathbb{C}$$

on generators as follows: If $x \in K_\xi(H)$, $x' \in K_\xi(H')$ represent links L, L' in H, H' then $\langle x, x' \rangle$ is given by the value of $L \cup L'$ in $K(S^3)$, considering H and H' as sitting in S^3. Taking the quotient by the left kernel in $K_\xi(H)$ and right kernel in $K_\xi(H')$ we obtain a non-degenerate pairing

$$\langle \, , \, \rangle : V_k(\Sigma) \times V_k'(\Sigma) \to \mathbb{C}.$$

It turns out that the $V_k(\Sigma)$ are finite-dimensional vector spaces. These will be the vector spaces the mapping class group of Σ acts on.

We now proceed to describe the action of the Dehn twists on $V_k(\Sigma)$. Let K denote the set of Dehn twists about curves in Σ bounding a disc in H. These extend in a unique way to diffeomorphisms of H, giving rise to an action by such Dehn twists on $K_\xi(H)$ preserving the left kernel of the above form. Therefore, the group generated by the Dehn twists act on $V_k(\Sigma)$. Denote this action by ρ_k. To describe the action of any element of the mapping class group, it now suffices to describe the action by Dehn twists about curves in Σ bounding a disc in H', since $K \cup K'$ generate the mapping class group. For an element $f' \in K'$ define $\rho_k(f')$ by

$$\langle \rho_k(f')(x), y \rangle = \langle x, (f')^{-1}(y) \rangle,$$

for $x \in V_k(\Sigma)$, $y' \in V_k'(\Sigma)$. Since the form is non-degenerate, this determines ρ_k on the group generated by K'.

An element of the mapping class group could be written as a word in Dehn twists in more than one way, and for a mapping class $\phi = f_1 \dots f_n$, $f_i \in K \cup K'$, one should verify that $\rho_k(\phi) := \rho_k(f_1) \cdots \rho_k(f_n)$, is well-defined – at least up to a scalar factor, giving rise once again to a projective representation. An argument for this is given in [Mas03] and [Rob94].

3.3.3 Geometric quantization approach

Our final approach is based on ideas from geometric quantization. Let in the following $\Sigma = \Sigma_{g,0}$ and, for simplicity, $G = SU(2)$ – as before generalizations to other groups exist. The space of irreducible representations of the fundamental group considered up to conjugation,

$$M' = \mathrm{Hom}(\pi_1(\Sigma), G)^{\mathrm{irr}}/G,$$

is a smooth symplectic manifold of dimension $6g - 6$. It can be identified with the space of irreducible flat connections on the trivial G-bundle $P = \Sigma \times G$ considered up to gauge equivalence. For a flat connection A, the tangent space $T_{[A]}M'$ is isomorphic to the Lie algebra valued 1-forms on Σ, denoted $\Omega^1(\Sigma; \mathfrak{g})$. For two such 1-forms α_1, α_2, the symplectic form on M' is given by

$$\omega(\alpha_1, \alpha_2) = -\int_\Sigma \mathrm{tr}(\alpha_1 \wedge \alpha_2).$$

Normalizing the trace appropriately, one can consider ω as the curvature form of a certain connection on a line bundle $\mathcal{L}$ on M' with first Chern class $c_1(\mathcal{L}) = [\omega]/2\pi$. Fixing a complex structure σ on Σ, M' gets the structure of a complex manifold M'_σ. Its Picard group is isomorphic to $\mathbb{Z}$, generated by a holomorphic line bundle $\mathcal{L}_\sigma$ whose underlying smooth line bundle is $\mathcal{L}$. Multiplying ω by a positive integer k corresponds to replacing $\mathcal{L}_\sigma$ by $\mathcal{L}_\sigma^{\otimes k}$.

Let $V_k(\Sigma, \sigma) = H^0(M'_\sigma, \mathcal{L}_\sigma^{\otimes k})$ be the space of holomorphic sections of $\mathcal{L}_\sigma^{\otimes k}$. This is the vector space we will use for our quantum representations, but there is a technical problem: The map induced by a homeomorphism $f : \Sigma \to \Sigma$ might not preserve the complex structure in question, and a priori we only obtain a map $f : V_k(\Sigma, \sigma) \to V_k(\Sigma, f^*(\sigma))$. We thus need to compare these two vector spaces. Such a comparison can be obtained as follows: Denote by $\mathcal{T}_g$ the space of complex structures on Σ, called Teichmüller space – in Teichmüller space, two complex structures are identified, if there exists a biholomorphic map between them, isotopic to the identity. The $V_k(\Sigma, \sigma)$ form a vector bundle $\mathcal{V}_k(\Sigma)$ over $\mathcal{T}_g$ called the Verlinde bundle. There exists on this bundle a projectively flat connection, called the Hitchin connection, giving rise to linear isomorphisms

$$P_{\sigma_1, \sigma_2} : V_k(\Sigma, \sigma_1) \to V_k(\Sigma, \sigma_2)$$

by parallel transport along any path between σ_1 and σ_2 in $\mathcal{T}_g$. That the connection is projectively flat means that the above parallel transport operator is independent of the path only up to multiplication by a non-zero scalar. Fixing a complex structure σ, we now define a projective representation

$$\rho_k(f) = P_{f^*(\sigma), \sigma} \circ f^*$$

of the mapping class group on $V_k(\Sigma, \sigma)$.

3.4 Properties and conjectures

It is known that the quantum representations coming from quantum groups and skein theory are equivalent, but it is still an open question whether or not they are equivalent to the ones coming from the geometric approach. They do, however, have many properties in common, suggesting that this is indeed the case.

First off, the spaces V_k in either approach have the same dimensions

$$d_g(k) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g}.$$

This is the so-called Verlinde formula. Note that $d_g(k)$ approaches infinity, when k goes to infinity (in what is sometimes called the semi-classical limit), suggesting that the representations should capture more and more information about the representations, as the level grows. In either approach, it is known that at any given level k, certain powers (depending only on k) of Dehn twists are in the kernels of the representations (meaning that they are mapped to a scalar multiple of the identity), meaning that none of the individual quantum representations are faithful. But in fact, the collection of all quantum representations completely determine the mapping class group when $g > 2$; we have the following:

Theorem 4 (Asymptotic faithfulness)**.** *We have*

$$\bigcap_{k=1}^{\infty} \ker(\rho_k) = \begin{cases} \{1, H\} & \text{if } g = 1,2 \\ \{1\} & \text{otherwise} \end{cases},$$

where H is hyperelliptic involution.

This was proven in the geometric case in [And06a], in the skein theory case in [FWW02], and more recently in [MN08]. It is natural to state the following conjecture.

Conjecture 5. For a given k, the kernel of ρ_k is generated by the above mentioned powers of Dehn twists.

Another natural question is whether or not one is able to extract information about the Nielsen–Thurston classification of mapping classes; the conjecture is the following.

Conjecture 6 (AMU)**.** Assume $\chi(\Sigma) < 0$. If ϕ is pseudo-Anosov, then there exists k_0 such that $\rho_k(\phi)$ has infinite order for $k \geq k_0$. Furthermore, the quantum representations determine the stretching factors for pseudo-Anosov mapping classes.

See for example [AMU06] for an analysis of $M_{0,4}$.

References

[AMU06] J. E. Andersen, G. Masbaum, and K. Ueno. Topological quantum field theory and the Nielsen-Thurston classification of $M(0,4)$. *Math. Proc. Cambridge Philos. Soc.*, 141 no. 3:477–488, 2006.

[And06a] J. E. Andersen. Asymptotic faithfulness of the $SU(n)$ representations. *Annals of Mathematics*, 163:347–368, 2006.

[And06b] J. E. Andersen. The Nielsen-Thurston classification of mapping classes is determined by TQFT. *To appear in Journal of Mathematics of Kyoto University*, 2006.

[Ati88] M. Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel. Topological quantum field theories derived from the kauffman bracket. *Topology*, 34:883–927, 1995.

[FLP79] A. Fathi, F. Laudenbach, and V. Poénaru. Travaux des thurston sur les surfaces. *Astérisque*, 66-67, 1979.

[FWW02] M. Freedman, K. Walker, and Z. Wang. Quantum $SU(2)$ faithfully detects mapping class groups modulo center. *Geom. Topol.*, 6:523–539, 2002.

[Hum79] S. P. Humpries. Generators for the mapping class group. *Topology of Low-Dimensional Manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, Lecture Notes in Math., Vol. 722 (1979), Springer:44–47, 1979.

[Mas03] G. Masbaum. Quantum representations of mapping class groups. *Groupes et Géométrie. Journée anuelle 2003 de la SMF*, pages 19–36, 2003.

[MN08] J. Marché and M. Narimannejad. Some asymptotics of TQFT via skein theory. *Duke Math. J*, 141, Number 3:573–587, 2008.

[Pen88] R. Penner. A Construction of Pseudo-Anosov Homeomorphisms. *Transactions of the American Mathematical Society*, 310:179–197, 1988.

[Rob94] J. Roberts. Skeins and mapping classes. *Math. Proc. Camb. Phil. Soc.*, 115:53–77, 1994.

[RT90] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Comm. Math. Phys*, 127:1–26, 1990.

[RT91] N. Yu. Reshetikhin and V. G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Invent. Math.*, 103:547–598, 1991.

[Tur94] V. G. Turaev. *Quantum Invariants of Knots and 3-Manifolds.* Walter de Gruyter & Co., Berlin, 1994.

[Wit88] E. Witten. Topological quantum field theory. *Comm. Math. Phys.*, 117:353–386, 1988.